

Dumitru Baleanu
Ziya Burhanettin Güvenç
J.A. Tenreiro Machado
Editors

New Trends in Nanotechnology and Fractional Calculus Applications

 Springer

New Trends in Nanotechnology and Fractional Calculus Applications

New Trends in Nanotechnology and Fractional Calculus Applications

Edited by

D. BALEANU

*Çankaya University,
Balgat-Ankara, Turkey*

Z.B. GÜVENÇ

*Çankaya University,
Balgat-Ankara, Turkey*

and

J.A. TENREIRO MACHADO

*Institute of Engineering of Porto,
Porto, Portugal*



Springer

Editors

Dumitru Baleanu
Çankaya University
Fac. Art and Sciences
Oğretmenler Cad. 14
06530 Ankara
Yüzüncü Yıl, Balgat
Turkey
dumitru@cankaya.edu.tr

J.A. Tenreiro Machado
Institute of Engineering
of the Polytechnic Institute of Porto
Dept. Electrotechnical Engineering
Rua Dr. Antonio Bernardino de Almeida
4200-072 Postage
Portugal
jtm@isep.ipp.pt

Ziya B. Güvenç
Çankaya University
Fac. Engineering & Architecture
Oğretmenler Cad. 14
06530 Ankara
Yüzüncü Yıl, Balgat
Turkey
guvenc@cankaya.edu.tr

ISBN 978-90-481-3292-8 e-ISBN 978-90-481-3293-5

DOI 10.1007/978-90-481-3293-5

Springer Dordrecht Heidelberg London New York

Library of Congress Control Number: 2009942132

©Springer Science+Business Media B.V. 2010

No part of this work may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission from the Publisher, with the exception of any material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work.

Cover design: eStudio Calamar S.L.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

By the beginning of November 2008, the International Workshops on New Trends in Science and Technology (NTST 08) and Fractional Differentiation and its Applications (FDA08) were held at Çankaya University, Ankara, Turkey. These events provided a place to exchange recent developments and progresses in several emerging scientific areas, namely nanoscience, nonlinear science and complexity, symmetries and integrability, and application of fractional calculus in science, engineering, economics and finance.

The organizing committees have invited presentations from experts representing the international community of scholars and welcomed contributions from the growing number of researchers who are applying these tools to solve complex technical problems. Unlike the more established techniques of physics and engineering, the new methods are still under development and modern work is proceeding by both expanding the capabilities of these approaches and by widening their range of applications. Hence, the interested reader will find papers here that focus on the underlying mathematics and physics that extend the ideas into new domains, and that apply well established methods to experimental and to theoretical problems.

This book contains some of the contributions that were presented at NTST08 and FDA08 and, after being carefully selected and peer-reviewed, were expanded and grouped into five main sections entitled “New Trends in Nanotechnology”, “Techniques and Applications”, “Mathematical Tools”, “Fractional Modelling” and “Fractional Control Systems”.

The selection of improved papers for publication in this book reflects the success of the workshops, with the emergence of a variety of novel areas of applications. Bearing these ideas in mind the guest editors would like to honor many distinguished scientists that have promoted the development of nanoscience and fractional calculus and, in particular, Prof. George M. Zaslavsky that supported early this special issue and passed away recently.

The organizing committees wish to express their thanks to Cem Ozdogan, Adnan Bilgen, Ozlem Defterli, Burcin Tuna, Nazmi Battal as well as to our students for their assistance.

The Editors would like to thank to Ozlem Defterli for helping in preparation of this book.

The organizing committees wish to thank the sponsors and supporters of NTST08 and FDA08, namely Çankaya University represented by the President of the Board of Trustees Sıtkı Alp, the Rector Professor Ziya B. Güvenç, TUBITAK (The Scientific and Technological Research Council of Turkey), and the IFAC, for providing the resources needed to hold this conference, the invited speakers for sharing their expertise and knowledge, and the participants for their enthusiastic contributions to the discussions and debates.

Ankara
March 31, 2009

Dumitru Baleanu
Ziya B. Güvenç
J.A. Tenreiro Machado

Contents

Part I New Trends in Nanotechnology

Novel Molecular Diodes Developed by Chemical Conjugation of Carbon Nanotubes with Peptide Nucleic Acid	3
Krishna V. Singh, Miroslav Penchev, Xiaoye Jing, Alfredo A. Martinez–Morales, Cengiz S. Ozkan, and Mihri Ozkan	
Hybrid Single Walled Carbon Nanotube FETs for High Fidelity DNA Detection	17
Xu Wang, Mihri Ozkan, Gurer Budak, Ziya B. Güvenç, and Cengiz S. Ozkan	
Towards Integrated Nanoelectronic and Photonic Devices	25
Alexander Quandt, Maurizio Ferrari, and Giancarlo C. Righini	
New Noninvasive Methods for ‘Reading’ of Random Sequences and Their Applications in Nanotechnology	43
Raoul R. Nigmatullin	
Quantum Confinement in Nanometric Structures	57
Magdalena L. Ciurea and Vladimir Iancu	

Part II Techniques and Applications

Air-Fuel Ratio Control of an Internal Combustion Engine Using CRONE Control Extended to LPV Systems	71
Mathieu Moze, Jocelyn Sabatier, and Alain Oustaloup	
Non Integer Order Operators Implementation via Switched Capacitors Technology	87
Riccardo Caponetto, Giovanni Dongola, Luigi Fortuna, and Antonio Gallo	

Analysis of the Fractional Dynamics of an Ultracapacitor and Its Application to a Buck-Boost Converter	97
A. Parreño, P. Roncero-Sánchez, X. del Toro García, V. Feliu, and F. Castillo	
Approximation of a Fractance by a Network of Four Identical RC Cells Arranged in Gamma and a Purely Capacitive Cell	107
Xavier Moreau, Firas Khemane, Rachid Malti, and Pascal Serrier	
Part III Mathematical Tools	
On Deterministic Fractional Models	123
Margarita Rivero, Juan J. Trujillo, and M. Pilar Velasco	
A New Approach for Stability Analysis of Linear Discrete-Time Fractional-Order Systems	151
Said Guermah, Said Djennoune, and Maamar Bettayeb	
Stability of Fractional-Delay Systems: A Practical Approach	163
Farshad Merrikh-Bayat	
Comparing Numerical Methods for Solving Nonlinear Fractional Order Differential Equations	171
Farhad Farokhi, Mohammad Haeri, and Mohammad Saleh Tavazoei	
Fractional-Order Backward-Difference Definition Formula Analysis	181
Piotr Ostalczyk	
Fractional Differential Equations on Algebroids and Fractional Algebroids	193
Oana Chiş, Ioan Despi, and Dumitru Opreş	
Generalized Hankel Transform and Fractional Integrals on the Spaces of Generalized Functions	203
Kuldeep Singh Gehlot and Dinesh N. Vyas	
Some Bounds on Maximum Number of Frequencies Existing in Oscillations Produced by Linear Fractional Order Systems	213
Sadeh Bolouki, Mohammad Haeri, Mohammad Saleh Tavazoei, and Milad Siامي	
Fractional Derivatives with Fuzzy Exponent	221
Witold Kosiński	
Game Problems for Fractional-Order Systems	233
Arkadii Chikrii and Ivan Matychyn	

Synchronization Analysis of Two Networks	243
Changpin Li and Weigang Sun	
Part IV Fractional Modelling	
Modeling Ultracapacitors as Fractional-Order Systems	257
Yang Wang, Tom T. Hartley, Carl F. Lorenzo, Jay L. Adams, Joan E. Carletta, and Robert J. Veillette	
IPMC Actuators Non Integer Order Models	263
Riccardo Caponetto, Giovanni Dongola, Luigi Fortuna, Antonio Gallo, and Salvatore Graziani	
On the Implementation of a Limited Frequency Band Integrator and Application to Energetic Material Ignition Prediction	273
Jocelyn Sabatier, Mathieu Merveillaut, Alain Oustaloup, Cyril Gruau, and Hervé Trumel	
Fractional Order Model of Beam Heating Process and Its Experimental Verification	287
Andrzej Dzieliński and Dominik Sierociuk	
Analytical Design Method for Fractional Order Controller Using Fractional Reference Model	295
Badreddine Boudjehem, Djalil Boudjehem, and Hicham Tebbikh	
On Observability of Nonlinear Discrete-Time Fractional-Order Control Systems	305
Dorota Mozyrska and Zbigniew Bartosiewicz	
Chaotic Fractional Order Delayed Cellular Neural Network	313
Vedat Çelik and Yakup Demir	
Fractional Wavelet Transform for the Quantitative Spectral Analysis of Two-Component System	321
Murat Kanbur, Ibrahim Narin, Esra Özdemir, Erdal Dinç, and Dumitru Baleanu	
Fractional Wavelet Transform and Chemometric Calibrations for the Simultaneous Determination of Amlodipine and Valsartan in Their Complex Mixture	333
Mustafa Çelebier, Sacide Altınöz, and Erdal Dinç	

Part V Fractional Control Systems

Analytical Impulse Response of Third Generation CRONE Control	343
Rim Jallouli-Khlif, Pierre Melchior, F. Levron, Nabil Derbel, and Alain Oustaloup	
Stability Analysis of Fractional Order Universal Adaptive Stabilization	357
Yan Li and YangQuan Chen	
Position and Velocity Control of a Servo by Using GPC of Arbitrary Real Order	369
Miguel Romero Hortelano, Inés Tejado Balsera, Blas Manuel Vinagre Jara, and Ángel Pérez de Madrid y Pablo	
Decentralized CRONE Control of $m \times n$ Multivariable System with Time-Delay	377
Dominique Nelson-Gruel, Patrick Lanusse, and Alain Oustaloup	
Fractional Order Adaptive Control for Cogging Effect Compensation	393
Ying Luo, YangQuan Chen, and Hyo-Sung Ahn	
Generalized Predictive Control of Arbitrary Real Order	411
Miguel Romero Hortelano, Ángel Pérez de Madrid y Pablo, Carolina Mañoso Hierro, and Roberto Hernández Berlinches	
Frequency Response Based CACSD for Fractional Order Systems	419
Robin De Keyser, Clara Ionescu, and Corneliu Lazar	
Resonance and Stability Conditions for Fractional Transfer Functions of the Second Kind	429
Rachid Malti, Xavier Moreau, and Firas Khemani	
Synchronization of Fractional-Order Chaotic System via Adaptive PID Controller	445
Mohammad Mahmoudian, Reza Ghaderi, Abolfazl Ranjbar, Jalil Sadati, Seyed Hassan Hosseinnia, and Shaher Momani	
On Fractional Control Strategy for Four-Wheel-Steering Vehicle	453
Ning Chen, Nan Chen, and Ye Chen	
Fractional Order Sliding Mode Controller Design for Fractional Order Dynamic Systems	463
Mehmet Önder Efe	

A Fractional Order Adaptation Law for Integer Order Sliding Mode Control of a 2DOF Robot 471
Mehmet Önder Efe

Synchronization of Chaotic Nonlinear Gyros Using Fractional Order Controller 479
Hadi Delavari, Reza Ghaderi, Abolfazl Ranjbar, and Shaher Momani

Nyquist Envelope of Fractional Order Transfer Functions with Parametric Uncertainty 487
Nusret Tan, M. Mine Ozyetkin, and Celaledin Yeroglu

Synchronization of Gyro Systems via Fractional-Order Adaptive Controller 495
Seyed Hassan Hosseinnia, Reza Ghaderi, Abolfazl Ranjbar, Jalil Sadati, and Shaher Momani

Controllability and Minimum Energy Control Problem of Fractional Discrete-Time Systems 503
Jerzy Klamka

Control of Chaos via Fractional-Order State Feedback Controller 511
Seyed Hassan Hosseinnia, Reza Ghaderi, Abolfazl Ranjbar, Farzad Abdous, and Shaher Momani

Index 521

Part I
New Trends in Nanotechnology

Novel Molecular Diodes Developed by Chemical Conjugation of Carbon Nanotubes with Peptide Nucleic Acid

Krishna V. Singh, Miroslav Penchev, Xiaoye Jing,
Alfredo A. Martinez–Morales, Cengiz S. Ozkan, and Mihri Ozkan

Abstract In this work single walled carbon nanotube (SWNT)-peptide nucleic acid (PNA) conjugates are synthesized and their electrical properties are characterized. Metal contacts to SWNT-PNA-SWNT conjugates, used for current–voltage (I – V) measurements, are fabricated by two different methods: direct placement on pre-patterned gold electrodes and metal deposition using focused ion beam (FIB). Back-gated I – V measurements are used to determine the electronic properties of these conjugates. Additionally, conductive atomic force microscopy (C-AFM) is used to characterize the intrinsic charge transport characteristics of individual PNA clusters.

As electronic devices scale down, traditional lithography-based fabrication methods face unprecedented challenges more than ever before [1,2]. The need for novel bottom up techniques to get over the hurdle posed by downscaling is getting increasingly urgent [3–5]. Molecular electronics, based on the unique self-assembly capabilities of molecules, exemplifies the idea of bottom-up fabrication approach [6,7]. Therefore, the study of the electrical properties of single molecular components, can serve as a starting point for the study and realization of molecular electronics. Carbon nanotubes (CNTs) based bioconjugates are a suitable candidate for molecular electronics as they incorporate the excellent electrical and structural properties of CNTs [8,9] and the self assembly properties of bio-molecules [10–12]. In our previous work, we have synthesized single walled carbon nanotube (SWNT)-peptide nucleic acid (PNA) conjugates [13]. The main aim behind this work is to test these conjugates for their future use in molecular electronics applications.

The as-synthesized conjugates have the following structure: two SWNT ropes joined by a PNA cluster, where PNA acts as a linker to bring two SWNT ropes

K.V. Singh

Department of Chemical and Environmental Engineering, University of California, Riverside, CA 92521

M. Penchev, X. Jing, A.A. Martinez–Morales, and M. Ozkan (✉)

Department of Electrical Engineering, University of California, Riverside, CA 92521
e-mail: cozkan@engr.ucr.edu; mihri@ee.ucr.edu

C.S. Ozkan

Department of Mechanical Engineering, University of California, Riverside, CA 92521

together. Due to their unique structure these conjugates can serve a twofolded purpose. On one hand, they can be used to develop CNT based molecular devices as SWNTs are functionalized and conjugated with a molecule. On the other hand, CNTs can act as electrodes to electrically characterize and test the functionality of PNA. In fact, till date there is no report on electrical transport through PNA. Using this approach of conjugating SWNTs with PNA, provides us with a tool to test for such electrical characteristics. Hence this work also reports the use of single-walled carbon nanotubes (SWNTs) as a wiring alternative for molecular-scale devices. The appropriate nanometer dimensions, chemical and mechanical stability, and high carrier mobility make SWNTs an ideal candidate for the same [14]. Due to these advantages provided by SWNTs as components for molecular devices, lots of advances have been made to incorporate them into molecular device platform [15–17]. These include the development of high quality nanotube syntheses and integrated molecular-SWNT chemical and biological sensors [18]. The biggest challenge in using SWNTs as wires for molecular circuits is to engineer synthesis techniques of combining molecules with SWNTs in a way that it will not affect the intrinsic electrical transport properties of SWNTs. This work also overcome this challenge by optimizing the functionalization of SWNTs which result in predominant end oxidation and hence incorporation of PNA molecules at the tip of tubes [13].

The major challenge in electrically characterizing these conjugates was fabricating electrodes/contacts to measure their electrical transport. Two different techniques: direct placement on pre-patterned gold electrodes and focused ion beam (FIB) were utilized according to the available resources and technology to develop these contacts. In addition, individual PNA clusters were also characterized by conductive atomic force microscopy (C-AFM). The electrical transport results present very interesting phenomena for these conjugates. The conjugates have asymmetrical electrical transport, allowing current to flow only in one direction, at room temperature which corresponds to diodic or rectifying behavior. In addition some conjugates also show characteristics of negative differential resistance (NDR) [19]. In this work, back-gated measurements on conjugates were also performed, allowing us to determine the transconductance and mobility of the conjugates. Therefore, this work presents electrical properties of novel SWNT-PNA-SWNT conjugates and in addition also comments on the conductivity of PNA.

The synthesis route for producing these conjugates is given in detail in our previous report [13]. Differently to our previous work, here we have used highly pure HiPCO SWNTs [20] to increase the reliability of electrical transport results as the SWNTs conduct through their surface [21]. Due to decrease in the impurities in SWNT structure, which contribute towards faster oxidation of SWNTs, we have to modify oxidation conditions. The new optimized oxidation conditions for predominantly end functionalization (as required) [13] of SWNTs are 14 h of acid reflux in 2.4 M of HNO_3 . Increase in oxidation time and also the strength of acid used in this work (previously 12 h and 1 M HNO_3) is a strong indicative of the purity of SWNTs employed in the synthesis of these SWNT-PNA conjugates. After oxidation and subsequent sonication of SWNTs, SWNT bearing NHS esters were prepared by coupling with EDC and NHS [13]. Both end functionalization of PNA (AcLys-GTGCTCATGGTG-Lys-NH₂) led to formation of SWNT-PNA-SWNT conjugates

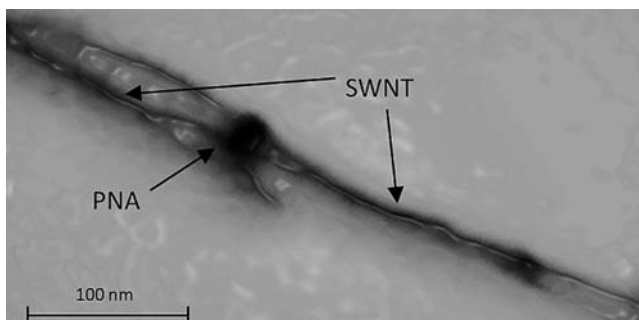


Fig. 1 SEM micrograph of a SWNT-PNA-SWNT conjugate

as an amide bond is formed between the amine of the amino-acid residue on the PNA backbone and SWNT-bearing NHS esters [13]. A typical scanning electron microscopy (SEM) image of a SWNT-PNA-SWNT conjugate is shown in Fig. 1. In this work we also modified the amino acid residue on the PNA backbone to Lysine to improve the solubility of PNA in water.

After synthesis of the SWNT-PNA-SWNT conjugates, electrical contacts were fabricated at the ends of individual conjugates by the following methods. The first method consists of a direct placement on pre-patterned gold electrodes. One block of four gold electrodes was patterned on Si/SiO₂ chips. The structure of one electrode consist of a large square pad ($L \sim 125 \mu\text{m}$) which is connected to a long metal strip approximately $80 \mu\text{m}$ long. In one block there were four such electrodes and in the center of the block the separation between the metal strips is around $1 \mu\text{m}$. On one single chip there were 289 such blocks. In the direct placement method the conjugates are deposited by drop casting; bridging across the metal strips due to the length of SWNTs. After locating the connected strips on a particular block, the electrical measurements are done by connecting the bigger pads of the corresponding metal strips to external probes (tip diameter $\sim 1 \mu\text{m}$) in a probe station (Signatone). Using an Agilent 4155 C semiconductor parameter analyzer the $I-V$ characteristics of these conjugates were obtained.

The major advantage of this method is the simplicity and less time consumption in preparing the sample for electrical characterization. But the major drawback is that this method works on “hit and trial” basis and locating a single conjugate connected across two metal strips is a time consuming step. In addition, the contact between the conjugate and the electrode is not necessarily good (as the conjugate is sitting on top of the electrode) and can create artifacts during the measurements. Sometimes it is also possible that whole chip does not have the required connection or electrodes are not connected by the right conjugates.

The second method used for fabricating the contacts employs the use of focused ion beam (FIB). It consists of an electron beam (SEM) as well as an ion beam (Gallium ions). This technique provides us the opportunity to visualize the conjugates (by SEM) and develop the contacts directly on the conjugates by metal deposition assisted by the ion beam (Leo XB1540). The required conjugate is

located on the pre-patterned electrode system (as discussed above) by SEM. The measurements are made at the same time for the contacts. Then the deposition of metal (i.e. Platinum) takes place by the following procedure. A gas containing metal ions is introduced into the system and allowed to chemisorb onto the sample. By scanning an area with the ion beam, the precursor gas is decomposed into volatile and non-volatile components; the non-volatile component (platinum metal) remains on the surface as a deposition while the volatile component is vaporized. One major advantage of this system is that one can monitor the formation of contacts in real time under SEM.

This technique overcomes the disadvantages of less control, lack of precision and “hit and trial” approach of the previous technique discussed above. But this technique has its own set of issues, which mainly include the destruction of sample by ion beam and shifting (if the system is not calibrated precisely). In order to avoid damaging the SWNT-PNA conjugates, the following parameters were chosen: deposition current of $2\ \mu\text{A}$ and scanning frequency of $0.1\ \text{Hz}$, which worked well for our conjugates. In addition, this technique can also be used to repair the damaged electrodes after measurements and the same conjugate can be reused, which is not possible by the other technique. Moreover, destructive ion milling can also be used as means to isolate the conjugate from other materials. For this purpose currents higher than $50\ \mu\text{A}$ were used.

In order to report the first electrical conductivity measurements of PNA molecules, we prepared samples for C-AFM analysis (Fig. 3) by drop casting a solution of PNA ($100\ \mu\text{M}$ concentration) on an oxygen plasma cleaned n-type Si substrate. Oxygen plasma cleaning ensured the removal of any carbonaceous impurities as they might interfere with the final results since PNA is also carbonaceous in nature. During CAFM measurements a Pt/Ir coated AFM tip ($\sim 20\ \text{nm}$ radius of curvature) was used as a top contact to measure the current with respect to an applied bias voltage. The electrical measurements were taken by first performing a morphology scan in contact mode and then driving the tip by a point and shoot method to the top of a specific PNA cluster.

After contact fabrication the SWNT-PNA-SWNT conjugates were tested by different methods as described above. Most of the conjugates show asymmetrical current–voltage (I – V) characteristic. Most of which show a rectifying or diodic behavior. This behavior was independent of the method used to fabricate the contacts. Typical diodic behavior is shown in Fig. 2a, c. In addition some conjugates also show negative differential resistance, which is characteristic of resonance tunneling diode (RTD). Figure 2b, d represent the NDR characteristic of few conjugates. Control devices based on SWNT-only samples were also fabricated and the results are shown in Fig. 2e, f.

Additionally, the intrinsic charge transport characteristics of individual PNA clusters (Fig. 3 inset) were also studied by C-AFM measurements. As shown in Fig. 3, typical PNA current–voltage measurements at the nanoscale exhibit a rectifying behavior analogous to the I – V curves observed for the SWNT-PNA conjugates. For the negative tip bias voltages, a steep and exponential increase of the tunneling current occurs beyond a threshold voltage of $-6\ \text{V}$. The turn-on voltage

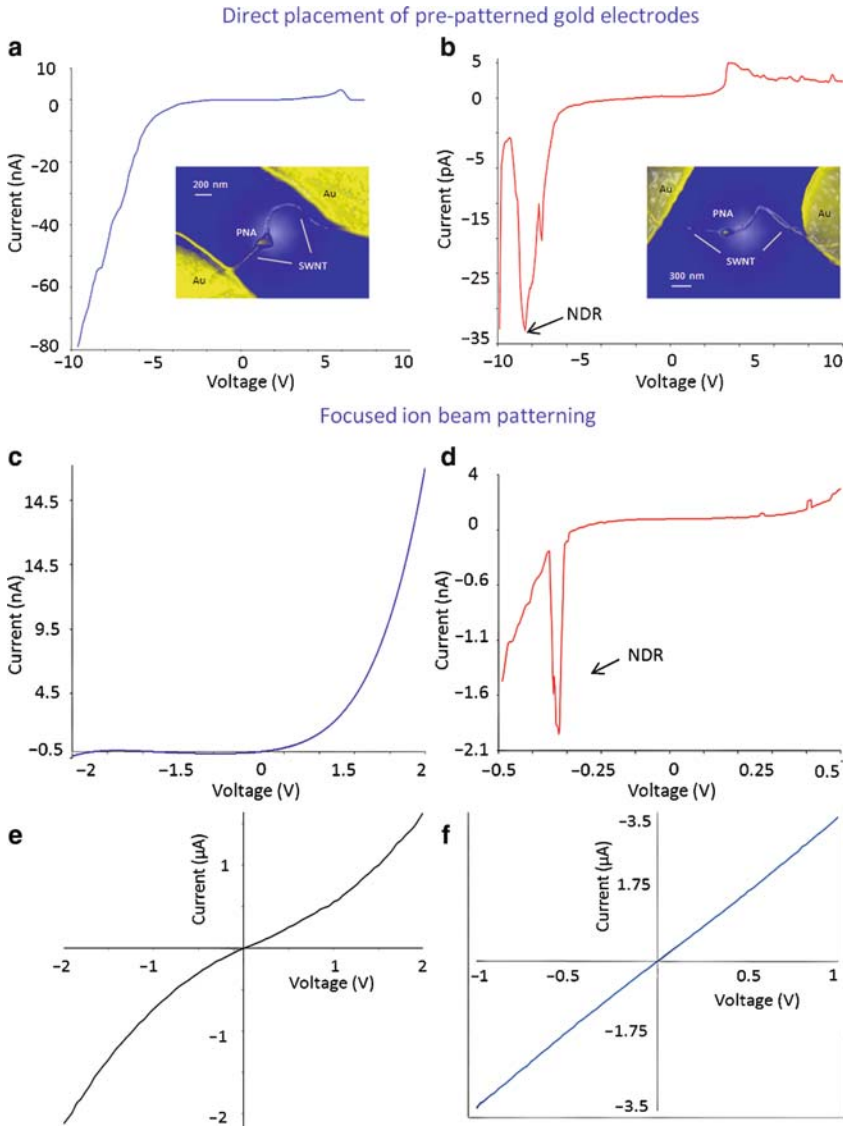


Fig. 2 Two terminal electrical characterization of SWNT-PNA-SWNT conjugates. (a) and (c) Diodic behavior is observed for both direct placement and focused ion beam (FIB) method. (b) and (d) Similarly, negative differential resistance behavior was observed in few conjugates for both methods. (e) and (f) SWNTs-only samples show symmetric behavior with high conductivity irrespective of method

observed in the PNA cluster is in good agreement with the measurements made on the SWNT-PNA conjugates (Fig. 2a). It is also interesting to point out that PNA shows extremely good current-blocking behavior under positive tip bias voltage of up to 10 V.

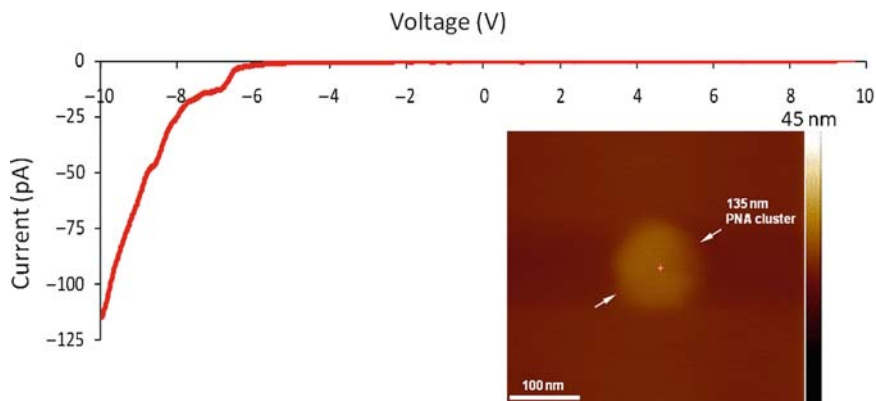


Fig. 3 Charge transport characterization by C-AFM. The $I-V$ curve shows a characteristic diodic behavior for a single PNA cluster. Inset: AFM topography image of a single PNA cluster

Field effect transistors (FETs) were fabricated on single conjugates by using a Si/SiO₂ substrate as the back-gate, as the back gate and insulator respectively, during the electrical measurements. A representative $I-V$ curve for these gated studies is represented in Fig. 4a showing that the SWNT-PNA-based FETs behave as ‘p’ type conjugates. Few conjugates did not show any change in conductivity on applying a gate voltage (Fig. 4b). To further test the electrical properties of our device structure control devices based on SWNT ropes alone (Fig. 4c, d) were also fabricated. The ropes which were semiconducting were found to be ‘p’ type while metallic ropes do not show any semiconducting behavior. The back-gated measurements were used to determine transconductance and mobility of the SWNT-PNA-SWNT conjugate FET device (Fig. 4e, f).

The diodic behavior observed in the SWNT-PNA-SWNT conjugates is not a new phenomenon in molecular electronics. In 1974 Aviram and Ratner proposed a molecule based rectifying behavior [22]. That work was one of the pioneers in the field of molecular electronics. Since then there have been numerous efforts to develop AR theory based rectifiers. In the literature, there are several molecules which have shown this rectifying behavior [23–25] but this kind of observation is made here for the first time for PNA. The mechanism for this rectifying behavior is explained in detail elsewhere [23–25]. In short, for an ideal AR molecular diode, the rectifying molecule has a D- σ -A structure, where D is a good electron donor, σ is the insulating bridge and A is the good electron acceptor. The rectifying behavior of the molecule is observed when this molecule is connected to the conductors (Conductor (C1)-Molecule (M)-Conductor (C2)) on both ends. The mechanism involves two molecular orbitals, the highest occupied molecular orbital (HOMO), mainly localized on D, which would be filled, and the lowest unoccupied molecular orbital (LUMO), mainly localized on A. Electrons transfer from one contact to the other contact by tunneling through the D- σ -A molecule which forms the preferentially excited electronic state. D⁺- σ -A⁻. Inelastic “downhill” tunneling within the molecule (involving either phonon emission or photon emission) then would reset

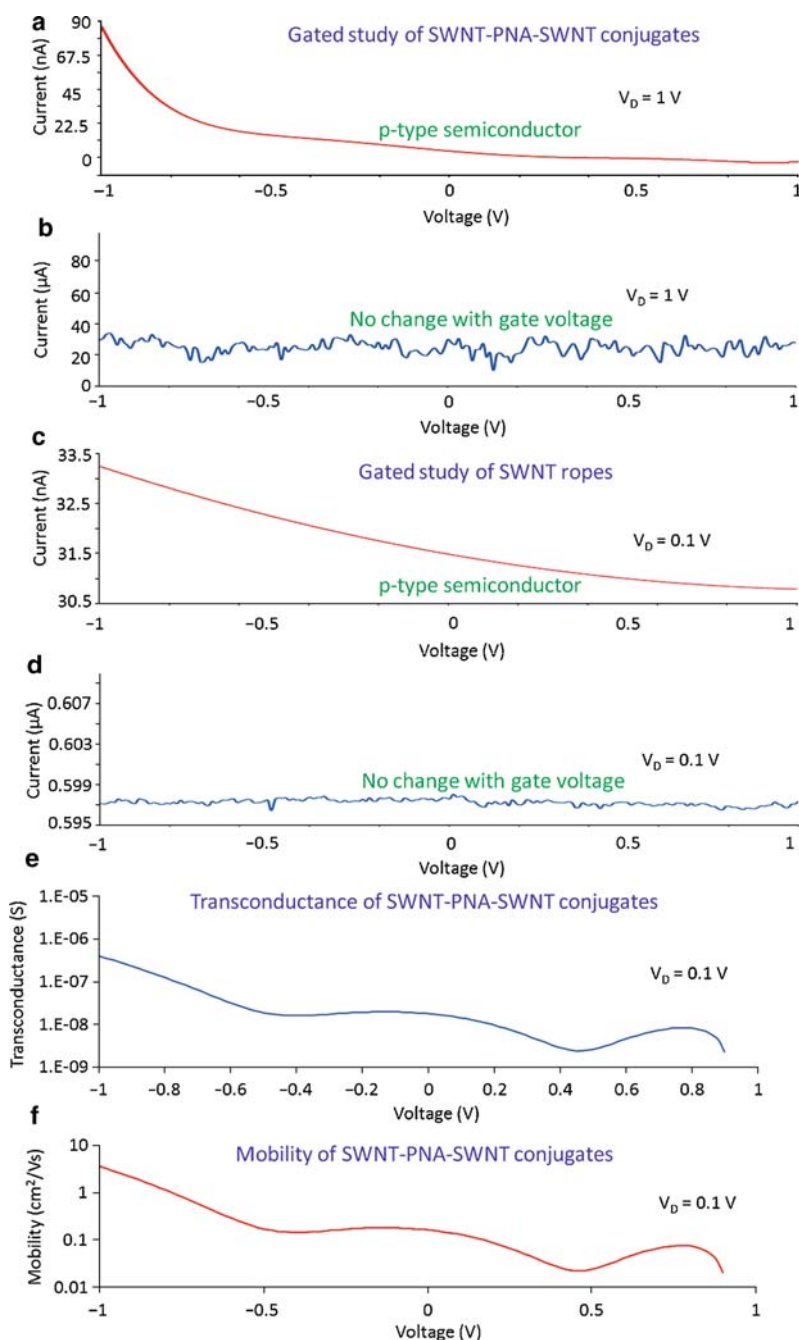


Fig. 4 FET characterization. (a) and (b) Gated study of SWNT-PNA-SWNT conjugates. The electrical behavior of the conjugates is modulated by the type of SWNTs connecting the conjugate. (c) and (d) SWNT ropes are shown to behave either as 'p' type semiconductors or metallic, respectively. (e) Transconductance of SWNT-PNA-SWNT conjugates. (f) Mobility of SWNT-PNA-SWNT conjugates

the ground state D- σ -A, but an electron would have been moved from metal electrode C1 to metal C2; hence the rectifying effect [11]. The proposed molecule was never synthesized, but helped in developing the theory behind the rectifying behavior of molecules in molecular electronics. The observance of diodic behavior is also due to the chemical structure of the molecule. When the relevant molecular energy is in resonance with the Fermi level of the metal electrode, there is a dramatic increase in the current through the molecule, and a dramatic selectivity of electron transport through the C1-M-C2 sandwich [26]. Hence when the molecular orbitals of PNA come to resonance with the molecular orbitals of SWNTs attached to them, there is an observed increase in the current. But this phenomenon is not reversible and the current can only be conducted in one direction only. Therefore, both the structure and contact of PNA with conductors (i.e. SWNTs) on both ends are responsible for the diodic behavior observed in the conjugates (Fig. 2a, c). Furthermore, the observance of diodic behavior of the PNA clusters in conductive AFM (Fig. 3) is also due to this C1-M-C2 sandwich. In this case the conductors are the AFM tip (metal) on the top and Silicon (semiconductor) on the bottom. This observation further supports the fact that the observed diodic behavior of SWNT-PNA-SWNT conjugates is not because of SWNTs contact with PNA but rather due to the PNA itself.

The exact mechanism of transfer between SWNT-PNA-SWNT will require extensive modeling based on molecular dynamics. Only then we can locate the various molecular orbitals in the conjugates and their behavior under an external electric field. But the mechanism explained above gives us a right start in this direction.

As far as NDR effect is concerned, there are many reports on observation of NDR in molecular electronics [27,28]. Many mechanisms have been proposed for the same but there is no consensus in the literature. In fact, we have previously observed a similar NDR behavior in our earlier work related to SWNT-DNA-SWNT conjugates [29]. Since the bonding between SWNT and DNA is analogous to the one between SWNT and PNA, we propose the following qualitative explanation for the observance of NDR effect in SWNT-PNA-SWNT conjugates (Fig. 5) [29]. At zero bias voltage, chains of SWNT-PNA-SWNT conjugates have uniform Fermi



Fig. 5 Schematic illustration of electrons transferring through energy barriers of PNA molecules

energy levels. When applied voltage increases, energy levels tilt and electrons start tunneling from the voltage source through the energy barriers of PNA molecules. Correspondently, current increases until the localized energy band inside quantum well shifts to below Fermi energy from the voltage source, leaving no corresponding energy levels for after-tunneling electrons to stay. As a result, current starts to decrease. As the applied voltage continues to increase, the higher unoccupied energy levels in PNA shift down to the energy level which are in alignment with the Fermi energy from the energy source and current starts to increase again. Since our conjugates consists of SWNT ropes formed by many intertwined tubes and in consequence of numerous PNA molecules, alignment and misalignment do not happen at the same voltage, it is reasonable that we get multiple current peaks for different SWNT-PNA conjugates.

In addition, Lake et al. [30] have postulated SWNT-pseudo peptide-SWNT nanostructure could exhibit RTD $I-V$ response via computations based on the density functional theory (DFT) and non-equilibrium Green function (NEGF) approach. Our results are in accordance with these theoretical and experimental analyses.

Control measurements were done on SWNT ropes alone with a two-fold purpose. Firstly, to differentiate the electrical characteristics obtained for the conjugates versus the electrical properties of SWNT ropes. Secondly, to indirectly prove that PNA is indeed joining two different SWNT ropes. Representative $I-V$ curves of SWNT ropes (Fig. 2e, f) show a symmetrical nature and higher conductivity for the ropes. The electrical characteristics clearly show that the ropes are fundamentally a different system from that of the conjugates.

The gated study presented in this work is the first of its kind for PNA based carbon nanotube conjugates. It was found that the conjugates were semiconducting as well as metallic (Fig. 4a, b). A control study was also performed on SWNT ropes alone (Fig. 4c, d). Few of the ropes were found to be metallic and some to be semiconducting, as expected. But the difference in the nature of SWNT-PNA-SWNT conjugates can also be explained on the basis of SWNTs. Since PNA is very small compared to SWNTs and also much less conductive than SWNTs (as shown in $I-V$ characteristics); the influence on total gated behavior will be modulated by the SWNTs of the conjugate. If PNA is attached to semiconducting SWNTs on both the ends, the conjugate will behave as semiconductor but if either or both of the SWNTs are metallic the conjugate will then behave as a metallic component.

Overall, the gated study confirms that PNA behaves as a hole conducting molecule. This study also confirms the theoretical model explained elsewhere for CNT-Peptide-CNT system [31]. Lake et al. modeled the peptide molecule and found out that peptide linker acts as a good bridge for hole transmission in the CNT valence band and strongly suppresses electron transmission in the CNT conduction band [31].

During the electrical characterization of these conjugates, the biggest challenge was to understand the difference in behavior observed among different conjugates. The reason for this variation could be result of the following three main factors: variation in number of SWNTs, variation of number of PNAs, and variation of the type of SWNTs in the conjugates.

The SWNTs used here are ropes and these ropes attach themselves to PNA molecules by covalent coupling as described above. But all the ropes are not of same diameter and hence do not contain the same number of tubes. Therefore, this variation in number of tubes will always be observed from one conjugate to another. This variation in number of tubes on both sides of a PNA cluster will also change the number of PNAs from one conjugate to another. But the number of PNAs can be estimated by the following methodology.

The number of PNA attached in one conjugate can be calculated from the diameter of the SWNT rope in that sample. Haddon et al. reported that the efficiency of the oxidation process for carbon nanotubes (tubes @ Rice are approximated for HiPCO tubes) is around 2% [32]. In this work we have preferentially oxidized the tips of SWNTs. Therefore, we can estimate the number of oxidized carbon atoms at the tip by this formula:

$$\begin{aligned}\emptyset &= \text{Number of oxidized carbon atoms in on tube} \\ &= \left\{ \frac{\pi * d_{\text{tube}}}{\text{Length of C-C bond in SWNT (nm)}} \right\} \times (\text{Efficiency of oxidation process})\end{aligned}$$

where, d_{tube} : Diameter of single tube (nm)

It is estimated that on an average in a SWNT rope of 20 nm diameter there are around 500 tubes [33]. To get the number of oxidized sites in a rope (Ω), we can multiply $\emptyset$ with rope correction factor Ψ (which gives the number of SWNTs in one rope of diameter d_{avg} .)

$$\text{where } \varphi = \left(\frac{d_{\text{avg}}}{20(\text{nm})} \right) \times 500$$

Therefore,

$$\Omega = \text{Number of oxidized carbon atoms in one rope} = \emptyset * \varphi$$

The efficiency of esterification (formation of SWNT-NHS esters) is nearly 100% as the intermediates are in excess. As per the chemistry, we also keep the amines (in our case PNA) in excess. Therefore, all the oxidation sites on the SWNT ropes will be utilized by PNA molecules. Since for one site we can only have one PNA molecule attached the number of PNA molecules attached will be equal to Ω .

A major challenge of using SWNTs in bulk or in solution is that it contains both metallic and semiconducting tubes/ropes. There is no easy way to separate them and utilize them separately. Our conjugates also suffer from this inherent disadvantage. In the conjugates, three types of configuration are possible; metallic (M)-PNA-M, semiconducting (SC)-PNA-SC and SC-PNA-M; will occur. In fact, this variation is clearly verified by the gated study of these conjugates. This configuration will affect the shape, position of NDR peaks and nature of the current–voltage response for SWNT-PNA-SWNT conjugates since the resonance of energy levels between SWNTs and PNA is responsible for the rectifying nature of these conjugates.

As discussed above, in addition to developing SWNT based devices, this structure could also served as a way of utilizing SWNTs as electrodes for the characterization of molecular structures. The most common method of testing molecules electrically is by Langmuir-Blodgett (LB) thin films [26, 34, 35]. In contrast to the LB thin film technique there are several advantages in using the architecture presented in this work for characterization of molecules. First, the electrical transport is confined to one dimension along the molecules, whereas in the thin film approach, conduction also takes place along the latitudinal direction as well. Second, the number of molecules attached is restricted by the coupling sites available on the SWNTs, permitting high accuracy in calculating the number of molecules attached. The number of functionalized sites in a rope/tube can be estimated (as explained above). Therefore, from the number of these sites the number of attached molecules can also be calculated. Third, CNTs themselves have exceptional electronic properties and also have excellent mechanical and chemical properties as well that could be useful for the characterization of the intrinsic properties of molecules.

In summary, we have synthesized single walled carbon nanotube (SWNT)-peptide nucleic acid (PNA) conjugates, which are characterized by several different techniques to determine their electrical properties. Our results demonstrate that the conjugates exhibit rectifying and negative differential resistance $I-V$ characteristics, making them ideal candidates for future electronic applications [36] as molecular diodes. Furthermore, the excellent structural and electrical properties of SWNTs enable us to use them as test electrodes in order to study the electrical and electronic properties of PNA cluster.

Acknowledgements We gratefully acknowledge financial support from the Nanomanufacturing Program of the National Science Foundation (NSF) (grant no: 0800680), the FCRP Center on Functional Engineered Nano Architectonics funded by the SRC and DARPA, and the Center for Hierarchical Manufacturing (CHM) funded by the NSF.

References

1. Lin BJ (2006) The ending of optical lithography and the prospects of its successors. *Micro-electronic Engineering* 83(4-9):604-613
2. Lin BJ (2006) Optical lithography—present and future challenges. *Comptes Rendus Physique* 7(8):858-874
3. Huang Y, Duan XF, Cui Y, Lauhon LJ, Kim KH, Lieber CM (2001) Logic gates and computation from assembled nanowire building blocks. *Science* 294(5545): 1313-1317
4. Cui Y, Lieber CM (2001) Functional nanoscale electronic devices assembled using silicon nanowire building blocks. *Science* 291(5505):851-853
5. Lu W, Lieber CM (2007) Nanoelectronics from the bottom up. *Nature Materials* 6(11):841-850
6. Fu L, Cao L, Liu Y, Zhu D (2004) Molecular and nanoscale materials and devices in electronics. *Advances in Colloid and Interface Science* 111(3):133-157
7. Balzani V, Credi A, Venturi, M (2003) Molecular logic circuits. *Chemphyschem* 4(1):49-59
8. Baughman RH, Zakhidov AA, de Heer WA (2002) Carbon nanotubes – the route toward applications. *Science* 297(5582):787-792

9. Wildoer JWG, Venema LC, Rinzler AG, Smalley RE, Dekker C (1998) Electronic structure of atomically resolved carbon nanotubes. *Nature* 391(6662): 59–62
10. Bashir R (2001) DNA-mediated artificial nanobiostructures: state of the art and future directions. *Superlattices and Microstructures* 29(1):1–16
11. Chakrabarti R, Klibanov AM (2003) Nanocrystals modified with peptide nucleic acids (PNAs) for selective self-assembly and DNA detection. *Journal of the American Chemical Society* 125(41):12531–12540
12. Vernille JP, Kovell LC, Schneider JW (2004) Peptide nucleic acid (PNA) amphiphiles: Synthesis, self-assembly, and duplex stability. *Bioconjugate Chemistry* 15(6):1314–1321
13. Singh KV, Pandey RR, Wang X, Lake R, Ozkan CS, Wang K, Ozkan M (2006) Covalent functionalization of single walled carbon nanotubes with peptide nucleic acid: Nanocomponents for molecular level electronics. *Carbon* 44(9):1730–1739
14. Star A, Han TR, Gabriel JCP, Bradley K, Gruner G (2003) Interaction of aromatic compounds with carbon nanotubes: Correlation to the Hammett parameter of the substituent and measured carbon nanotube FET response. *Nano Letters* 3(10):1421–1423
15. Kong J, Chapline MG, Dai HJ (2001) Functionalized carbon nanotubes for molecular hydrogen sensors. *Advanced Materials* 13(18):1384–1386
16. Lin Y, Taylor S, Li HP, Fernando KAS, Qu LW, Wang W, Gu LR, Zhou B, Sun YP (2004). Advances toward bioapplications of carbon nanotubes. *J Mat Chem* 14(4):527–541
17. Tasis D, Tagmatarchis N, Bianco A, Prato M (2006) Chemistry of carbon nanotubes. *Chemical Reviews* 106(3):1105–1136
18. Kong J, Soh HT, Cassell AM, Quate CF, Dai HJ (1998) Synthesis of individual single-walled carbon nanotubes on patterned silicon wafers. *Nature* 395(6705):878–881
19. Sze SM, Ng KK (2006) Tunnel Devices. *Physics of Semiconductor Devices*. 3rd edn. Wiley, Hoboken p 417
20. Nikolaev P (2004) Gas-phase production of single-walled carbon nanotubes from carbon monoxide: A review of the HiPCO process. *Journal of Nanoscience and Nanotechnology*. 4(4):307–316
21. Odom TW, Huang JL, Kim P, Lieber CM (1998) Atomic structure and electronic properties of single-walled carbon nanotubes. *Nature* 391(6662):62–64
22. Aviram A, Ratner MA (1974) Molecular Rectifiers. *Chem Phys Lett* 29(2):277–283
23. Metzger RM (1999) Electrical rectification by a molecule: The advent of unimolecular electronic devices. *Acc Chem Res*. 32(11):950–957
24. Metzger RM (2003) Unimolecular electrical rectifiers. *Chem Rev*. 103(9):3803–3834
25. Metzger RM (2004) Electrical rectification by monolayers of three molecules. *Macromolecular Symposia* 212(1):63–72
26. Metzger RM (2006) Unimolecular rectifiers: Methods and challenges. *Analytica Chimica Acta* 568(1–2):146–155
27. Chen J, Reed MA, Rawlett AM, Tour JM (1999) Large on-off ratios and negative differential resistance in a molecular electronic device. *Science* 286(5444):1550–1552
28. James DK, Tour JM (2006) Organic synthesis and device testing for molecular electronics. *Aldrichimica Acta* 39(2):47–56
29. Wang X, Liu F, Andavan GTS, Jing XY, Singh K, Yazdanpanah VR, Bruque N, Pandey RR, Lake R, Ozkan M, Wang KL, Ozkan CS (2006) Carbon Nanotube-DNA nanoarchitectures and electronic functionality. *Small* 2(11):1356–1365
30. Pandey RR, Bruque N, Alam K, Lake RK (2006) Carbon nanotube – molecular resonant tunneling diode. *Physica Status Solidia – Applications and Materials Science* 203(2):R5–R7
31. Bruque N, Pandey RR, Lake RK, Wang H, Lewis JP (2005) Electronic transport through a CNT-Pseudopeptide-CNT hybrid material. *Molecular Simulation* 31(12):859–864
32. Hu H, Bhowmik P, Zhao B, Hamon MA, Itkis ME, Haddon RC (2001) End-group and defect analysis of soluble single-walled carbon nanotubes. *Chem Phys Lett* 345(1–2):25–28
33. Thess A, Lee R, Nikolaev P, Dai H, Petit P, Robert J, Xu C, Lee YH, Kim SG, Rinzler AG, Colbert DT, Scuseria GE, Tomanek D, Fischer JE, Smalley RE (1996) Crystalline ropes of metallic carbon nanotubes. *Science* 273(5274):483–487

34. Metzger RM, Panetta CA (1989) Langmuir-blodgett films of potential donor sigma acceptor organic rectifiers. *J Mol Elect* 5(1):1–17
35. Okazaki N, Sambles JR, Jory MJ, Ashwell GJ (2002) Molecular rectification at 8 K in an Au/C(16)H(33)Q-3CNQ LB film/Au structure. *Applied Physics Letters* 81(12): 2300–2302
36. Kumar MJ (2007) Molecular Diodes and Applications. *Recent Patents on Nanotechnology* 1:51–57

Hybrid Single Walled Carbon Nanotube FETs for High Fidelity DNA Detection

Xu Wang, Mihri Ozkan, Gurer Budak, Ziya B. Güvenç, and Cengiz S. Ozkan

Abstract A novel application for detecting specific biomolecules using SWNT-ssDNA nanohybrid is described. SWNT-ssDNA hybrid is formed by conjugating amino-ended single strand of DNA (ssDNA) with carboxylic group modified SWNTs through a straightforward EDC coupling reaction. ssDNA functionalized SWNT hybrids could be used as high fidelity sensors for biomolecules. The sensing capability is demonstrated by the change in the electronic properties of SWNT. Employing DNA functionalized SWNT FETs could lead to dramatically increased sensitivity in biochemical sensing and medical diagnostics applications.

1 Introduction

Carbon nanotubes (CNT) have been utilized widely in nanoelectronic devices such as field effect transistors (FET) [1], single-electron transistors [2], rectifying diodes [3] and logic circuits [4] due to its unique mechanical, thermal and electrical properties. They are chemically inert and it is difficult to conduct synthetic chemical treatment on them because they are resistant to wetting and indissolvable in water and organic solvents. In order to expand their potential applications in biomedical and optoelectronic devices, surface functionalization strategies have been explored by many research groups within recent years. The attachment of chemical functional

X. Wang

Department of Chemical Engineering, University of California, Riverside, CA 92521

M. Ozkan

Department of Electrical Engineering, University of California, Riverside, CA 92521

e-mail: mihri@ee.ucr.edu

G. Budak

Nanomedicine Research Laboratory, Gazi University, Besevler, Ankara, Turkey 06510

Z.B. Güvenç

Electronic and Communication Engineering, Cankaya University, Ankara, Turkey 06530

e-mail: guven@cankaya.edu.tr

C.S. Ozkan (✉)

Department of Mechanical Engineering, University of California, Riverside, CA 92521

e-mail: cozkan@engr.ucr.edu

groups represents a strategy for overcoming the disadvantages of CNTs and has become attractive for synthetic chemists and materials scientists. Functionalization can improve CNTs solubility and processability, and will allow combination of unique properties of CNTs with those of other types of materials. The functionalization of CNTs can be divided into covalent and noncovalent types. Covalent functionalization is based on covalent linkage of functional entities onto CNTs ends and/or sidewall. Non-covalent functionalization is mainly based on the adsorption forces between functional entities and CNTs, such as van der Waals and π -stacking interaction. With the successful surface functionalization of CNTs, various strategies of forming CNT hybrids with chemicals, polymers, and biological species have been developed, including fluorination of nanotubes [5], chlorination of nanotubes [6], formation of carbon nanotube-acyl amides [7], and carbon nanotube-esters [8]. The integration of biomaterials, such as proteins, enzymes, antigens, antibodies, and nucleic acids with CNTs would combine the conductive or semiconductive properties of CNTs with recognition or catalytic properties of biomaterials. A number of researchers focus on DNA assemblies with CNTs because of the molecular recognition capability and high aspect ratio nanostructures. DNA has been utilized as scaffolding materials or fabrics with applications in electronics; such constructs include DNA lattices [9], grids [10], tiles [10], ribbons [10], tubes [10], and origami [11] for organizing components of electronics.

CNT-DNA complexes have been assembled via different methods. DNA's interaction with CNT through the physical binding has been explored. DNA's non-specific binding to CNT wall has been visualized by high resolution transmission electron microscopy [12].

DNA transport through a single MWNT cavity has been directly observed by fluorescence microscopy [13]. During the process, both Van der Waals and hydrophobic forces are found to be important, with the former playing a more dominant role on CNT-DNA interactions [14]. DNA interaction with CNT through chemical covalent binding has also been described [15]. The amide linkage is formed by the reaction of carboxylic groups on CNT with the amine groups of ssDNA in a solution. Such heterostructures indicated a negative differential resistance (NDR) effect indicating a biomimetic route to forming resonant tunneling diodes (RTD). CNT-DNA assemblies have been applied into detection of biomaterials and chemical species. Label free detection of DNA hybridization using carbon nanotube network field-effect transistors [16] has been demonstrated. DNA functionalized single wall carbon nanotubes for electrochemical detection has been reported [17]. Most of this prior work presents the sensing capability of CNT networks or CNT film structures. In this work, the detection of specific sequences of DNA using a single SWNT field effect transistor is described. SWNTs are purified and dispersed in *o*-dichlorobenzene (ortho-dichlorobenzene) solvent before functionalized by ssDNA. The functionalization is completed by forming an amide linkage between carboxylic groups of SWNT and amine groups of ssDNA via the EDC coupling method. Modified SWNT based biosensor in the configuration of a field effect transistor (FET) is fabricated using electron beam lithography (EBL). When specific sequences of ssDNA which are complementary to the ssDNA covalently bound on

the SWNT surface are exposed to the device, modulation of the current-voltage characteristics demonstrate the capability of SWNT-ssDNA nanohybrids for applications in high fidelity biosensing.

2 Experimental Section

2.1 SWNT Purification and Dispersion

SWNTs with carboxylic functional groups in 2.73 wt% were purchased from Cheap Tubes, Inc. They were first purified and dispersed following a previously defined procedure [18] as follows: SWNT-COOH (1 mg) was added in *o*-dichlorobenzene (*o*-DCB) solvent (10 mL), followed by sonication in an ice bath for 10 min. Sonication usually generates a lot of heat, therefore, an ice bath is used for protecting the SWNTs from physical damage. After sonication, the mixture solution was centrifuged for 90 min at 13,000 rpm. The supernatant was then further centrifuged at 55,000 rpm for 2 h. The resulting supernatant solution is almost transparent, and the resulting functionalized SWNTs are shown in Fig. 1.

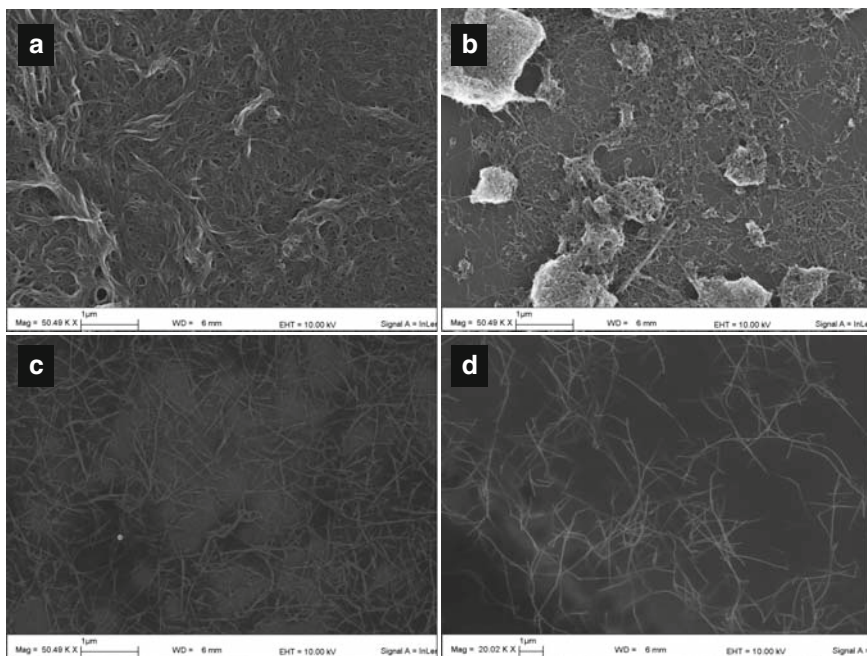


Fig. 1 SWNT purification and dispersion process. (a) SEM image of commercial carboxylic group functionalized SWNTs. (b) SWNTs sonicated in ODCB for 5 min. (c) Supernatant of SWNT solution collected after centrifugation at 13,000 rpm for 90 min. (d) Supernatant of SWNT solution collected after centrifugation at 55,000 rpm for 2 h

2.2 Device Fabrication

A drop of purified SWNT dispersion solution was deposited on a marked heavily doped p + Si/SiO₂ (300 nm) substrate. After the solution was dried at room temperature, discrete SWNTs and groups were left on the surface of the substrate. Metal electrode contacts were deposited at the ends of a single SWNT by using electron beam lithography and lift-off patterning (Fig. 2). Initial electrical testing was carried out by sweeping the back-gate voltage from -10 to $+10$ V under a fixed source-drain voltage at 1 V using an Agilent 4155C semiconductor parametric analyzer. Current–voltage (I – V) measurements indicated that the SWNT was of p-type (Fig. 3).

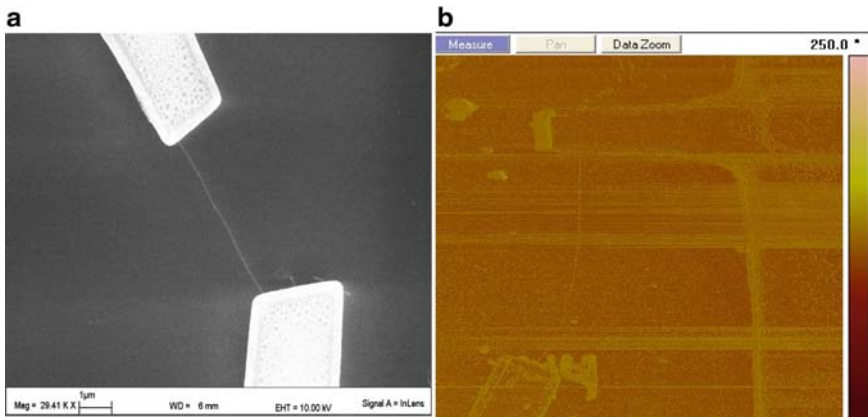


Fig. 2 (a) SEM image of SWNT field effect transistor fabricated with electron beam lithography. (b) AFM image of another SWNT FET device

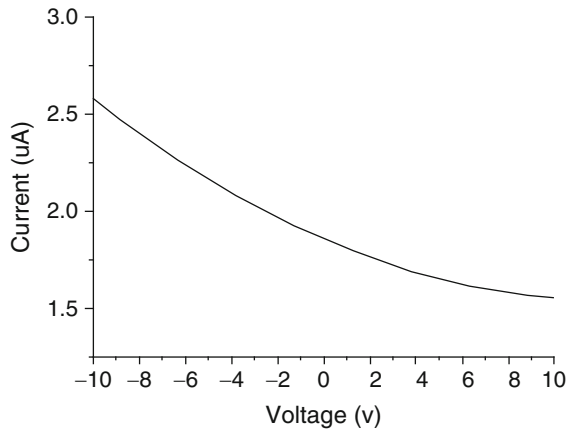


Fig. 3 I – V_g measurements of the SWNT FET for $V_{ds} = 1$ V with a gate oxide thickness of 500 nm

2.3 *Synthesis of SWNT-ssDNA Conjugations and Detection of Specific DNA Sequences*

SWNT-ssDNA conjugations were formed by reacting the amine group at the end of a single strand DNA with the carboxylic group on the surface of SWNTs via the EDC coupling reagent. Since SWNTs were fixed by the metal electrodes on the substrate, the substrate was immersed into the EDC solution for 30 minutes. Amine functional group modified ssDNA (sequence: 5'-CTCTCTCTC-NH₂-3', from Sigma-Gynosis) and NHS-sulfo reagent were added to the solution. After incubating for 12 h, the sample was dried at room temperature. During the incubation process, ssDNA molecules bound to the SWNT surfaces via amide linkage. After obtaining an initial $I-V$ measurement of the SWNT-ssDNA FET structure, it was then immersed into a complementary strand DNA (cDNA) solution where fragments with the complementary sequence of 5'-GAGAGAGAG-3' were hybridized to the ssDNA at 42 C for 4 h. $I-V$ measurements were conducted and the modulation of the conductivity was recorded.

3 Results and Discussion

Commercial SWNTs were dispersed in dionized water, and a drop of dispersion solution was dried on a silicon substrate and imaged as reference (Fig. 1a). A lot of impurities, such as carbonaceous graphite particles, sonopolymers that were involved during SWNT fabrication and acid oxidization are observed. Most of SWNTs bundle together due to van der waals interactions between SWNTs. After sonication in *o*-DCB, a drop of sample was taken for SEM imaging (Fig. 1b), indicating the dispersion of SWNTs becoming much better although impurities still existed. According to our experience, *o*-DCB exhibits stronger π -orbital interaction with the sidewalls of SWNTs. During a sonication process, *o*-DCB molecules penetrate SWNT bundles by overcoming the van der waals interaction [18]. Therefore, sonication of SWNTs in *o*-DCB is critical to obtain well dispersed SWNTs. In order to remove the impurities, centrifuging with different speeds conducted. Centrifuging under low speed was performed first, followed by ultra-centrifugation under high speed. Larger impurities settled down and were excluded after the first centrifugation step (Fig. 1c). With the centrifugation speed increasing, a decreasing number of SWNTs with an increase in quality (much less impurities) as shown in Fig. 1d.

Purified SWNTs were deposited on a p⁺ doped silicon substrate capped with 500 nm SiO₂. SWNT field effect transistors were fabricated via electron beam lithography. Figure 2a shows the configuration of the device. A single SWNT was fixed at both ends by metal electrode contacts patterned by electron beam evaporation. The contacts made in this way are reliable for a long time and can withstand immersion in water bath [19]. Another sample is presented by AFM imaging in Fig. 2b. Most of SWNTs after dispersion have a diameter of 15–20 nm, and are

isolated from each other in a well dispersed manner. SWNT FET characterization was carried out by measuring the current between source and drain electrodes under gate voltage sweeping. $I-V$ curve in Fig. 3 shows that the current is decreasing with applying a positive voltage, which demonstrates that the SWNT in the FET is of a p-type semiconductor.

Due to the carboxylic groups of SWNT, amino ended ssDNA readily binds to SWNT under EDC coupling and NHS-sulfo reagents acting in the solution. After ssDNA attach to the carboxylic group sites on the surface of SWNT, the functionalized SWNT was immersed into a target DNA (cDNA) solution. SWNT serves as the semiconductor, and ssDNA bound along the surface of SWNT serves as the receptors for the target DNA fragments. $I-V$ measurements of SWNT, SWNT-ssDNA hybrids and SWNT-ssDNA-cDNA hybrids were recorded respectively. From the $I-V$ curves (Fig. 4), after ssDNA fragments covalently bind to the SWNT, the conductivity of the SWNT is reduced (Fig. 4, red) compared to that of before binding (Fig. 4, black). We suggest that upon SWNT-ssDNA binding, geometric deformations occurs, leading to charge carrier scattering sites in the SWNT, hence the reduced conductivity [20]. With the target DNA hybridizing with ssDNA, the conductivity increases (Fig. 4, green). The increase in conductivity is due to an increase in the density of negative charges at the SWNT surface associated with the binding of cDNA. In the sensor device, ssDNA serves not only as receptors for targets, but also as the gate dielectric. When cDNA is added, ssDNA hybridizes with cDNA instead of binding to SWNT directly. cDNA molecules bear negative charges on their backbone. Even though cDNA is dried during the measurements, residual water molecules from the buffer solution are still adsorbed on DNA's hydrophilic phosphoric acid backbone by forming hydrogen bonds [21], together with the cations counterbalancing the negative charge of DNA [22]. Also, the effect of measurement environment after DNA molecules dried could not be ignored [23, 24]. Under a high humidity level, water molecules would accumulate at the phosphate backbone of DNA [24]. The electrical measurements in this paper are conducted under an ambient humidity level of 40%. Therefore, cDNA molecules bear negative charges with

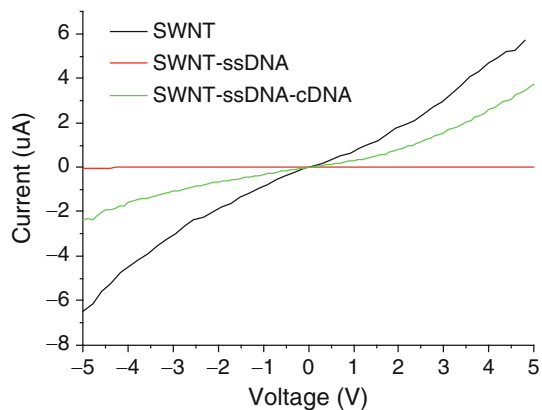


Fig. 4 $I-V$ curves of SWNT before and after ssDNA covalent binding (black and red). $I-V$ measurements of ssDNA-SWNT nanohybrids detecting the target DNA (cDNA) is shown in green

water molecules surrounding them. cDNA hybridization with ssDNA is consistent with applying a negative gate voltage on SWNT FET. Thus, the conductivity of p-type SWNT increases when cDNA fragments hybridize to the ssDNA receptors.

4 Conclusion

SWNT-ssDNA based hybrid biosensor for the detection specific sequences of DNA has been developed. SWNT is purified and well dispersed before conjugating with ssDNA. SWNT FET measurements indicate a p-type semiconductor behavior. After functionalized by amino-ended ssDNA, the SWNT FET is used for detecting target DNA molecules. Adding target DNA molecules, which hybridize with ssDNA molecules on the surface of SWNT results in a significant modulation of SWNT conductivity. The bio-sensing process is analogous to applying a negative bias voltage on the gate of SWNT FET. Therefore, the conductivity of SWNT increases. Our results illustrate the promise of hybrid SWNT FETs for detecting a broad range of biological and chemical species.

Acknowledgement The authors gratefully acknowledge financial support of this work by the Center for Nanotechnology for the Treatment, Understanding and Monitoring of Cancer (Nano-Tumor) funded by the National Cancer Institute, and the Center for Hierarchical Manufacturing (CHM) funded by the National Science Foundation.

References

1. Jimenez D, Cartoixa X, Miranda E et al (2007) A simple drain current model for Schottky-barrier carbon nanotube field effect transistors. *Nanotechnology* 18(2):Article No. 025201
2. Li H, Zhang Q, Li JQ (2006) Carbon-nanotube-based single-electron/hole transistors. *Appl Phys Lett* 88(1):Article No. 013508
3. Kim BK, Kim JJ, So HM et al (2006) Carbon nanotube diode fabricated by contact engineering with self-assembled molecules. *Appl Phys Lett* 89(24):Article No. 243115
4. Raychowdhury A, Roy K (2007) Carbon nanotube electronics: Design of high-performance and low-power digital circuits. *IEEE Trans Circ Syst I-Reg Pap* 54:2391–2401
5. Kelly KF, Chiang IW, Mickelson ET et al (1999) Insight into the mechanism of sidewall functionalization of single-walled nanotubes: an STM study. *Chem Phys Lett* 313(3–4):445–450
6. Fagan SB, da Silva AJR, Mota R et al (2003) Functionalization of carbon nanotubes through the chemical binding of atoms and molecules. *Phys Rev B* 67(3):Article No. 033405, 4 pages
7. Liu J, Rinzler AG, Dai HJ et al (1998) Fullerene pipes. *Science* 280(5367):1253–1256
8. Qu LW, Martin RB, Huang WJ et al (2002) Interactions of functionalized carbon nanotubes with tethered pyrenes in solution. *J Chem Phys* 117(17):8089–8094
9. Dwyer C, Johri V, Cheung M et al (2004) Design tools for a DNA-guided self-assembling carbon nanotube technology. *Nanotechnology* 15(9):1240–1245
10. Zhang JP, Liu Y, Ke YG et al (2006) Periodic square-like gold nanoparticle arrays templated by self-assembled 2D DNA nanogrids on a surface. *Nano Lett* 6(2):248–251
11. Rothmund PWK (2006) Folding DNA to create nanoscale shapes and patterns. *Nature* 440(7082):297–302

12. Guo ZJ, Sadler PJ, Tsang SC (1998) Immobilization and visualization of DNA and proteins on carbon nanotubes. *Adv Mat* 10(9):701–703
13. Ito T, Sun L, Crooks RM (2003) Observation of DNA transport through a single carbon nanotube channel using fluorescence microscopy. *Chem Commun* (13):1482–1483
14. Gao HJ, Kong Y, Cui DX et al (2003) Spontaneous insertion of DNA oligonucleotides into carbon nanotubes. *Nano Lett* 3(4):471–473
15. X. Wang, Liu F, Andavan GTS et al (2006) Carbon Nanotube-DNA nanoarchitectures and electronic functionality. *Small* 2:1356–1365
16. Star A, Tu E, Niemann J et al (2006) Label-free detection of DNA hybridization using carbon nanotube network field-effect transistors. *Proceedings of the National Academy of Sciences of the United States of America* 103(4):921–926
17. Hu CG, Zhang YY, Bao G et al (2005) DNA functionalized single-walled carbon nanotubes for electrochemical detection. *J Phys Chem B* 109(43):20072–20076
18. Kim DS, Nepal D, Geckeler KE (2005) Individualization of single-walled carbon nanotubes: is the solvent important? *Small* 1(11):1117–1124
19. Cengiz Ozkan, Xu Wang (2008) Multisegment Nanowire Sensors for the Detection of DNA Molecules. *Nano Lett* 8(2):398–404
20. Star A, Gabriel JCP, Bradley K et al (2003) Electronic detection of specific protein binding using nanotube FET devices. *Nano Lett* 3(4):459–463
21. Lee Otsuka Y, Gu H-Y, Lee J.-H, Yoo J-O, Tanaka K-H, Kawai H, Tabata T (2002) *Jpn J Appl Phys* 41:891; Ha DH, Nham H, Yoo KH et al (2002) Humidity effects on the conductance of the assembly of DNA molecules. *Chem Phys Lett* 355(5–6):405–409
22. Otto P, Clementi E, Ladik J (1983) The electronic-structure of DNA related periodic polymers. *J Chem Phys* 78(7):4547–4551; Lewis JP, Ordejon P, Sankey OF (1997) Electronic-structure-based molecular-dynamics method for large biological systems: application to the 10 basepair poly(dG)center dot poly(dC) DNA double helix. *Phys Rev B* 55(11):6880–6887; York DM, Lee TS, Yang WT (1998) Quantum mechanical treatment of biological macromolecules in solution using linear-scaling electronic structure methods. *Phys Rev Lett* 80(22):5011–5014; Ye YJ, Jiang Y (2000) Electronic structures and long-range electron transfer through DNA molecules. *Int J Quant Chem* 78(2):112–130
23. Jo YS, Lee Y, Roh Y (2003) Effects of humidity on the electrical conduction of lambda-DNA trapped on a nano-gap Au electrode. *J Korean Phys Soc* 43:909–913
24. Kleine-Ostmann T, Jordens C, Baaske K et al (2006) Conductivity of single-stranded and double-stranded deoxyribose nucleic acid under ambient conditions: the dominance of water. *Appl Phys Lett* 88(10):Article No. 102102

Towards Integrated Nanoelectronic and Photonic Devices

Alexander Quandt, Maurizio Ferrari, and Giancarlo C. Righini

Abstract State of the art nanotechnology appears like a confusing patchwork of rather diverse approaches to manipulate matter at the nanometer scale. However, there are strong economic and technological driving forces behind those developments. One key technology consists of a rather dramatic shrinking of integrated electronic devices towards the very size limits of nanotechnology, just to satisfy the growing demand for commonly available computing power. Furthermore, the corresponding step from microelectronics to nanoelectronics pushes another important technological sector, which aims at the development of novel optical devices, that ought to furnish the bandwidth and speed to ship the plethora of accumulating processing bits. In the following, we point out some of the basic technological challenges involved, and present a selection of experimental and numerical approaches that aim at the development of novel types of optoelectronic nanodevices.

1 Introduction

Nanotechnology has become a common buzzword for a general technological development, that promises to make our lives easier and longer. It is based on our unique abilities to manipulate matter at the atomic scale. But even the biggest enthusiast of nanotechnology might become rather thoughtful, after putting away Drexler's *Engines of Creation* [11], and browsing in Hero of Alexandria's *Pneumatics* [43], which stems from the first century AD. How could it be, that it took almost 1,700 years until the steam engine finally initiated the industrial revolution,

A. Quandt (✉)

Institut für Physik, Universität Greifswald, Felix-Hausdorff-Str. 6, 17489 Greifswald, Germany
e-mail: alexander.quandt@physik.uni-greifswald.de

M. Ferrari

CSMFO Lab., CNR-IFN Trento, Via alla Cascata 56/C, 38100 Povo, Italy
e-mail: mferrari@science.unitn.it

G.C. Righini

CNR-Nello Carrara Institute of Applied Physics, MDF Lab, 50019 Sesto Fiorentino, Italy
e-mail: giancarlo.righini@cnr.it

although the basics of steam power had already been known at the time of Hero? The rather sobering answer might be that at the time of Hero, and for a long period of time thereafter, the huge amounts of power provided by the steam engine were simply not needed. Because manpower was abundant, due to slavery and serfdom.

In the dawning age of nanotechnology, it might be rather bewildering to see the same major driving forces at work, which actually led to slavery and serfdom at the time of Hero: comfort, health, business and entertainment. With the distinction that nanotechnology should make those things available to everyone. In fact, there is hardly any part of human life that would not sooner or later go online, and the corresponding need for bandwidth and higher bit rates is growing enormously. As a measure for the technological evolution of optical networks, one usually considers the product of the length L times the maximum bit rate B_0 of the communication link. It turns out that LB_0 is approximately increasing by a factor of ten every four years (optical Moore's law [39]).

Recently, a group of researchers at Nippon Telegraph and Telephone Corporation (NTT) carried out a study of cutting edge optical fiber communication technologies [17]. They reported a bandwidth of 14 TBits/s transmitted over 160 km of optical fiber, which involved wavelength and polarization multiplexing techniques [39], using 140 channels within a window of 1,450–1,650 nm wavelengths of the optical carrier waves. This amounts to a bit rate of 111 Gbit/s per channel, which is more than double the maximum line rate that is commercially available now. Those results are most likely the prelude to a novel 100 Gigabit Ethernet standard, which is badly needed, in order to satisfy the growing need for broadband-access lines. And to provide the necessary flexibility in handling novel types of internet based services like file swapping or video sharing, which are extremely bandwidth intensive, and rather unpredictable [17].

Attached to the nodes of a rapidly growing global communication internet are novel types of computers with rapidly shrinking processor units. This massive integration process more or less follows Moore's law [23], which states that the number of transistors per square centimeter doubles every 12 months. It is quite obvious that such a development will sooner or later hit the limits of nanotechnology itself, which are located in the Å domain ($1 \text{ Å} = 10^{-10} \text{ m}$), being the typical size range of single atoms.

By now the most recent processor generations are already based on transistor technologies with gate lengths in the range of several dozen nanometers ($1 \text{ nm} = 10^{-9} \text{ m}$). To maintain the reliability of established microfabrication techniques at such a tiny length scale represents a formidable technological challenge [35]. Furthermore the miniaturization of transistors implies a dramatic increase in switching speed, such that signal propagation delays in the interconnects between transistors become a real issue, and optical interconnects ought to come to the rescue of chip design [1].

The proper integration of optical interconnects will pose a serious problem for future integrated nanocircuits. Following Moore's law, the expected gate lengths of the electronic elements might rapidly drop below the 10 nm range [15], whereas optical interconnects might need to stay compatible with the standards of long-range

optical interconnects, and use standard wavelengths in the infrared domain around 1,550 nm. Optical devices that ought to interact with standard optical carrier waves must be of nearly the same size range, which will be orders of magnitudes larger than the electronic components of future integrated optoelectronic chips.

In the following, we will present a selection of experimental and numerical approaches to develop and characterize basic electronic and optical devices, which might be most valuable in the design of future integrated optoelectronic chips. In Sect. 2 we will discuss the practical limits of MOSFET design within the nano-domain, and present alternative design approaches based on carbon nanotubes [2] and graphene [12]. In this context, we will also illustrate the rather important role of numerical simulation methods [34]. In Sect. 3 we will study some of the key elements for the integration of optical devices like VCSEL photonic sources and photonic crystal waveguides, and present some experimental and numerical approaches [18, 39] to optimize their basic functionalities.

Finally, we will summarize our findings in Sect. 4. Note that our selection of topics is neither intended to be exhaustive nor representative. Nor might it be completely unbiased. Our main goal here is to point out some of the most promising starting points for one's own experimental or numerical access to the development of novel electronic and optical devices for the nanotechnology era.

2 Integrated Electronic Devices

Metal oxide semiconductor field-effect transistors (MOSFETs) as depicted in Fig. 1(a) are today's standard workhorses of integrated electronics. The layer-by-layer layout of complex networks containing billions of transistors on the area of a single chip requires a permanent refinement of Very Large Scale Integration (VLSI) techniques [26]. As VLSI is based on the optical projection of photomasks, state of the art VLSI already involves projection techniques using electron-beam lithography or illumination wavelengths within the deep UV range [16]. The latter may be

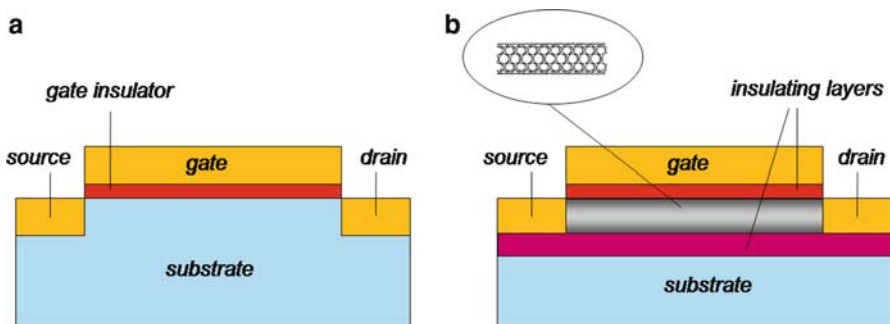


Fig. 1 Towards nanoelectronics. (a) Basic structure of a MOSFET transistor. (b) FET transistor, which involves semiconducting carbon nanotube channels

combined with immersion and double-projection techniques to create feature sizes of the order of a few dozen nanometers, which are well below the wavelengths of visible light, and close to the Rayleigh-limit of optical projection methods [41].

What about physical limits on nanometer sized MOSFETs? Thanks to Moore's law, there has always been a bulk of literature on the projected scaling of such devices [25], and about the technological problems involved in downsizing the basic MOSFET design [15, 27]. We will discuss some of those issues in Sect. 2.1. Then in Sect. 2.2 we will illustrate some of the improved device characteristics for FETs with nanotubular components. Furthermore, alternative semiconducting substrates like graphene [12, 31] might actually allow for a further extension of Moore's law towards the very size limits of nanotechnology, using a cluster based nano-patterning approach described in Sect. 2.3.

2.1 *Scaling of MOSFET Devices*

The basic layout of a MOSFET transistor is sketched in Fig. 1(a). It consists of source and drain contacts, which are doped, and conducting diffusion layers isolated by a semiconducting substrate. The third contact called the gate is also conducting, and it is separated from the other components by a thin insulating layer. As long as the gate voltage is low, there is no current flowing from the source to the drain, due to the semiconducting properties of the substrate. But once the gate voltage overcomes a certain threshold voltage, there will be current flowing, as soon as we apply an appropriate electric field between the source and the drain.

In order to run such a transistor with technologically appealing device characteristics, one has to dope it in a systematic fashion, such that the doping of the substrate shows an opposite polarity to the source and drain. This effectively creates two back-to-back junction diodes. A suitable voltage V_{gs} applied through the gate will pull mobile carriers (electrons or holes) to the underside of the metal oxide layer, thus opening a conducting channel through the substrate. Once the voltage is turned off, the surface under the gate will be depleted of carriers, and no current will be able to flow any more.

The gate/oxide/substrate sandwich of gate length L , width W and thickness D may be pictured as a capacitor with dielectric constant ε and capacitance $C_g = \varepsilon WL/D$. Thus there is a charge $Q_g = C_g V_{gs}$ accumulating in this conducting channel. Once a voltage V_{ds} is applied between the source and the drain, there is a current I_{ds} flowing that experiences a resistance $R = LD/\mu\varepsilon W V_{gs}$. An elementary derivation of these results can be found in [14]. A rough estimate of the device speed is related to the time constant τ of a model RC circuit with the same characteristics:

$$\tau = RC_g = \frac{L^2}{\mu V_{gs}} \quad (1)$$

Thus, shorter discharge times τ may be obtained by shrinking the gate length L . However, this implies a shorter distance between the source and the drain, which might lead to difficulties in switching off an operating device. This could still be avoided using massive doping. Another possibility to increase device speed would be through an increase of the mobility μ for the carriers that travel from the source to the drain, for example by straining the substrate [15]. A third possibility would be to increase the gate voltage V_{gs} . But the metal oxide layer is already close to its physical limit (around 1 nm), and increasing the gate voltage will lead to leakage currents. Furthermore the power consumption will sharply increase, as the switching power is proportional to the operating frequency f , and to the dynamic switching energy. The latter may be estimated from:

$$E = \frac{1}{2}(C_g + C_w)V_{gs}^2 \quad (2)$$

where C_w is the effective capacitance of the wiring, which is a rather complex metal-dielectric interconnect structure.

Let us consider a single metal line with contact capacitance C_w , and of length L and diameter A . Then we note that the corresponding resistance $R \sim L/A$ will obviously increase with shrinking line diameter A , leading to longer signal delay times related to $\tau = RC_w$. This signal delay will not be an issue for similarly shrinking local interconnects with small lengths L , but it will become a serious problem for the much longer global interconnects, which ought to join important parts of a processor [35].

Here we close our short discussion of basic design problems for MOSFET transistors within the nanodomain. Alternative design concepts like the FinFET transistor are shortly described in [23], and a more detailed description of ultimate device limits may be found in [27].

2.2 Nanotube Transistors and Interconnects

An alternative road to the design of nanoelectronic devices is the employment of semiconducting carbon nanotube (CNT) channels as integral part of a working FET, which is indicated in Fig. 1(b). Carbon nanotubes may be pictured as rolled up versions of rectangular strips cut out of a single layer of graphite called graphene [10]. A single graphene layer consists of carbon atoms located on the vertices of a honeycomb lattice. Depending on the direction of the cut, the resulting carbon nanotubes exhibit different chiralities, which influence their basic electronic properties quite strongly (i.e. metallic vs. semiconducting [10]). Unfortunately, it is hard to control this chirality during the synthesis of CNTs.

The major advantages of implementing semiconducting carbon nanotubes as FET channels have recently been pointed out in [2]: the nanotube channel is quite small (1–2 nm) and atomically smooth, the carrier mobilities are very high at low

gate voltage, and the capacitance of CNTs is rather low. The gap size of semiconducting CNTs is inversely proportional to their diameters, which allows for a rather flexible use of CNTs as basic nanoelectronic components [10]. Furthermore, CNTs dispose of a rather favorable optical properties, to be discussed in Sect. 3.

Therefore the integration of CNT components might allow for novel high-speed, low power and nanometer sized FET and optoelectronic devices discussed in [2]. However, major technological challenges are represented by a controlled layout, a method to separate metallic/semiconducting CNTs during synthesis, and a systematic control of contact barriers between CNTs and the source/drain of the MOSFET device shown in Fig. 1(b). Note that the contacts of a CNT based FET are usually made of metals.

Partial technological solutions for some of these problems are discussed in [2]. But progress may also be made through the employment of boron nanotubes (BNTs) [36]. Those materials are the brainchild of extensive *numerical* simulations on small boron clusters [5], which suggested [6] the existence of stable boron sheets (i.e. the boron analogue of graphene) and boron nanotubes (i.e. the boron analogue of carbon nanotubes shown in Fig. 2(a)).

Note that numerical simulations on unknown boron nanomaterials are far from trivial. Those materials were outside the horizon of standard textbook wisdom [33], and therefore the tedious identification of stable ground state configurations of planar and tubular boron clusters required the usage of ab initio simulation methods at the highest level of numerical accuracy (for a survey of such methods see [34]). Nevertheless, these earlier results not only stimulated the successful synthesis of BNTs [9], alongside a plethora of novel types of semiconducting boron nanowires (see [36]). But they were also the basis of recent refinements of the atomic structures of BNTs, based on a remarkable hole-doping scenario [40].

There is now some general consensus about a number of very favorable properties for nanotechnological implementations of boron nanotubes, as pointed out

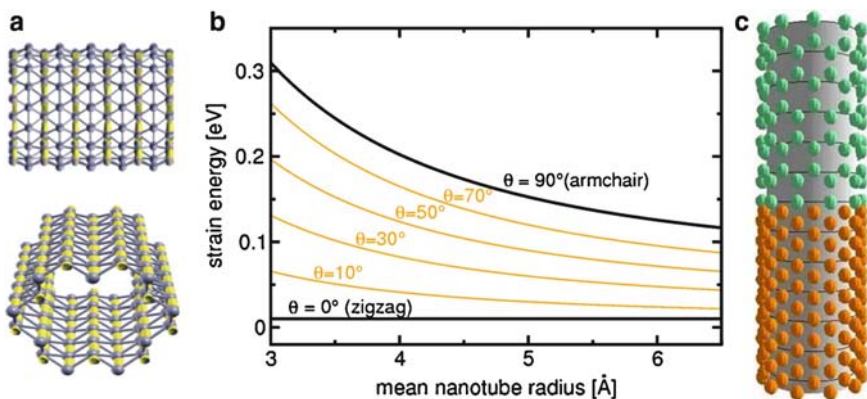


Fig. 2 Tubular carbon-boron interconnects. (a) Model armchair (top) and zigzag (bottom) boron nanotube (BNT). (b) Strong dependence of elastic properties on the chirality of various BNTs [22]. (c) Stable boron-carbon heterojunction (CNT at the top, BNT at the bottom) [20]

in [36]: first of all, BNTs should always be metallic, independent of their chirality, which would make them perfect conducting nanowires. On the other hand, the elastic properties of BNTs are strongly dependent on their chirality, as shown in Fig. 2(b) and pointed out in [22]. This is most obvious from the constricted nature of the zigzag BNT indicated at the bottom of Fig. 2(a), as compared to the stable round structure of the armchair BNT shown at the top of Fig. 2(a) (for details see [21]). In contrast to CNTs, the mechanical properties of BNTs might actually be controlled during synthesis, thus leading to some control over their chiralities [22].

Furthermore, similar bond lengths and the electron deficient nature of boron should make boron based nanomaterials largely compatible to carbon nanomaterials, to silicon substrates, and to all sorts of metallic wirings. The basic compatibility between carbon and boron nanomaterials has already been demonstrated in numerical simulations of nanotubular carbon–boron heterojunctions [20]. One exemplary metallic carbon–boron junction is shown in Fig. 2(c), and it consists of a CNT on top, and a BNT at the bottom. Another interesting feature of such junctions is the fact that they might easily be formed by excessive doping of CNTs with boron atoms: simulations revealed that boron atoms have a strong tendency to migrate towards the open ends of CNTs [13], where they grow BNT type of extensions.

Note that the formation of stable heterojunctions between BNTs and CNTs could actually induce a similar structure control over the CNT components, simply by controlling the BNT segments attached to them (see [22]). Therefore BNT–CNT based networks could become vital components of future nanotube based FET design, where the metallic boron component might be responsible for structure control, as well as for stable interconnects with the outside world.

2.3 *Ultimate Integrated Devices Based on Graphene*

The scaling of important device properties for integrated nanoelectronic circuits described in Sect. 2.1 points towards thinner and thinner MOSFET devices. One ultimate technological limit would be the controlled layout of integrated circuits on a 2D semiconducting substrate. Lucky enough, a suitable substrate material has already been identified in terms of graphene [31]. This term denotes a whole family of nanomaterials, which consist of (irregularly shaped) flakes of carbon monolayers, cut from a basic carbon honeycomb sheet sketched in Fig. 3(a).

Small amounts of graphene may be produced in a disarmingly simple fashion, using adhesive tape to gradually cleave small flakes of graphite into thinner and thinner fragments (for a Do It Yourself description of this process see [12]). The electronic properties of graphene flakes depend on the nature of their borders [30], but a safe bet is to either obtain semiconducting flakes from scratch, or otherwise turn a given flake into a semiconducting one by manipulating its borders. Note that the mobility of conducting electrons within graphene is very high. Furthermore, the conducting electrons seem to move ballistically, i.e. without being scattered by the carbon atoms of the underlying honeycomb lattice [12].

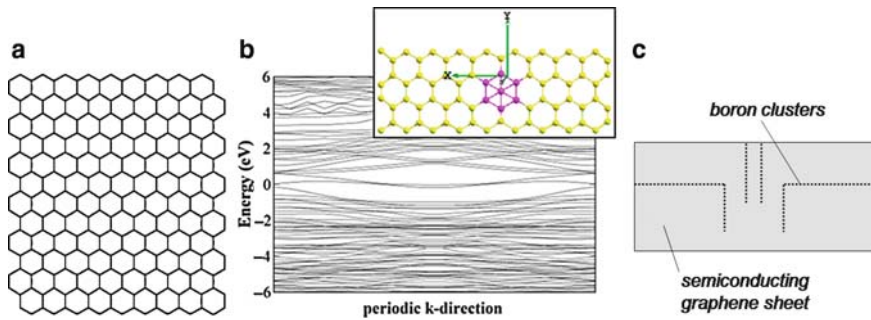


Fig. 3 Graphene based nanoelectronics. (a) Honeycomb lattice of single graphene layer. (b) Chain of B₇-clusters embedded into semiconducting armchair graphene nanoribbon (top). Note that this system is supposed to be periodic in the y -direction, such that the boron clusters are *not* directly connected, but separated by a full carbon honeycomb. This functionalization nevertheless induces conducting channels inside the gap of the undoped graphene substrate (bottom). (c) FET type of wiring and basic functionalization of a semiconducting graphene substrate, based on *unconnected* chains of embedded boron clusters

Recent numerical simulations [38] uncovered a way to functionalize semiconducting graphene sheets, based on conducting nanowires only a few atoms thick. The corresponding model system is shown at the top of Fig. 3(b). It consists of a small hexagonal B₇-clusters being embedded into a semiconducting rectangular graphene nanoribbon with armchair borders. Note that the structure shown in Fig. 3(b) is actually periodic in the y -direction, such that neighboring B₇-clusters are *not* directly connected, but separated by a full carbon honeycomb. Nevertheless, we notice the appearance of conducting channels in the gap of the original semiconducting substrate, as shown at the bottom of Fig. 3(b).

This points towards a very robust way of laying out a basic wiring within a semiconducting graphene substrate, like the basic FET type of blueprint shown in Fig. 3(c). The fact that the boron clusters were not directly connected in the model system of Fig. 3(b) also suggests that even a rather irregular and heterogeneous embedding of boron islands might work in practice. Therefore the cluster embedding could in principle be carried out with state of the art technological equipment, as described in [37].

3 Photonic Devices and Interconnects

Optical interconnects are a well-established high-bandwidth technology for long-distance communication. And the dreaded *internet bottlenecks* are not related to occasional cable accidents on the high seas, but rather to data traffic jams on the notorious *last mile*, where optical fiber technologies are only gradually replacing electrical or wireless carriers. On the other hand, due to Moore's law and modern multi-core processor architectures, the requirements for on-board IO bandwidth will

rapidly approach the range of 1 Tbit/s (and beyond), thus shifting the bottlenecks to the last (centi-)meter. As pointed out in [1], commercially available processors are now featuring data buses that operate at 1.3–1.4 GHz, and at bandwidths around 17 Gbit/s. Experimental electrical transmission lines slightly surpass this value, but for achieving bandwidths in the range of Tbit/s over the area of a microchip, electrical transmission looks rather insufficient.

In order to get more insight into this problem, we again pick up our theoretical considerations about the scaling of electrical interconnects from Sect. 2.1. But at this point, we would like to be a little bit more precise: let us once again consider a wire of length L and area A . Again we find that the resistance should scale as $R \sim L/A$. But for the capacitance C_w of such a wire, we will now assume a simple, but much more realistic linear dependance $C_w \sim L$, which presupposes a chosen standard geometry for such a wire. This leads to a time constant τ of

$$\tau = RC_w \sim \frac{L^2}{A} \quad (3)$$

A conservative estimation [28] for the corresponding bandwidth B , which ought to be inversely proportional to the time constant τ , amounts to

$$B \approx 10^{16} \frac{A}{L^2} \text{bit/s} \quad (4)$$

This result is rather interesting, for various reasons: if we play around with some realistic figures, where a global interconnect might have a cross-section of $1 \mu\text{m}^2$ and a length of 1 mm or less, we will see that we are talking about bandwidths in the range of MBit/s or Gbit/s, rather than the targeted Tbit/s. Even worse, the geometrical factor A/L^2 in Eq. 4 tells us that, as long as we have to stick to a certain standard wire geometry characterized by a fixed *aspect ratio* $L/\sqrt{A}$, we will never be able to increase the corresponding data bandwidth by only an overall miniaturization of such a wire.

According to [1], a conventional integrated photon source (see Sect. 3.1) will easily deliver a bandwidth of 10 Gbit/s and more, and that bandwidth will be largely independent of the length of the transmission line, at least on the length scale of on-chip optical interconnects. By combining whole arrays of such devices, it should be possible to achieve Tbit/s bandwidths with device technologies that might be implemented into the standard VLSI type of chip production. Other advantages of optical interconnects are the reduction or absence of inter-channel crosstalk and electromagnetic interference [1, 28]. It allows for much denser packings of integrated optical waveguides, thus leading to a dramatic increase in the available bandwidth. Furthermore, optical technologies may in principle be implemented with lower power expenditures. For a detailed discussion of the technological aspects of optical interconnects, we refer the interested reader to [28].

Nevertheless, it should not be concealed here that the basic technology for integrated optical interconnects is still in its infancy. And that the candidate materials, devices and fabrication processes still have to be explored and gradually

improved. Therefore we will only be able to sketch some blueprints for integrated optical interconnects in Sect. 3.1. In Sects. 3.2 and 3.3 we will present some basic photonic components, which could become key elements of future optical circuits. Section 3.2 will be devoted to photonic sources, where we will also present some experimental and numerical results about Bragg reflectors and microcavities, which are vital parts of such devices. Furthermore, in Sect. 3.3 we will illustrate optical waveguiding based on photonic crystals.

Finally we would like to point out that other promising technological solutions to achieve nanometer sized integrated optoelectronic circuits may be provided by *plasmonics* [3, 24, 32]. The latter denotes a very fascinating branch of photonics, that is dealing with processes that take place at the interface between a dielectric medium and a metallic medium. It basically turns out that light, which is hitting such a metal/dielectric interface, may stimulate coherent electron oscillations known as surface-plasmon polaritons (SPPs). It is interesting to note that even at the time of Hero, artists were already make use of plasmonics to achieve the most vibrant colors for their glass ware. A brief survey over the most recent technological applications of metal nanoparticles embedded into a surrounding dielectric medium is given by [24].

Interesting enough, the wavelengths of these SPPs are actually smaller than the wavelengths of the incident light, which offers the possibility to shrink optical technologies towards the size ranges of integrated electronic devices [3, 24]. The SPPs rapidly decay into both media, and unfortunately they will only be able to propagate for a limited range along the interface. Nevertheless, a lot of progress has been made to extend the range of SPPs towards technologically interesting distances, and the first working plasmonic devices even give reason to dream of future integrated all-photonic (or better: all-plasmonic) chip technologies [32].

3.1 Optical Interconnects

The application of optical technologies in the framework of nanoelectronics should mainly provide sufficient bandwidth for global data interconnects on a given chip. Like any conventional optical transmission system, on-board optical interconnects will be composed of three major components: photonic sources, photonic links and photonic detectors [39]. Photonic sources, like light emitting diodes (LEDs) or lasers (see Sect. 3.2), as well as the corresponding photo-detectors ought to be directly coupled to the electronic circuitry, with acceptable signal-to-noise ratios. The specific technological requirements for these components are summarized in [28].

Some all-silicon based solutions would certainly have the advantage to be easily integrable into conventional chip production processes, and some progress has already been made in that direction [1]. Nevertheless, the standard technology for photonic sources or detectors involves III–V semiconductors [1], and due to the lattice mismatches between those materials and silicon, the corresponding integrated optoelectronic chips have to be produced outside the silicon fabrication process

itself. Later on, the completed optoelectronic chips must be mounted on the electronic chips, in a process that is called flip-chip bonding [39].

Channels for optical data transmission might be conventional fibers, free-space transmission, or optical waveguide links directly built onto or into the substrate of the processor board. Optical fibers are most suitable for straight linear interconnects, but for microelectronic applications they have to be modified in order to sustain elevated temperatures for a long period of time [1]. For free space routing, there are interesting solutions, which involve external reflection holograms [39]. For on-board waveguiding, one faces the usual technical problems of confining an optical mode, which strongly depends on the refractive index of the available materials. Furthermore one might need to guide modes around sharp bends, which may actually be solved using photonic crystals, as described in Sect. 3.3.

3.2 Photonic sources and Bragg Reflectors

Light emission by solid-state semiconductor devices is based on electron-hole recombination. The perfect setup is a junction with a p -doped semiconductor on one side, and an n -doped semiconductor on the other side (p - n -junction). Operating such a device in a forward bias mode (i.e. positive voltage at the p -side, and negative voltage at the n -side), one can achieve a sufficiently high recombination rate to see more than just a feeble glow, which is called *injection electroluminescence*.

The standard light-emitting diode (LED) is operating on this principle of injection electroluminescence. The good news is that such devices may be miniaturized to a degree that will allow for their combination with microelectronic circuits, where electronic data streams will directly modulate the LED to allow for optical transmissions (electronic-to-optical transducers [39]).

Note that the radiation from a standard LED will be *incoherent*, as long as the recombination will only be caused by *spontaneous emission*. In order to achieve optical amplification or laser action, one has to accomplish some population inversion. The latter might be achieved by raising the forward voltage beyond a certain level, or by some optical pumping, for example when using rare-earth dopants like Er^{3+} as active components. The pumping will excite electrons from the valence band to the conduction band of the active component, where these electrons will undergo a fast non-radiative transition to a lower, and long-lived excited level. A photon of just the right energy will then be able to induce a recombination of the corresponding electron-hole pair, and thus clone a second photon of the same phase and energy. This process is called *stimulated emission*, and it leads to *coherent* radiation.

In order to achieve constant laser action, one has to provide a certain feedback, where photons are kept long enough inside the active zone to cause whole avalanches of photon clones. This may be achieved by using Bragg reflectors. The latter are stacks of dielectric layers i with effective thickness d_i , and with varying refractive index n_i , as shown in Fig. 4. When light of a certain wavelength λ

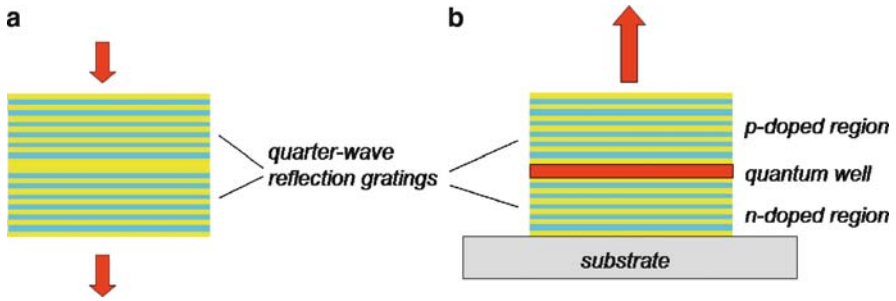


Fig. 4 Photonic devices. (a) Two Bragg reflectors and a larger spacer layer, which form a photonic microcavity. (b) Principal layout of a vertical-cavity surface-emitting laser (VCSEL). The whole system is mounted on a substrate, and the reflectance of the laser mode by the Bragg reflectors is rather high. Laser modes are generated through electron-hole recombinations inside the pumped spacer layer, and the resulting coherent light will escape vertically, as indicated

impacts on such a structure, partial reflectance and constructive interference may occur whenever $\lambda = 4n_i d_i$ (“quarter wave reflection grating”). This might add up to total reflectance [4], the so-called stop bands. The aim of a related optical microcavity shown in Fig. 4(a) is to trap standing waves inside a spacer layer between two Bragg reflectors.

Now we dispose of all the basic photonic components to devise a suitable coherent light source for nanotechnological purposes. It is called vertical-cavity surface-emitting laser (VCSEL), and the basic blueprint of such a device is shown in Fig. 4(b). A VCSEL is a sandwich of an active lasing area between two Bragg reflectors mounted on a suitable substrate, where one reflector is n -doped, and the other reflector is p -doped. Recombination takes place inside the active area, which may be pumped electrically. The microcavity provides the necessary feedback for laser action, and the coherent radiation escapes vertically, as indicated in Fig. 4(b).

Such VCSEL devices have the distinct advantage that they may be fabricated in a layer-by-layer fashion, with standard layout techniques known from microelectronic VLSI, and in a size range that will allow for their implementation as active parts of on-board optical interconnects [39]. Nevertheless, the optimization of the various photonic components of a VCSEL for microelectronic purposes is still the subject of ongoing research. In Fig. 5(a) we show a picture of a photonic microcavity made of SiO_2 and TiO_2 , which disposes of an Er^{3+} doped spacer layer [7]. Such a device could act as an optical filter and amplifier. The corresponding normal transmittance is shown in Fig. 5(a), where we recognize a broad stop band with a cavity mode at 1,544 nm.

Note that active VCSEL components may be analyzed in a similar fashion. Further improvement in the design of such devices might actually come from numerical simulations [39]. An example is shown in Fig. 5(b), and it refers to the model microcavity of Fig. 4(a). We just assumed that the index of refraction for the two media is $n = 1$ and $n = 2$, and that the thickness of the plates is $d = 2/3$ and $d = 1/3$ in arbitrary units. The spacer layer is characterized by $n = 2$ and

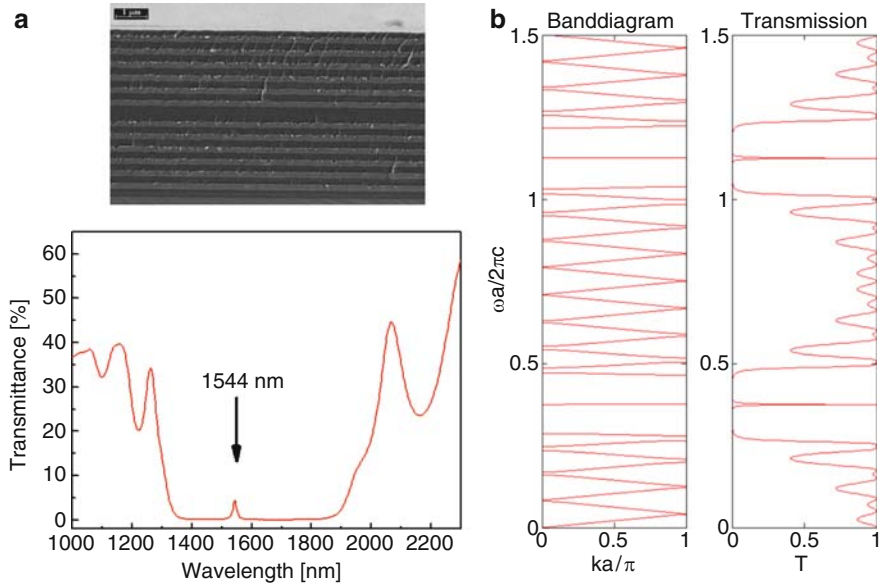


Fig. 5 Microcavity. (a) Bragg reflectors made from SiO_2 (black) and TiO_2 (grey), where the spacer layer is doped with Er^{3+} -ions. The corresponding normal transmittance shows an ample stop band around the cavity mode [7]. (b) A supercell simulation (left) of an infinite model Bragg reflector shows band gaps and cavity modes (horizontal line). The transmittance (right) for a finite and air terminated Bragg reflector is marked by a (near) perfect reflectance of the stop bands, and the appearance of a sharp cavity mode

$d = 2/3$. If we extend this model periodically in the framework of a supercell model, the propagating electromagnetic modes are *Bloch waves* with pseudomomentum $\mathbf{k}$ and frequency ω , and the corresponding band structure $\omega(\mathbf{k})$ is shown on the left side of Fig. 5(b) (for details see [18]). We may clearly identify photonic band gaps as the origins of the observed stop bands, and we also observe the appearance of horizontal lines inside those band gaps, which correspond to cavity modes. A similar picture is obtained after determining the normal transmission for an air terminated model Bragg reflector using the transfer matrix formalism (for details see [39]). The results are shown on the right side of Fig. 5(b), where we observe (near) total reflectance for the stop bands, and a sharp transmittance peak inside those stop bands, which corresponds to the cavity modes.

Finally we would like to point out another potential photonic source for basic nanoelectronic purposes [2]: it turns out that a CNT-FET can be operated in an ambipolar fashion, such that electrons and holes are transported simultaneously through such a device. Nevertheless the electroluminescence caused by the resulting electron-hole recombinations is rather broad and nondirectional. But a narrow and directional light beam may still be obtained, after embedding the radiating CNT into a suitable microcavity.

3.3 Photonic Crystal Waveguides

Photonic crystals are a new class of optical materials. They are artificially created periodic structures of dielectric materials, which hold promise for future all-optical circuits [19, 44]. A typical example is the 2D dielectric square lattice shown in Fig. 6(a), which may be realized as a 3D structure after embedding parallel cylindrical dielectric columns into a suitable dielectric host. It has been noted right from the beginning [19, 44], that some photonic crystals could give rise to omnidirectional stop bands, in analogy to the Bragg reflectors described in Sect. 3.2. However, as pointed out in [45], the formation of stop bands in 2D and 3D photonic crystals requires a high index contrast, which is quite different from 1D systems like the Bragg reflector described in Sect. 3.2, or from reflections by low-contrast lattice planes in the framework of x-ray diffraction.

Nevertheless, once there appears a stop band for a certain type of photonic crystal, the latter might serve as a cladding for the guiding of modes, whose frequencies fall into the corresponding band gap. A sample waveguide on the basis of the square lattice is indicated in Fig. 6(a), where modes might be guided along a rectangular bent, see [18].

A numerical determination of the photonic band structure for the dielectric square lattice shown in Fig. 6(b) reveals a complete band gap for one type of polarization,

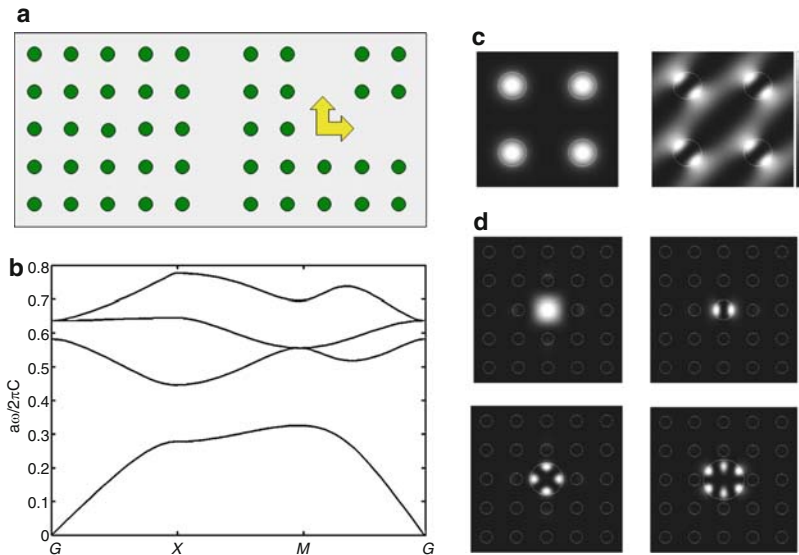


Fig. 6 Photonic waveguide. (a) Dielectric square lattice (left) and a rectangular defect line (right), which may be used for photonic waveguiding. (b) Photonic band structure of TM modes of square lattice is marked by a complete band gap. There is no such gap for the TE mode [18]. (c) Field intensities for the first two photonic modes at the Γ -point. (d) Various multipole distributions of field intensities, which correspond to point defects inside a square lattice (cavity modes)

which is called TM mode (see [18]). We assumed that the surrounding medium is air with dimensionless dielectric constant $\epsilon = 1$, and that the dielectric columns have a dielectric constant of $\epsilon = 8.9$, and a radius of $r = 0.2a$, where a is the lattice constant of the square lattice. Note that the band structure for the other polarization mode does *not* give rise to a similar stop band. Nevertheless, once we are able to encode a signal into a suitable TM mode, that signal may be guided by defect lines, as indicated in Fig. 6(a), and simulated in [18]. Other types of 2D dielectric lattices actually show a complete band gap for both types of polarizations. A detailed analysis of the corresponding band structures, and simulations of waveguiding along defect lines, may also be found in [18].

In Fig. 6(c), we also displayed the field intensities for a dielectric square lattice, which correspond to the lowest modes at the G -point of the band structure shown in Fig. 6(b). It is also very instructive to plot the field intensities for various defects with $r = 0.0a, 0.34a, 0.55a, 0.7a$, introduced right at the center of the lattice shown in Fig. 6(a). The corresponding cavity modes give rise to a multitude of multipole patterns, see Fig. 6(d). It is also possible to achieve laser action on the basis of such cavity modes [29].

Nevertheless the perfect solutions for problems related to on-board optical waveguiding must be sought among 3D, rather than 2D photonic crystals. One interesting example has been given in [8]. Another solution on the basis of the inverse opal structures is described in [42]. This solution has the distinct advantage that artificial opals and inverse opals may easily be assembled and structured, in particular on a silicon substrate.

4 Summary

We discussed various experimental and numerical approaches to explore basic nanoelectronic components. The advantages and shortcomings of the basic MOSFET design, which were discussed in Sect. 2.1, lead to a number of technological challenges for the integrated layout of future nanoelectronic circuits. We just emphasized two of them: first the urgent need for some key modifications of the basic MOSFET design. To this end, we identified carbon nanotubes as one of the most promising basic components of future MOSFETs in Sect. 2.2. And in Sect. 2.3 we described a much tinier technology, based on graphene and small boron clusters.

Another major technological challenge will be the gradual substitution of global electronic interconnects by optical interconnects, as required by a steep increase in the bandwidth demands for the rapid data exchange among widely separated parts of a processor. In Sect. 3.1 we briefly sketched the required optical technologies, followed by a discussion of integrated photonic sources in Sect. 3.2, and a discussion of optical waveguiding on the basis of photonic crystals in Sect. 3.3.

The continuous progress in the development of future nanoelectronic circuits implies close cooperations between *Theory* and *Experiment* on the fundamental side, and between *Science* and *Engineering* on the practical side. We hope that the present paper could deliver some insights into this complex, but rather fascinating process.

At the moment, nanotechnology has reached a stage that is marked by a dramatic increase in our fundamental knowledge of the electronic and optical properties of nanosystems. And we also observe the creation of a large variety of novel nanoengineering tools, very much in the spirit of *Hero's* original *pneumatics* [43] catalogue of engineering tools from the first century AD.

Of course, it might be rather unlikely that nanotechnology will face a similar historical development like steam power technology, and lie idle in the centuries to come. But inventions that might be the nanotechnological equivalent of *Savery's*, *Newcomen's* and *Watt's* ingenious steam engines are still to be made. Nevertheless, most likely it will also be rather practical inventions, that will finally bundle the enormous amount of knowledge gathered in the field of nanoelectronics and nanophotonics, and pave the way for the optoelectronic power engines of the future.

Acknowledgements The authors acknowledge significant and stimulating scientific discussion with Cem Özdoğan (Çankaya) and Jens Kunstmann (Dresden) about basic properties of boron-carbon nanomaterials, and the invaluable support of Alexander Leymann (Greifswald) for modeling of photonic structures. This research was performed in the framework of the project COST MP0702: Towards Functional Sub-Wavelength Photonic Structures.

References

1. Alduino, A., Paniccia, M.: Wiring electronics with light. *Nature Photonics* **1**, 153 (2007)
2. Avouris, P.: Carbon nanotube electronics and photonics. *Physics Today* **62**, 34 (2009)
3. Barnes, W., Dereux, A., Ebbesen, T.: Surface plasmon subwavelength optics. *Nature* **424**, 824 (2003)
4. Born, M., Wolf, E.: *Principles of Optics*, 6th Ed. Cambridge University Press, Cambridge (1998)
5. Boustani, I.: Systematic ab initio investigation of bare boron clusters: Determination of the geometry and the electronic structure of b_n ($n=2-14$). *Phys. Rev. B* **55**, 16,426 (1997)
6. Boustani, I., Quandt, A., Hernandez, E., Rubio, A.: New boron based nanostructured materials. *J. Chem. Phys.* **110**, 3176 (1999)
7. Chiasera, A., Belli, R., Bhaktha, S., Chiappini, A., Ferrari, M., Jestin, Y., Moser, E., Righini, G., Tosello, C.: High quality factor er^{3+} -activated dielectric microcavity fabricated by rf sputtering. *Appl. Phys. Lett.* **89**, 171,910–1 (2006)
8. Chutinan, A., John, S.: Light localization for broadband integrated optics in three dimensions. *Phys. Rev. B* **72**, 161,316–1 (2005)
9. Ciuparu, D., Klie, R.F., Zhu, Y., Pfefferle, L.: Synthesis of pure boron single-wall nanotubes. *J. Phys. Chem. B* **108**, 3967 (2004)
10. Dresselhaus, M.S., Dresselhaus, G., Eklund, P.: *Science of Fullerenes and Carbon Nanotubes*. Academic Press, San Diego (1996)
11. Drexler, K.: *Engines of Creation: The Coming Era of Nanotechnology* (Reprint). Anchor Books, New York (1987)
12. Geim, A.K., Kim, P.: Carbon wonderland. *Sci. Am.* **298**, 68 (2008)
13. Hernandez, E., Ordejon, P., Boustani, I., Rubio, A., Alonso, J.A.: Tight binding molecular dynamics studies of boron assisted nanotube growth. *J. Chem. Phys.* **113**, 3814 (2000)
14. Hey, A.: *Feynman and Computation*. Perseus Books, Cambridge (1999)
15. Jeong, M., Doris, B., Kedzierski, J., Rim, K., Yang, M.: Silicon device scaling to the sub-10-nm regime. *Science* **306**, 2057 (2004)
16. Ito, T., Okazaki, S.: Pushing the limits of lithography. *Nature* **406**, 1027 (2000)

17. Jinno, M., Miyamoto, Y., Hibino, Y.: Optical-transport networks in 2015. *Nature Photonics* **1**, 157 (2007)
18. Joannopoulos, J., Johnson, S., Winn, J., Meade, R.: *Photonic Crystals*, 2nd Ed. Princeton University Press, Princeton (2008)
19. John, S.: Strong localization of photons in certain disordered dielectric superlattices. *Phys. Rev. Lett.* **58**, 2486 (1987)
20. Kunstmann, J., Quandt, A.: Nanotubular boron-carbon heterojunctions. *J. Chem. Phys.* **121**, 10,680 (2004)
21. Kunstmann, J., Quandt, A.: Broad boron sheets and boron nanotubes: an ab initio study of structural, electronic and mechanical properties. *Phys. Rev. B* **47**, 035,413 (2006)
22. Kunstmann, J., Quandt, A., Boustani, I.: An approach to control the radius and the chirality of nanotubes. *Nanotechnology* **18**, 155,703 (2007)
23. Lundstrom, M.: Moore's law forever? *Science* **299**, 210 (2003)
24. Maier, S., Atwater, H.: Plasmonics: Localization and guiding of electromagnetic energy in metal/dielectric structures. *J. Appl. Phys.* **98**, 011,101 (2005)
25. Mead, C.: Scaling of mos technology to submicrometer feature sizes. *J. VLSI Sig. Proc.* **8**, 9 (1994)
26. Mead, C., Conway, L.: *Introduction to VLSI Systems*. Addison-Wesley, Reading (1980)
27. Meindl, J., Chen, Q., Davis, J.: Limits on silicon nanoelectronics for terascale integration. *Science* **293**, 2044 (2001)
28. Miller, D.: Rationale and challenges for optical interconnects to electronic chips. *Proceedings of the IEEE* **88**, 728 (2000)
29. Miyai, E., Sakai, K., Okano, T., Kunishi, W., Ohnishi, D., Noda, S.: Photonics: Lasers producing tailored beams. *Nature* **441**, 946 (2006)
30. Nakada, K., Fujita, M., Dresselhaus, G., Dresselhaus, M.S.: Edge states in graphene ribbons: Nanometer size effect and edge shape dependence. *Phys. Rev. B* **54**, 17,954 (1996)
31. Novoselov, K., Geim, A.K., Morozov, S.V., Jiang, D., Zhang, Y., Dubonos, S.V., Grigorieva, I.V., Firsov, A.A.: Electric field effect in atomically thin carbon films. *Science* **306**, 666 (2004)
32. Ozbay, E.: Plasmonics: Merging photonics and electronics at nanoscale dimensions. *Science* **311**, 189 (2006)
33. Pauling, L.: *Nature of the Chemical Bond*. Cornell University Press, Ithaca (1960)
34. Payne, M., Teter, M., Allan, D., Arias, T., Joannopoulos, J.: Iterative minimization techniques for ab initio total-energy calculations: molecular dynamics and conjugate gradients. *Rev. Mod. Phys.* **64**, 1045 (1992)
35. Peercy, P.: The drive to miniaturization. *Nature* **406**, 1023 (2000)
36. Quandt, A., Boustani, I.: Boron nanotubes. *ChemPhysChem* **6**, 2001 (2005)
37. Quandt, A., Ferrari, M.: Low dimensional composite nanomaterials: Theory and applications. *Adv. Sci. and Tech.* **55**, 74 (2008)
38. Quandt, A., Özdoğan, C., Kunstmann, J., Fehske, H.: Functionalizing graphene by embedded boron clusters. *Nanotechnology* **19**, 335,707 (2008)
39. Saleh, B., Teich, M.: *Fundamentals of Photonics*, 2nd Ed. Wiley-Interscience, Hoboken (2007)
40. Tang, H., Ismail-Beigi, S.: Novel precursor for boron nanotubes: the competition of two-center and three-center bonding in boron sheets. *Phys. Rev. Lett.* **99**, 115,501 (2007)
41. Totzeck, M., Ulrich, W., Göhnermeier, A., Kaiser, W.: Semiconductor fabrication: Pushing deep ultraviolet lithography to its limits. *Nature Photonics* **1**, 629 (2007)
42. Vlasov, Y., Bo, X., Sturm, J., Norris, D.: On-chip natural assembly of silicon photonic bandgap crystals. *Nature* **414**, 289 (2001)
43. Woodcroft, B.: *The Pneumatics of Hero of Alexandria*. Taylor Walton and Maberly, London (1851)
44. Yablonovitch, E.: Inhibited spontaneous emission in solid-state physics and electronics. *Phys. Rev. Lett.* **58**, 2059 (1987)
45. Yablonovitch, E.: Photonic band-gap structures. *J. Opt. Soc. Am. B* **10**, 283 (1999)

New Noninvasive Methods for ‘Reading’ of Random Sequences and Their Applications in Nanotechnology

Raoul R. Nigmatullin

Abstract A brief description of new *noninvasive* methods that can be applied for ‘reading’ of different random sequences (including quantum fluctuations registered on scales 10^{-9} – 10^{-6} m) is presented. The author expects that these methods will find wide applications in quantitative description of different random sequences. All methods described in this paper can be divided on three parts: POLS-the Procedure of the Optimal Linear Smoothing. This procedure helps to find an optimal trend and divide *separately* the possible trend of the random sequence considered and its relative fluctuations. The second method is related to the generalization of the conventional statistics and consideration of the total set of moments, including their non integer and even complex values. This statistics is extremely helpful in determination of the statistical proximity of different sequences compared. The third method is related to consideration of the detrended sequences (i.e. relative fluctuations). Based on the Linear Principle of the Strongly Correlated Variables (LPSCV) it becomes possible to prove the Universal Distribution Function for the Relative Fluctuations (UDFRF) for the strongly correlated variables *does* exist that opens quite new possibilities for quantitative ‘reading’ of a wide class of detrended sequences and their possible classifications.

1 Introduction

At analysis of any random sequence we usually have three types of errors:

1. The measurement errors related to equipment and methodology used
2. The uncontrollable errors related to the model chosen
3. The uncontrollable errors related to treatment procedure

Is it possible to eliminate the errors related to points 2 and 3?

Recent investigations of the author of this paper show that answer can be *positive*.

R.R. Nigmatullin (✉)
Kazan State University, Kremlevskaya str.18, 420008, Kazan, 420008, Russia
e-mail: nigmat@knet.ru

The author of this paper wants to outline for potential reader the unique and rather general methods related to quantitative “reading” of arbitrary random sequences having different (technical, geological, economical, medical, etc.) origin. These suggested methods have the following remarkable properties:

P1. These methods are *noninvasive*, i.e. they contain only *controllable* errors related to the chosen treatment procedure and transformations of the random sequences considered. Thanks to this property they have doubtless advantages before invasive methods as: the classical Fourier-transform (the well-known Gibbs oscillations), wavelet transformations and other recently developed methods [1–6] that contain uncontrollable errors and correlations [7].

P2. Using new methods any random sequence can be read ‘quantitatively’ and, if it is necessary, can be compared with another sequence with the usage of a ‘*universal*’ set of the reduced (fitting) parameters.

P3. These suggested methods are completely free from any *a priori* (model) suggestion related to statistical nature of the random sequence analyzed.

These three properties make these methods indispensable and universal for analysis of random sequences having different statistical nature.

At present time in accordance with ideology of their construction they can be divided on three independent parts. Not showing here the specific real data related to different technologies and branches of human activity we show the possibilities of each method on typical model file that explains the essence of each method. The selection of model data is related also with the author’s ethic position: not to touch the interests of his potential partners that give him an admission to treat of different real data.

Let us consider two ‘ideal’ functions

$$y_j \equiv F(x_j) = A \cdot [x_j^v \sin(a \cdot \ln(x_j) + \varphi_0)], \quad (1)$$

where the parameters for both chosen functions accept the following values ($A = 20, 25$); ($v = 0.7, 0.8$); ($a = 2.5, 3.0$); $\varphi_0 = (5, 6)$, correspondingly. Two functions are supposed to be located in discrete points: $x_1, x_2, \dots, x_N$ ($N = 500$). ($N = 500$). With the help of two random functions $\text{Pr}_{1,2}(x_j)$ which are *not* equaled to the Gaussian distributions and generating random numbers in the desired interval $[0, 1]$ we created two random sequences of the type

$$\text{nin}(1, 2)_j = y_j + (-\Delta + 2\Delta \cdot \text{Pr}_{1,2}(x_j)) \max(y), \quad \Delta = 0.5. \quad (2)$$

These initial sequences are shown on Fig. 1.

2 Procedure of the Optimal Linear Smoothing (POLS)

This method helps to find the optimal and smoothed trend (pseudo-fitting) function and separate it from their detrended relative fluctuations. Let us suppose that the random sequence considered contains large-scale fluctuations (trend) and high-frequency fluctuations, which are usually determined as a “high-frequency

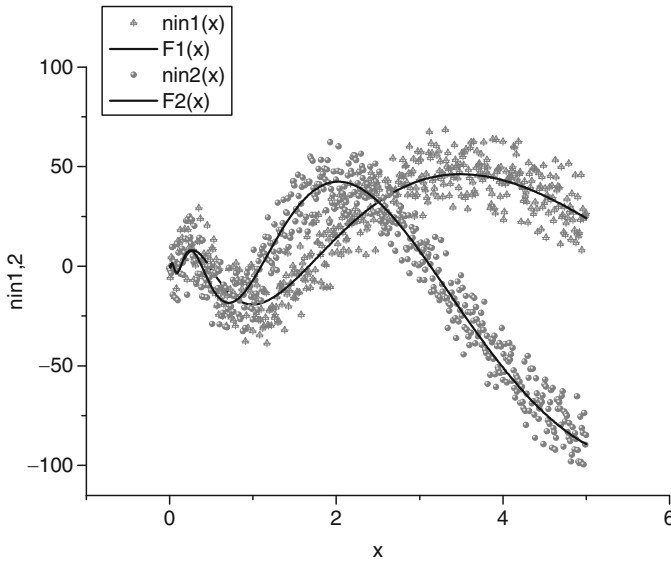


Fig. 1 Two random sequences created by algorithm presented in Eq. 1. The functions (mimicking trend) are given by bolded *solid lines*. The first and the second noises are presented by *grey crossed triangles and balls*, correspondingly

noise”. In order to separate those from each other we use the procedure of the optimal *linear* smoothing (POLS) based on the Gaussian kernel. This procedure is defined as

$$\tilde{y} = Gsm(x, y, w) = \frac{\sum_{j=1}^N K\left(\frac{x_i - x_j}{w}\right) nin_j}{\sum_{j=1}^N K\left(\frac{x_i - x_j}{w}\right)}, \quad K(t) = \exp(-t^2/2). \quad (3)$$

Here the function $K(t)$ defines the Gaussian kernel, the value w defines the current width of the smoothing window. The set $nin_j (j = 1, 2, \dots, N)$ defines the initial random (“noisy”) sequence. In spite of the fact that there are many smoothing functions imbedded in many mathematical programs this chosen function has two important features: (a) the transformed smoothed function (1) is obtained in the result of *linear* transformation and does not have uncontrollable error; (b) the value of the smoothing window (w) is adjustable (fitting) parameter and accept *any* value. This function in a certain sense can be considered as a *pseudo-fitting function*, which is not associated directly with a specific model describing the desired process. The value of the optimal window w_{opt} is chosen from the condition

$$\begin{aligned} \Delta n_j &= y_j - Gsm(x, \tilde{y}, w), \\ \tilde{y}_{w'} &= Gsm(x, y_w, w'), w' < w, \\ \min(\text{RelErr}) &= \left(\frac{\text{stdev}(\Delta n)}{\text{mean}(y)} \right) \cdot 100\%. \end{aligned} \quad (4)$$

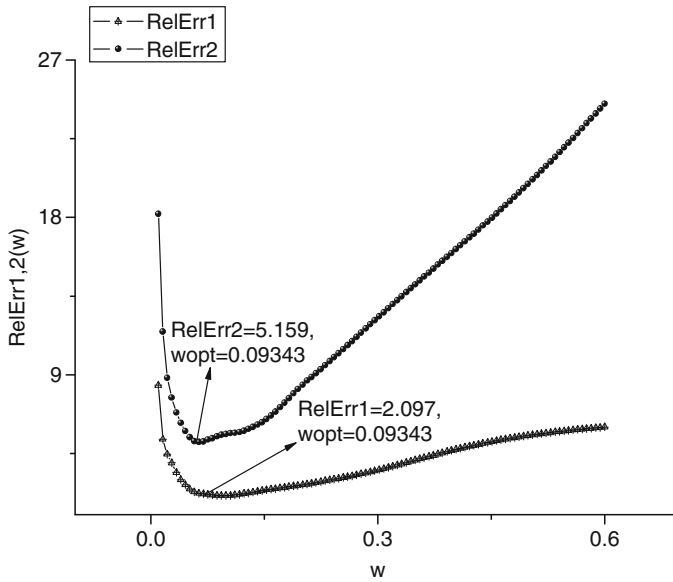


Fig. 2 Behavior of the functions for relative errors with respect to the current value of smoothing window (w). The first local minima are shown

This procedure automatically decreases the value of the initial fluctuations and helps to find the optimal value of the parameter w minimizing the value of the relative error. In many model calculations realized this optimal value w_{opt} does exist and in many cases considered corresponds to the first local minima. These minimal values for relative errors associated with random functions (2) are shown on Fig. 2. It helps to find the optimal smoothed curve (trend or pseudo-fitting function) describing the large-scale fluctuations. The desired trend minimizing the value of the relative error is found from expression

$$Tr = Gsm(x, nin, \tilde{w}), \quad \tilde{w} \equiv w_{opt}. \quad (5)$$

After calculation of the optimal trend it becomes possible to divide initial random sequence on two independent parts: (a) the optimal trend expressed by relationship (5) and (b) detrended sequence representing the values of the *relative fluctuations*, which, in turn, is expressed as

$$srf = nin - Tr. \quad (6)$$

Here srf defines the detrended sequence of the relative fluctuations. One can apply the POLS for relative fluctuations in order to calculate the optimal deviations that are closely associated with the found trend. These optimal deviations are found from expressions

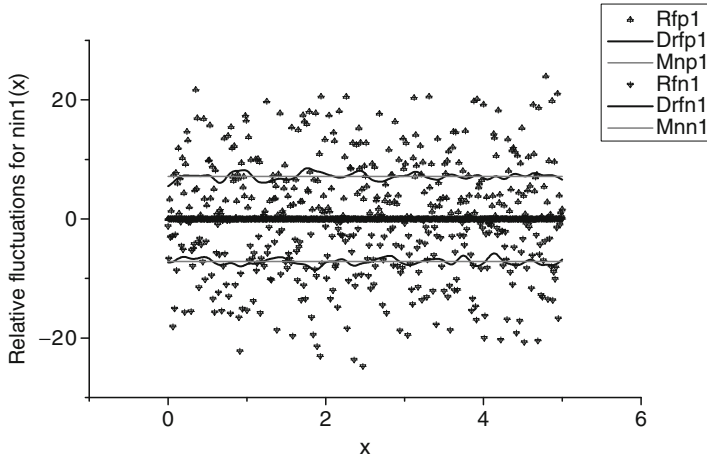


Fig. 3 The relative fluctuations (*crossed up and down triangles*) and their smoothed curves $Dtr^{\pm}$ (*solid waving lines*) realized with the help of expressions (7) are shown for the first sequence. Mean values $mn^{\pm}$ are shown by horizontal lines. For the second noise the plot looks similar and it is not shown

$$\begin{aligned}
 rf^{\pm} &= mn^{\pm} + (M^{\pm} - mn^{\pm}) \cdot (Pr^{\pm}), Dtr^{\pm} = Gsm(x, rf^{\pm}, w_{opt}), \\
 mn^{\pm} &= mean(rf^{\pm}), M^{+} = \max(rf^{+}), Pr^{+} = \frac{Srf^{+}}{\max(Srf^{+})}, \\
 M^{-} &= \min(srf^{-}), Pr^{-} = \frac{Srf^{-}}{\min(Srf^{-})}, Srf^{\pm} = \frac{1}{2} (srf \pm |srf|). \quad (7)
 \end{aligned}$$

These expressions help to find the optimal deviations $Dtr^{\pm}$ that in many cases are more precise in comparison with simple mean values $mn^{\pm}$ showing the symmetrical deviations from the calculated trend. The basic calculated values following from expression (7) for the first function is shown on Fig. 3. The similar plot is calculated for the second function but it is not shown. The true deviations as functions of the variable x are calculated as

$$Tr^{\pm} = Tr + Dtr^{\pm}, \quad (8)$$

and shown for the first function on Fig. 4. The remained relative fluctuations determined by expression (6) can be sorted out in decreasing order forming the sequence of the relative fluctuations (SRA). In fact, it can be considered as the inverted histogram function. After numerical integration this ordered sequence of the relative fluctuations can be transformed to the bell-like curve, which can be determined as the cumulative integral curve. The sequences of random fluctuations with their corresponding SRAs are shown on Fig. 5. The subsequent analysis of the cumulative integral curves is considered in Section 4.

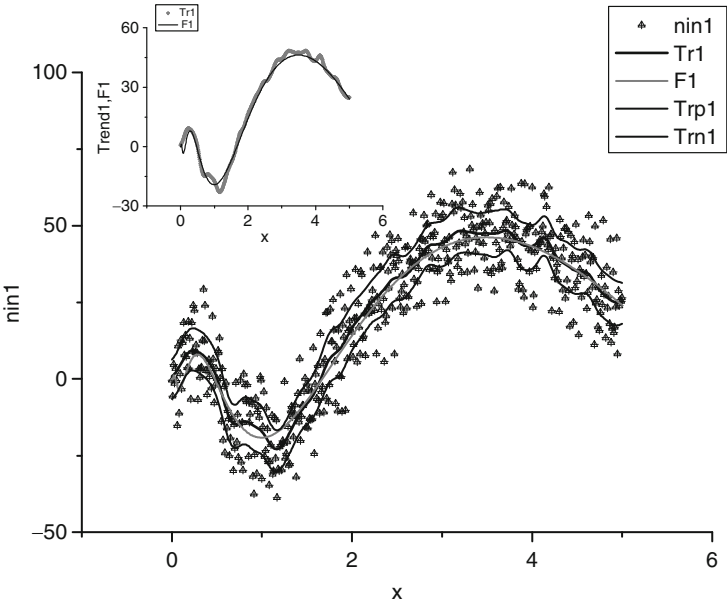


Fig. 4 Here we show the realization of expressions (7) for the first sequence. Separately, in the small frame above we show the calculated trend (grey symbols) calculated with the help of POLS and true function (black solid line). They are very close to each other. For the second sequence the plots are similar and not shown

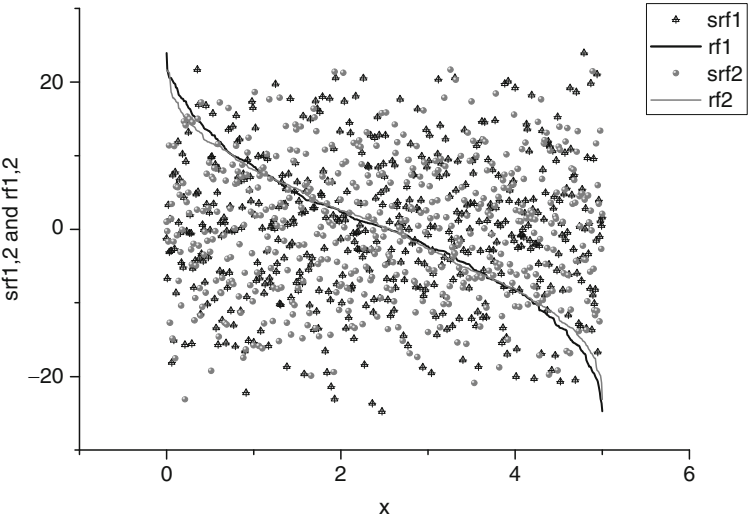


Fig. 5 Sequences of the relative fluctuations and ranged fluctuations (marked by solid lines) for both sequences after elimination of the corresponding trends are shown here. The corresponding cumulative curves $CI(x)$ and their fit by Eq. 12 realized with the help of the ECs method are shown below on Fig. 8

3 The Statistics of the Fractional Moments and Detection of the Statistical Proximity

The second method [8,9] is based on the generalization of the conception of integer moments. One can define the generalized moments of *any* (real or complex) order and operate with a function depending on the index of a moment p , which is chosen as an *independent* parameter. In order to consider all moments in the dimensionless units one can define the *generalized mean value* (GMV) function. This function has one important property. It can be fitted (with controllable error) by means of a set of exponential functions, entering in the GMV-function and so the latter one can be approximately presented in the following form

$$G_p = \left(\frac{1}{N} \sum_{j=1}^N nin_j^p \right) \equiv (\Delta_p)^{1/p} \cong g_{mx} \left(1 + \sum_{k=1}^s a_k \exp(-\lambda_k p) \right)^{1/p}. \quad (9)$$

The first sum in (9) determines the value of the moment Δ_p of the p th order.

($0 < p < \infty$), the second sum corresponds to the approximate values of the moment Δ_p and GMV-function, which is determined by the intermediate expression in (9). The relationship (9) does not contain uncontrollable transformations. Its second part can be fitted by the eigen-coordinates (ECs) method (described in detail in papers [10–12]) under approximate function with the value of the fitting error that can be always evaluated. Relationship (9) shows also that any random sequence nin_j ($j = 1, 2, \dots, N$) can be presented in the form of the monotonically increasing GMV-function in the space of the fractional moments. This space can be one-dimensional (real moments) or multidimensional (if the sequences are two-dimensional or in the case, when the values of moments are expressed by complex values). These generalizations are considered in detail in papers of the author [8,9]. Application of the GMV-functions for construction of calibration curves [9,13,14], and for detailed analysis of complex optical spectra [15] proved its high effectiveness. Recently the author of this paper has discovered that GMV-function can be used for detection of the *statistical proximity* of two or more random sequences compared. If two GMV-functions being plotted with respect to each other form a segment of the straight line, then two initial sequences are determined as statistically *close* to each other. So, the fitting parameters to the straight line can be used as a quantitative measure of the statistical proximity. At least one can use four parameters: Tg (the value of the slope, which should be located in the interval: $0 < Tg \leq 1$), B (intercept), $Relerr(\%)$ (relative error showing the accuracy to the straight line fitting) and the PCC (Pearson correlation coefficient).

Moving in this constructive direction related to the generalization of the concept of the moments “tuned” for calculation of different discrete sequences, one can define the *generalized Pearson correlation function* (GPCF), which enables to

evaluate true correlations between two or more random sequences on all set of the given moments. The GPCF is determined as [15, 16]:

$$GP_p = \frac{G_p(1, 2)}{\sqrt{G_p(1, 1)}\sqrt{G_p(2, 2)}}, G_p(1, 2) = \left(\frac{1}{N} \sum_{j=1}^N |nin(1)_j \cdot nin(2)_j|^p \right)^{1/p}. \quad (10)$$

Expression (10) at $p = 1$ coincides with the usual Pearson correlation coefficient (PCC) [17, 18], which describes only partly of possible correlations existing between two random sequences compared. Analysis of the GPCF (10) shows that this function has clearly pronounced asymptotic region at large values of the moments $p(p \rightarrow \infty)$ and passes through a possible minimal value starting from the unit value at $p = 0$. In order to satisfy the Coshy inequality the random sequences compared $nin(1, 2)_j$ should be *strictly* positive. This asymptotic value GP_p at large values of $p \rightarrow \infty$ can be associated with *true* value of correlations arising between two random sequences. The model calculations show that function (10) has unique sensitivity to the presence of a small signal (perturbation) and enables to fix the presence of one “strange” point among million(!) “native” points, i.e. the ratio S/N can constitute the value 10^{-6} ! One can stress here that correlations evaluated with the help of the GMV and GPC-functions are *different*. The first function takes into account the correlations that do *not* depend on position of the points considered while the latter one (10) takes into account the *intercorrelations* between the random points of the two sequences compared.

The statistics of the fractional moments is in the state of its development but the first nontrivial results shown in the recent publications of the author [9, 13–15] demonstrates the high effectiveness of this new statistics especially in evaluation of the statistical proximity between different trends. These methods allows also to recognize the Tsallis distribution that appears in creation of non-extensive statistical mechanics [19]. Figure 6 demonstrates the comparison of the calculated trends for both functions in the space of the fractional moments. The corresponding derivatives calculated in the space of the fractional moments can be also informative. They can reflect possible peculiarities that are appeared between two plateau regions. It is convenient to present the function (9) in the logarithmic scale. This presentation gives a possibility to detect two plateau regions and determine the true interval for the calculated values of the chosen GMV functions. In order to perform this operation we choose the moment index p in the following form

$$p = \exp \left(p \min + \frac{k}{K} (pmx - p \min) \right), \quad k = 0, 1, \dots, K$$

$$p \min = -20, \quad pmx = +20, \quad K = 150. \quad (11)$$

Figure 7 demonstrates the behavior of the GPCF in logarithmic scale (Eq. 10) and comparison of the GMV functions with respect to each other. The fitting parameters characterizing the statistical proximity phenomenon are shown inside Fig. 7. One can notice that the compared curves are close to the straight line and so one can conclude that they might have functional dependences close to each other.

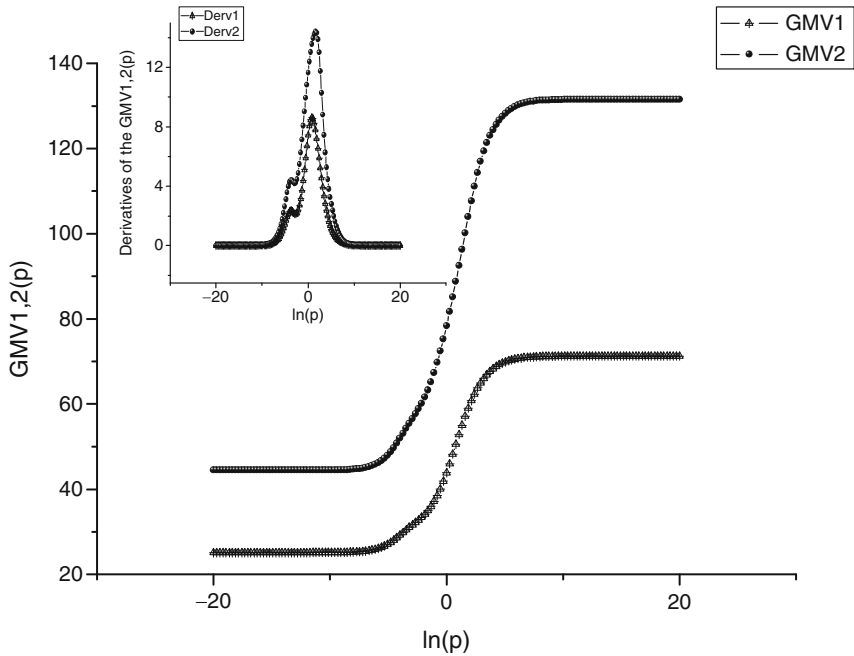


Fig. 6 The GMV functions calculated with the help of Eqs. 9 and 11 for two trends. In the small frame above the corresponding derivatives are shown

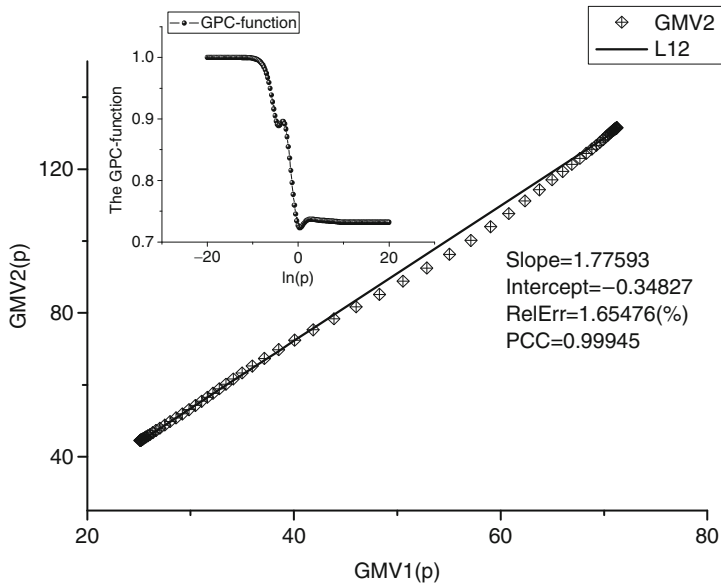


Fig. 7 Analysis of trends in the space of the fractional moments. The comparison of the calculated trends shows that they are statistically close to each other. The GPCF calculated with the help of Eq. 10 for both trends is shown on the *small figure above*

The GPCF is shown on the small figure above. This function shows the true level of correlations of the compared trends. This level is located in the vicinity of the minimal point or at the limiting values (when $\ln(p) \gg 1$). In our case $\max(\ln(p)) = 20$.

4 The Universal Distribution Function of the Relative Fluctuations

The third method is under preparation but the grounds of it have been described in detail in the recent publications [20, 21]. It was related with proving of existence of a *universal* distribution function of the relative fluctuations (UDFRF) for different detrended sequences. It becomes possible to formulate the *linear principle of the strongly-correlated variables* (LPSCV) as an alternative of the Gaussian central limiting theorem and obtain the analytical form of the desired distribution function that describes the envelope of the sequence of the ranged amplitudes (SRA) of any detrended random sequence in the case of their strong correlations. The verification on available data (optical data, EPR data, meteo- and radiation data, economical data, etc.) showed that this linear principle is realized definitely in reality and the desired distribution function, applied to the description of the envelope of the ranged amplitudes, describes very well the verified data. This distribution function contains a set of exponential or power-law functions and follows as a solution of the *functional* equation that expresses in the mathematical form the essence of the LPSCV. The detrended sequences of the ranged amplitudes (SRA) do *not* contain uncontrollable errors and represent themselves the inverted histogram. This function *cannot* be transformed to the desired histogram because the analytical form of the fitting function does *not* have the inverse function expressed in the analytical form.

The fitting parameters of the linear combination of the exponential or power-law functions can be used as a ‘universal’ noise label for description of envelopes of the sequences of the ranged amplitudes of some detrended noises or for comparison of two or more detrended sequences with each other. It is easy to prove that the cumulative integral curve obtained numerically by trapezoid method from the SRA and corresponding to the relative fluctuations (defined by Eq. 6) can be expressed approximately as

$$CI(x) \cong b(x - x_0)^\alpha \cdot (x_N - x)^\beta. \quad (12)$$

Here b, α, β form a set of the fitting parameters that are calculated easily by the ECs method. Figure 8 demonstrates the results of the fitting procedure to Eqn. 12. It is necessary to note that for the *uniform* distribution of the relative fluctuations the SRA forms a segment of the straight line and $CJ(x)$ is reduced to parabolic curve with $\alpha = \beta = 1$. For heterogeneous distributions of SRA more precise fitting functions are needed. They are considered and justified in papers [20, 21] (Fig. 8).

These three methods combined together will be extremely effective in *quantitative* description of *any* random sequence (with trend or without one) in order to construct the statistically homogeneous cluster (containing a set of the reduced

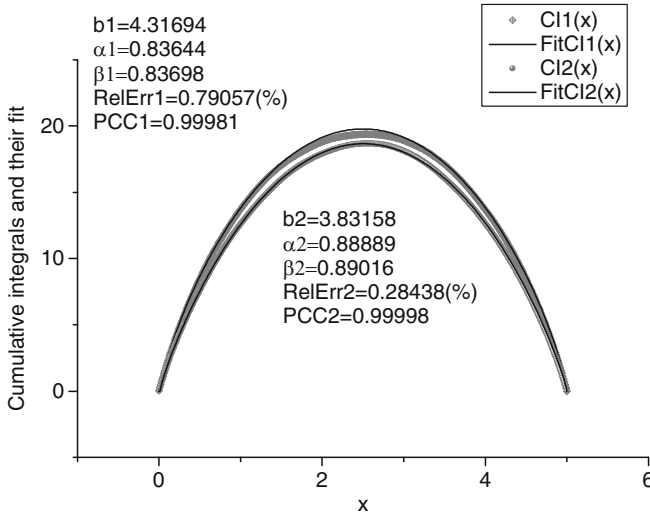


Fig. 8 The fit of the cumulative integrals to Eq. 12 realized with the help of the ECs method. The fitting parameters are shown inside the figure. The fitting curves are shown by *solid lines*

parameters) or for detection of their unusual (marginal) properties. Recently another important result (it will be published in the *Journal of Wave Phenomena*) has been obtained: one can prove that any random sequence contains a set of the so-called *noise invariants* (two of them are well-known: arithmetic mean and standard deviation) and the distribution of these invariants also forms a quantitative *universal* label (QUL) that is described by the generalized Gaussian distribution function containing in its argument the polynomial of the fourth order. So, there is a possibility to describe accurately any random function in terms of distribution of the found noise invariants. This distribution contains in general only *nine* quantitative values and no more.

5 Brief Description of Problems that Can Be Solved by These Noninvasive Methods

New methods verified on available data allowed in solution of some important problems that *never* considered before or were solved on intuitive level only.

5.1 Comparison with Pattern Equipment and Self-Verification of the Readiness of Complex Equipment to Measurements

Let us suppose that some company produces expensive equipment for accurate measurements. A group of the qualified personnel prepares the pattern equipment for

the measurements of such kind calibrating and verifying the most important parts: sensors, detectors, measurement cells, etc. that will participate in the process of the measurement. It becomes possible to record of different noises (heat, acoustic, vibro-mechanical, electromagnetic, etc.) from the most sensitive (and expensive) parts of the pattern equipment with the help of new methods “to read” them quantitatively and find the desired range of the parameters corresponding to its optimal work. Then one can determine the class of its accuracy and find the range of the fitting parameters together with their fluctuations covering the possible range of its work together with limiting boundaries. This optimal set of the fitting parameters with their deviations is kept in the memory of personal computer (PC) for comparison. The diagnosis (calibration) of the second, third and subsequent equipment is considerably simplified and it is becoming cheaper, because the PC realizes the comparison of the “noisy” sensors of the verified equipment with the pattern one. Where the comparison of such kind does not pass the personnel tries to find the reasons of this discrepancy and inconsistency.

If the usage of the verified equipment takes place during long time then based on the fitting parameters recalculated and obtained from “noisy” sensors one can determine the *ageing tendency* and replace the most fragile part of the verified equipment having the strong tendency to this ageing process. This questioning process helps to find an optimal regime of *the readiness* of complex equipment for further working. Before the verification of the statement of the “readiness of the complex equipment for working” was totally based on the intuition of the service personnel.

One can produce of this diagnostic equipment independently from the concrete functions of the equipment verified and thereby to find an optimal regime of its functioning and further usage. As such high-expensive equipment that needs to be tested one can use turbines of power stations, engines of airplanes, helicopters, rockets and other expensive equipment requiring careful fulfillment of the regimes of their optimal functioning and control of the components having the high probability of their failure. Recently the author took part in the work related to improving the calibration of the Russian standard gas sensor and proved the effectiveness of the new methods. It became possible to find the optimal smoothing window minimizing the value of the relative error and increase this value as minimum to 1.5–2.0 times.

5.2 The Diagnosis of Deceases Based on Quantitative “Reading” of Noises, Which Are Registered Inside a Human Body

Based on analogy that high-expensive equipment reminds a human body and his organs, they are similar to sensors, detectors and registered cells, which give random signals of different nature one can essentially improve and develop new methods in diagnosis of badly-detectable deceases. Here the new methods described above can be tuned for increasing of the sensitivity of the existing methods of diagnosis or for creation of new diagnostic methods based on the quantitative reading of the most informative noises (heat, acoustical, vibro-mechanical, electromagnetic, etc.), which are registered from the human working organ (heart, stomach, leaver, etc.).

5.3 *The Increasing of Sensitivity of Existing Spectrometers and Creation of Supersensitive Gas-Sensors and Chromatographers*

The preliminary investigations show that thanks to accurate reading of the fluctuation noises which are always appeared at small concentrations of the substance registered one can increase the sensitivity of the conventional spectrometers approximately on one order of magnitude with respect to its limiting value. At these detailed investigations the constructed calibration curve with respect to external value of the varied concentration can have *nonlinear* dependence [13, 14]. The most perspective application of new methods suggested can be creation of supersensitive gas- and liquid-sensors tuned for detection of different explosives, alcohol and drugs.

5.4 *Applications to Different Nanotechnologies*

These methods can prove of their effectiveness in analysis of different nanonoises that arises in the scale ($10^{-9} \div 10^{-6}$) m. In this range of scales all equipment works on the limits of its sensitivity and all spectra (optical, dielectrics, mechanical, etc.) measured are becoming noisy, i.e. affected by unpredictable and uncontrollable factors presenting during the experiment. In this case the methods suggested above can improve the sensitivity of the equipment used and divide the noise coming from the substance and equipment. It helps also to form statistically homogeneous clusters of the reduced (fitting) parameters, which differentiate the influence of different external factors. The author of these suggestions is ready to consider and discuss with potential experimentalists and ambitious groups of researchers some interesting and significant class of noises (random sequences). In the end of this subsection we want to stress *again* that the methods suggested are non-invasive (they contain) only controllable errors that helps to increase the sensitivity of different detectors and sensors tuned for measurement of “randomness” in this challengeable range of scales. Now some papers related to the direct treatment of infra-red spectra and related to detection of chemical reactions on nano(micro)electrodes surfaces are under preparation.

Acknowledgements The author wants to express his sincere thanks for the Russian Ministry of Science and Education for their financial support in the frame of grant “Russian Scientific Potential of High Schools”.

References

1. Yulmetyev R, Hanggi P, Gafarov F (2000) Stochastic dynamics of time correlation in complex systems with discrete time. *Phys Rev E* 62:6178–6194
2. Yulmetyev R, Hanggi P, Gafarov F (2002) Quantification of heart rate variability by discrete nonstationary non-Markov stochastic processes. *Phys Rev E* 65:046107

3. Yulmetyev RM, Gafarov FM, Yulmetyeva DG, Emelyanova NA (2002) Intensity approximation of random fluctuation in complex systems. *Physica A* 303:427–438
4. Timashev SF (2000) A new dialogue with nature. Stochastic and chaotic dynamics in the lakes. In Broomhead DS, Luchinskaya EA, McClintock PVE, Mulin T (eds) *STOCHAOS*. AIP Conference Proceedings. Melville, New York, pp 238–243
5. Timashev SF (2000) A new dialogue with Nature. *Ibid*:562–566
6. Timashev SF (2001) Flicker-noise spectroscopy as a tool for analysis of fluctuations in physical systems in noise in physical systems and $1/f$ fluctuations – ICNF 2001. In: Bosman G (ed) World Scientific, New Jersey, London, pp 775–778
7. Dyakonov VP (2002) Wavelets. From theory to practice. Moscow, Solon, p 448 (in Russian)
8. Nigmatullin RR (2005) The statistics of the fractional moments: new method of quantitative reading of random sequence. *Scientific notes of KSU* 147:129–161 (in Russian)
9. Nigmatullin RR (2006) The statistics of the fractional moments: Is there any chance to read “quantitatively” any randomness? *J Signal Process* 86:2529–2547
10. Nigmatullin RR (1998) Eigen-coordinates: new method of identification of analytical functions in experimental measurements. *Appl Magnet Reson*, 14:601–633
11. Nigmatullin RR (2000) Recognition of nonextensive statistic distribution by the eigen-coordinates method. *Physica A* 285:547–565
12. Abdul-Gader Jafar MM, Nigmatullin RR (2001) Identification of a new function model for the AC-impedance of thermally evaporated (undoped) selenium films using the eigen-coordinates method. *Thin Solid Films* 396:280–294
13. Nigmatullin RR, Smith G (2005) The generalized mean value function approach: new statistical tool for the detection of weak signals in spectroscopy. *J Phys D: Appl Phys* 38:328–337
14. Nigmatullin RR, Moroz A, Smith G (2005) Application of the generalized mean value function to the statistical detection of water in Decane by near-infrared spectroscopy. *Physica A* 352:379–396
15. Pershin SM, Bunkin AF, Lukyanchenko VA, Nigmatullin RR (2007) Detection of the OH band fine structure in liquid water by means of new treatment procedure based on the statistics of the fractional moments. *Laser Phys Lett* 4:808–813
16. Nigmatullin RR, Arbuzov AA, Nelson SO (2006) Dielectric relaxation of complex systems: quality sensing and dielectric properties of honeydew melons from 10 MHz to 1.8 GHz. *J Inst, JINST*, N1-P10002
17. Kendall MG, Stuart A (1962) The advanced theory of statistics, vol.1. Ch. Griffin, NY, London, Sydney, Toronto
18. Elsyasberg PE (1983) Measurable information: how much? How to treat it? Moscow, “Science”. The publishing house of physical and mathematical literature (in Russian).
19. Tsallis C (1999) Nonextensive statistics: theoretical, experimental and computational evidences and connections. *Braz J Phys* 29:1–35
20. Nigmatullin RR (2007) Universal distribution function for fluctuations of the strongly-correlated systems. *Nonlinear World* 5:572–602 (in Russian)
21. Nigmatullin RR (2008) Strongly correlated variables and existence of the universal distribution function for relative fluctuations. *J Wave Phenom* 16:119–145

Quantum Confinement in Nanometric Structures

Magdalena L. Ciurea and Vladimir Iancu

Abstract This paper discusses the quantum confinement effects in nanometric structures that form low dimensional systems. In such systems, each surface/interface acts like a potential barrier, i.e. the wall of a quantum well, generating new energy levels. These levels are computed in a model that uses the approximation of the infinite rectangular quantum wells. The model is adapted for 2D, 1D and 0D systems, respectively. Different applications are discussed. The differences between the model results and the experimental data are proved to be of the same order of magnitude as the differences between the levels computed within the frame of infinite and finite quantum well approximations.

1 Introduction

The study of the nanometric structures, i.e. low dimensional structures (LDS), presents a great interest for both its fundamental aspects and its numerous applications [1–10]. A structure is considered as a LDS if it has nanometric size on *at least* one direction. In fact, all structures have three dimensions. However, if their size on at least one direction is small enough (no more than one order of magnitude greater than the interatomic distance on that direction), the structure can be considered as quasi-low dimensional. In these structures, the ratio between the number of atoms located at the surface/interface $N_S^{(\delta)}$ and the total number of atoms N can be expressed as

$$N_S^{(\delta)}/N \approx 2(3 - \delta)a/d_\delta, \quad (1)$$

M.L. Ciurea (✉)

National Institute of Materials Physics, Bucharest-Magurele 77125, Romania
e-mail: ciurea@infim.ro

V. Iancu

University “Politehnica” of Bucharest, Bucharest 060042, Romania
e-mail: codas@physics.pub.ro

where δ is the dimensionality of the structure, a the (mean) interatomic distance, and d_δ the (minimum) LDS size. The relation is exact for 2D plane, 1D cylindrical, and 0D spherical symmetry, respectively. If one takes $a \approx 0.25$ nm, then $N_S^{(\delta)}/N = 0.5$ for $d_0 = 3$ nm, $d_1 = 2$ nm, and $d_2 = 1$ nm. If one takes $d_0 = 5$ nm (20 interatomic distances), $N_S^{(\delta)}/N = 0.3$. One can see that the surface/interface plays a very important role in nanometric structures.

On the other hand, each surface/interface acts like a potential barrier. Therefore, one can consider that they represent the walls of quantum wells, inducing quantum confinement (QC) effects, and in particular generating new energy levels. There are two aspects that arise from this interpretation: the depth and the shape of the quantum well. It was proved [11] that the QC represents a zero order effect, while the nature of the material represents only a first order effect. Consequently, the infinite quantum well must be a good first approximation. The shape of the quantum well determines the series of ratios of the differences between the QC levels (corresponding to the possible transitions). By comparing the theoretical ratios (computed for rectangular, parabolic and Woods–Saxon quantum wells) with the experimental ones, we have reached the conclusion that the rectangular quantum well is the best approximation [12]. Therefore, we will use in the following the infinite rectangular quantum well (IRQW) model.

The present paper applies this model to the study of the nanometric structures. Section 2 deals with the 2D systems, Sect. 3 with the 1D and Sect. 4 with the 0D ones. The last section presents the conclusions.

2 2D Structures

The 2D structures are layers of nanometric thickness. The best ones are monoatomic (e.g. monoatomic graphene). In the most cases, these layers are plane, parallel with the crystalline planes. Then, the electron Hamiltonian can be exactly split as the sum of two parts:

- A parallel part (i.e. parallel with the layer surface), which is Bloch-type, leading to a 2D band structure
- An orthogonal part, which can be approximated with an IRQW, leading to QC levels

Therefore, the electron energy has the form

$$\varepsilon^{(2)} = \varepsilon_n(k_x, k_y) + [(2\pi^2\hbar^2)/(m_\perp^* d^2)] p^2, \quad (2)$$

where $\varepsilon_n(k_x, k_y)$ is the 2D band energy, $m_\perp^*$ is the effective mass on the confinement direction, and $p > 0$ is a natural number.

In order to locate properly the QC levels, let us consider the case of absolute zero temperature. Then, the highest electron energy is at the top of the valence band and on the fundamental QC level. This means that *the fundamental QC level is located*

at the top of the valence band and the other QC levels are located in the band gap. To mark this, we will shift the zero of the QC energy and measure it from the top of the valence band:

$$\varepsilon^{(2)} = [\varepsilon_n(k_x, k_y) + (2\pi^2\hbar^2)/(m_\perp^*d^2)] + [(2\pi^2\hbar^2)/(m_\perp^*d^2)](p^2 - 1) \equiv \varepsilon_n^{(s)}(k_x, k_y) + \varepsilon_{p-1}. \quad (3)$$

Here $\varepsilon_n^{(s)}(k_x, k_y)$ is the shifted band energy and ε_{p-1} the QC level energy ($\varepsilon_0 \equiv 0$).

The modelling of the 2D structures consider the change of the band gap from one layer to the following one as possible quantum wells and introduce QC levels in the conduction and valence bands. However, these are not proper QC levels, but resonant levels. They do not contribute to the transport properties, but to the optical ones. To have a complete analysis of the behaviour of a LDS, one has to take both QC and resonant levels into account. As the resonant levels are well known (see [13]), in this paper we are concerned only with the proper QC levels.

As an application, let us consider the contribution of the QC levels to the functioning of quantum well solar cells [14]. In order to evaluate the internal quantum efficiency for the absorption on the QC levels we need to evaluate the matrix element of the electric dipole interaction Hamiltonian

$$H_{fi} = -\frac{2}{d}e \int_0^d \vec{E} \cdot \vec{r} \sin \frac{p_f \pi z}{d} \sin \frac{p_i \pi z}{d} dz. \quad (4)$$

To have absorption, H_{fi} must be different from zero. If the electric field is parallel with the layer (e.g. at normal incidence), the quantum selection rule is $p_f = p_i$. Such a case involves only resonant levels. If the field is orthogonal on the layer, $p_f - p_i = 2p - 1$. This corresponds to the QC levels. Obviously, the absorbed wavelengths are different. Then the quantum efficiencies and the corresponding transition energies are

$$\eta_{||} = (8\pi^2 e^2)/(c\hbar\varepsilon_0\varepsilon_r), \quad (5)$$

$$\Delta E \equiv hc/\lambda = E_g + \pi^2\hbar^2/2m_\parallel^*d^2 \quad (6)$$

(where $1/m_\parallel^* = 1/m_{e||}^* + 1/m_{g||}^*$ is the exciton mass) for the transition between the first symmetric resonant levels at normal incidence, and

$$\eta_\perp = [(4096e^2)/(\pi^2 c\hbar\varepsilon_0\varepsilon_r^2)] \cdot [p^2/(4p^2 - 1)^4], \quad (7)$$

$$\Delta E \equiv hc/\lambda = [(\pi^2\hbar^2)/2m_\perp^*d^2] \cdot (4p^2 - 1), \quad (8)$$

for the transitions between QC levels.

It has to be remarked that the resonant levels appear only in multilayer structures, namely in the layers with smaller gap, while the QC levels appear in all the layers.

3 1D Structures

In the following, we will consider only cylindrical nanowires, to facilitate the identification of the orbital magnetic quantum number. Because of this choice, the splitting of the Hamiltonian as the sum of a longitudinal part and a transversal one is no longer exact. However, this splitting is a good approximation. Then the electron energy is

$$\varepsilon^{(1)} = \varepsilon_n(k_z) + [(2\pi^2\hbar^2)/(m_\perp^*d^2)] \cdot x_{p,l}^2, \quad (9)$$

where $\pi \cdot x_{p,l}$ is the p -th non-null zero of the cylindrical Bessel function $J_l(x)$.

Once again we can analyze the case of absolute zero temperature to find out that the QC levels are located in the band gap, so that we can shift the zero of the QC energy and measure it from the top of the valence band:

$$\begin{aligned} \varepsilon^{(1)} = \{ & \varepsilon_n(k_z) + [(2\pi^2\hbar^2)/(m_\perp^*d^2)] \cdot x_{1,0}^2 \} \\ & + [(2\pi^2\hbar^2)/(m_\perp^*d^2)] \cdot (x_{p,l}^2 - x_{1,0}^2) \equiv \varepsilon_n^{(s)}(k_z) + \varepsilon_{p-1,l}, \end{aligned} \quad (10)$$

where $\varepsilon_n^{(s)}(k_z)$ is the shifted band energy and $\varepsilon_{p-1,l}$ the QC level energy ($\varepsilon_{0,0} \equiv 0$).

If we try to analyze different excitation transitions, we have to remember that the valence band acts like an infinite particle reservoir, so that all excitations start from the fundamental QC level. Then, the ratio between consecutive transition energies (playing the role of activation energies) is

$$R^{(1)} = \varepsilon_{p'',l''}/\varepsilon_{p',l'} = (x_{p''+1,l''}^2 - x_{1,0}^2)/(x_{p'+1,l'}^2 - x_{1,0}^2). \quad (11)$$

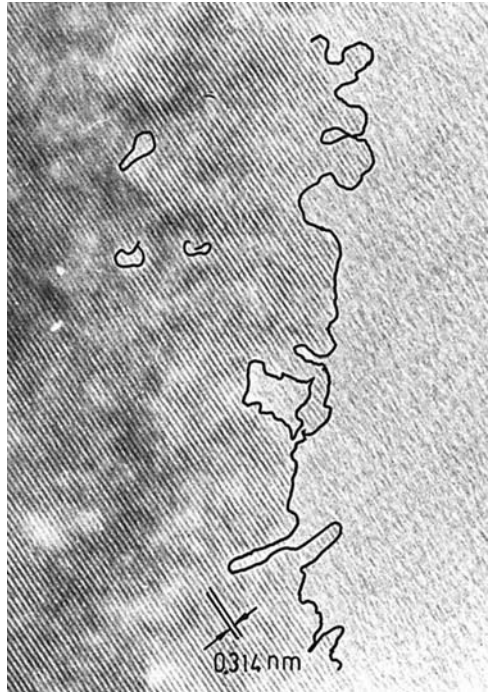
The choice of the quantum selection rules depends on the kind of excitation we have. In the case of thermal excitation, the condition is that the energy variation should be minimum. In the case of an electrical transition (i.e. under high electric field, $eU \gg k_B T$), $\Delta l = 0$. In the case of an optical transition, $\Delta l = \pm 1$.

As an application, we will analyze the case of nanocrystalline porous silicon (nc-PS). In a previous paper [15], we have discussed the microstructure of nc-PS films that present a double scale of porosity: an alveolar columnar microporous structure (pore diameters of 1.5–3 μm), and a nanoporous structure of the alveolar walls (100–200 nm thickness). High resolution transmission electron microscopy (HRTEM) images proved that these walls form a nanowire network, with nanowire diameter of 1–5 nm (see Fig. 1).

The investigation of the temperature dependence of the dark current in these nc-PS films [16] proved that the characteristics were of Arrhenius type. For fresh samples, only one activation energy, $E_1 = (0.52 \pm 0.03) \text{ eV}$, was observed (see Fig. 2a). For samples stabilized by controlled oxidation, two activation energies, $E_1 = (0.55 \pm 0.05) \text{ eV}$, $E_2 = (1.50 \pm 0.30) \text{ eV}$, appeared, the change occurring rather abruptly at $T \approx 280 \text{ K}$ (see Fig. 2b).

The ratio of the two energy values is $E_2/E_1 = 2.727$. From this ratio and (11), we can identify $E_1 \equiv \varepsilon_{1,0}$ and $E_2 \equiv \varepsilon_{2,0}$. Using (10), we then find

Fig. 1 HRTEM detail of the alveolar wall of nc-PS, shown by lattice fringes contrast with respect to amorphous silicon oxide and glue [15] (Reused with permission from ML. Ciurea, V. Iancu, V.S. Teodorescu, L.C. Nistor, and M.-G. Blanchin, "Journal of Electrochemical Society 146, 2517, 1999". Copyright © 1999, The Electrochemical Society, Inc.)



$d_s = (3.22 \pm 0.05)$ nm, in agreement with the microstructure investigations. If we use the same identification for the fresh samples, we find $d_f = (3.31 \pm 0.03)$ nm. This means that by oxidation the diameter decreased with less than $1 \text{ \AA}$, which is absurd. This discrepancy arose from the fact that we have used the effective mass approximation (EMA), which is no longer valid at nanometric scale. In this approximation, the energy is inversely proportional with the square of the diameter. A thorough analysis, performed by using the linear combination of atomic orbitals (LCAO) method [17], proved that $\varepsilon \sim d^{-\alpha}$, with $\alpha = 1.02$ for cylindrical nanowires, leading to $d_f = (3.40 \pm 0.03)$ nm. This means that the oxide layer at the surface of the nanowires is monoatomic.

The phototransport (PT) in nc-PS was studied by tracing the spectral dependence of the photocurrent [18]. Several maxima and shoulders were identified in the $I-\lambda$ characteristics (see Fig. 3) and all but one could be identified with transitions between QC levels (see Table 1). The maximum No. 6, at 873 nm (1.42 eV), as attributed to surface states. The relative errors made by the model in both cases ($I-T$ and $I-\lambda$ characteristics) were under 3%.

Fig. 2 $I-T$ characteristics taken in dark on (a) fresh and (b) stabilized nc-PS [16] (Reprinted from Thin Solid Films 325, M.L. Ciurea, I. Baltog, M. Lazar, V. Iancu, S. Lazanu, and E. Pentia, “Electrical behaviour of fresh and stored porous silicon films”, 271. Copyright © 1998, with permission from Elsevier.)

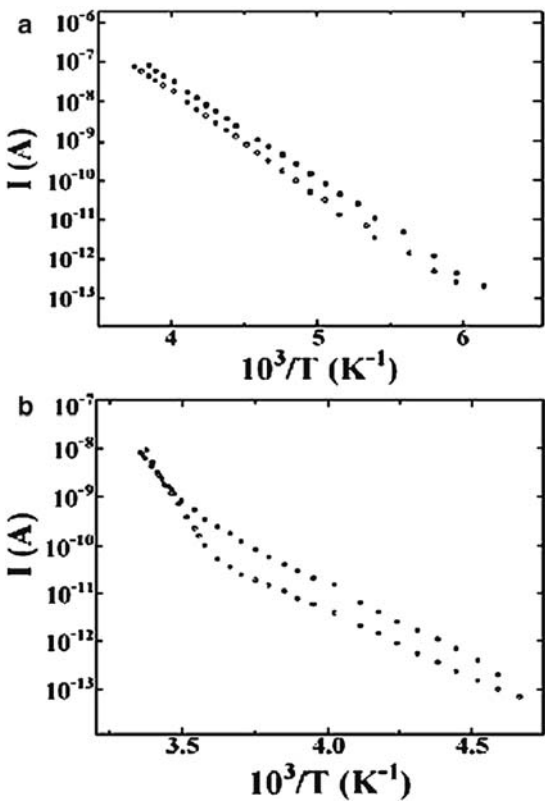


Fig. 3 $I-\lambda$ characteristics taken at 20 V on stabilized nc-PS [18] (Reprinted with permission from V. Iancu, M.L. Ciurea, I. Stavarache, and V.S. Teodorescu (2007), “Journal of Optoelectronics and Advanced Materials 9, 2638”. Copyright © 2007.)

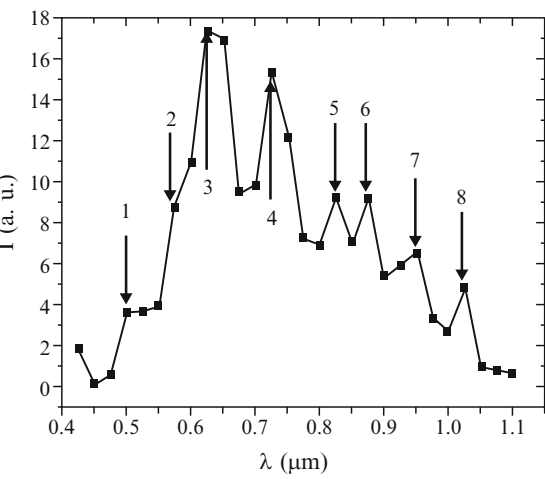


Table 1 QC transitions identified in PT measurements on nc-PS [18] (Reprinted with permission from V. Iancu, M.L. Ciurea, I. Stavarache, and V.S. Teodorescu (2007), “Journal of Optoelectronics and Advanced Materials 9, 2638”. Copyright © 2007.)

No.	λ (nm)	E_{exp} (eV)	Transition
1.	506	2.45	$(1, 2) \rightarrow (2, 3)$
2.	575	2.16	$(0, 0) \rightarrow (1, 2)$
3.	631	1.96	$(1, 1) \rightarrow (0, 3)$
4.	719	1.72	$(2, 0) \rightarrow (1, 2)$
5.	825	1.50	$(2, 0) \rightarrow (3, 1)$
6.	875	1.42	–
7.	935	1.33	$(1, 0) \rightarrow (0, 2)$
8.	1025	1.21	$(1, 0) \rightarrow (2, 1)$

4 0D Structures

As 0D structures, we will consider only spherical dots, for similar reasons as for the nanowires. In the case of dots, there appears a specific behaviour [12]. When the diameter is small enough (under about 20 interatomic distances, i.e. about 5 nm), one has no longer a proper band structure, but sets of levels forming quasibands, separated by rather large intervals (quasigaps). More than that, the momentum conservation law no longer applies [19]. Such dots are usually called “quantum dots”.

In a quantum dot, the energy is simply

$$\varepsilon^{(0)} = [(2\pi^2\hbar^2)/(m_e d^2)] \cdot x_{p,l}^2, \quad (12)$$

where $\pi \cdot x_{p,l}$ is the p-th non-null zero of the spherical Bessel function $j_l(x)$ and the effective mass is replaced by the free electron mass (without a band structure, the concept of effective mass becomes meaningless). Indeed, from the LCAO computations [17], we have $\varepsilon \sim d^{-\alpha}$, with $\alpha = 1.39$ for spherical dots. This means that we can approximate the effective mass as

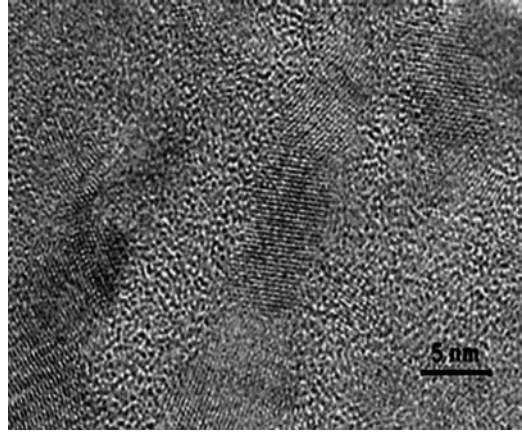
$$m^* \approx m_\infty^* + (a/d)^\beta (m_e - m_\infty^*), \quad (13)$$

with $\beta \approx 1$. It is easy to see that $m^* \approx m_e$ for quantum dots. Once again, the QC levels are located in the quasiband gap; and once again we can measure the QC energy from the fundamental state ($\varepsilon_{0,0} \equiv 0$), by writing

$$\begin{aligned} \varepsilon^{(0)} &= [(2\pi^2\hbar^2)/(m_e d^2)] \cdot x_{1,0}^2 \\ &+ [(2\pi^2\hbar^2)/(m_e d^2)] \cdot (x_{p,l}^2 - x_{1,0}^2) \equiv E_V + \varepsilon_{p-1,l}. \end{aligned} \quad (14)$$

When one considers the transitions, one has the same selection rules as in the case of nanowires. However, as there is no more valence band, i.e. no more particle reservoir, the transitions are made from one QC level to the next permitted one (following the selection rules), and (11) becomes

Fig. 4 HRTEM image of a Si-SiO₂ sample with $x = 66\%$ [12] (Reprinted from Chemical Physics Letters 423, 225, M.L. Ciurea, V.S. Teodorescu, V. Iancu, and I. Balberg, “Electrical transport in Si-SiO₂ nanocomposite films”, 225. Copyright © 2006, with permission from Elsevier.)



$$R^{(0)} = (\varepsilon_{p'',l''} - \varepsilon_{p',l'}) / (\varepsilon_{p',l'} - \varepsilon_{p,l}) \\ = (x_{p''+1,l''}^2 - x_{p'+1,l'}^2) / (x_{p'+1,l'}^2 - x_{p+1,l}^2). \quad (15)$$

If the dots are bigger, one has a proper band structure and (15) takes the same form as (11).

We will apply these results to a Si-SiO₂ nanocomposite, formed by nanocrystalline silicon (nc-Si) quantum dots embedded in an amorphous silicon dioxide (a-SiO₂) matrix. The microstructure investigations proved that, for nc-Si volume concentration x in the interval 50–75%, most of the dots have diameters around 5 nm [12], [20], as it can be seen from Fig. 4.

The I - T characteristics, measured at different voltages on a sample with the nc-Si volume concentration $x = 66\%$ [12] are presented in Fig. 5. One can see that, at low voltages, there are three activation energies, $E_1 = (0.22 \pm 0.02)$ eV, $E_2 = (0.32 \pm 0.02)$ eV, and $E_3 = (0.44 \pm 0.02)$ eV. By using (11) and (15), one obtains the confirmation that the nc-Si form quantum dots. At the same time, one can identify the transitions between QC levels, by taking $E_1 = \varepsilon_{1,1} - \varepsilon_{0,1}$, $E_2 = \varepsilon_{2,1} - \varepsilon_{1,1}$, and $E_3 = \varepsilon_{3,1} - \varepsilon_{2,1}$. From (14) one obtains $d = (5.2 \pm 0.4)$ nm, in agreement with the microstructure measurements. Then, the model errors are smaller than 3%.

It can be observed in Fig. 5 that the first activation energy appears only at low voltages. This fact was explained by studying the I - V characteristics. The characteristic taken at the same concentration is presented in Fig. 6 [12]. From Fig. 4 one can see that the quantum dots form chains, but these chains are not long enough to reach from one electrode to the other one (separated by 1 mm distance). Therefore, the carriers tunnel through the a-SiO₂ regions. The height of the potential barrier was estimated from the nc-PS measurements to be 2.2 eV [16].

The number of tunnelled barriers must be of the order of hundreds or more, so that the barrier becomes trapezoidal under the applied field. Consequently, the high field-assisted tunnelling is described by the Simmons formula (see [12])

Fig. 5 $I-T$ characteristics taken in dark on a Si-SiO₂ sample with $x = 66\%$ [12] (Reprinted from Chemical Physics Letters 423, 225, M.L. Ciurea, V.S. Teodorescu, V. Iancu, and I. Balberg, “Electrical transport in Si-SiO₂ nanocomposite films”, 225. Copyright © 2006, with permission from Elsevier.)

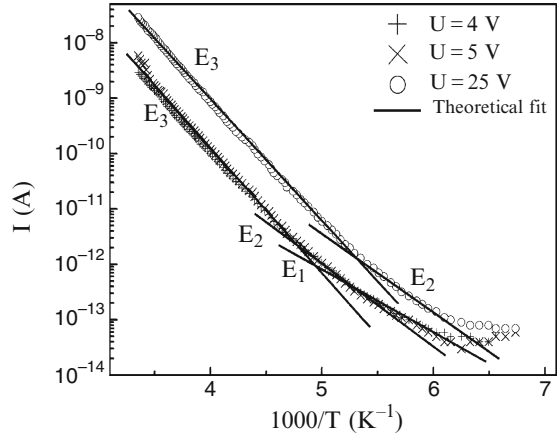
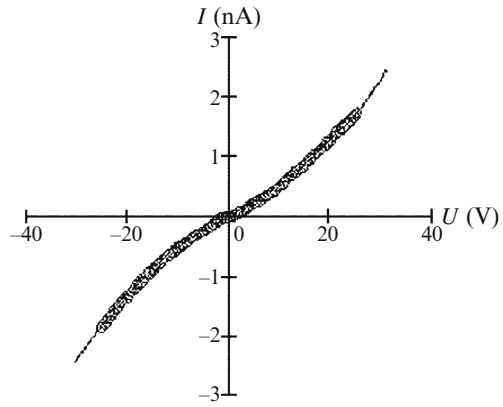


Fig. 6 $I-V$ characteristic taken in dark on a Si-SiO₂ sample with $x = 66\%$ [12] (Reprinted from Chemical Physics Letters 423, 225, M.L. Ciurea, V.S. Teodorescu, V. Iancu, and I. Balberg, “Electrical transport in Si-SiO₂ nanocomposite films”, 225. Copyright © 2006, with permission from Elsevier.)



$$I = a \left[\varphi \exp(-\delta\chi\sqrt{\varphi}) - (\varphi + qU_b) \exp(-\delta\chi\sqrt{\varphi + qU_b}) \right], \quad (16)$$

where a is a constant proportional with the number of equivalent paths for the carriers, q is the carrier charge ($|q| = e$), $\chi = (8m^*/\hbar^2)^{1/2}$ (remember that, for quantum dots, $m^* \approx m_e$), $U_b = U/N$ is the mean bias applied on a barrier of height φ and width δ , and N is the number of barriers.

One can see that there are only three fit parameters, as (16) can be put in the form [12]

$$I = I_0 \text{sign}(U) \left[(1 - |U|/U_0) \exp(-\alpha\sqrt{1 - |U|/U_0}) - \exp(-\alpha) \right]. \quad (17)$$

Here $I_0 = |a|\varphi$, $U_0 = N\varphi/e$, $\alpha = \delta\chi\varphi^{1/2}$, and $q = -e$. By fitting the experimental curve and using the value $\varphi = 2.2 \pm 0.1$ eV, we have obtained $\delta = (0.97 \pm 0.05)$ nm and $N = 87 \pm 4$. Then $eU_{\text{max}}/N\varphi \approx 1/6$ (the barrier is

indeed trapezoidal) and for $U = 25$ V, $U_b = (0.29 \pm 0.01)$ V $> E_1/e$ (the first level is already excited by the applied field).

Another application consists in evaluating the internal quantum efficiency for the quantum dot solar cells [21]. Inside the dot, the wavefunction is

$$\psi_{n,l,m}(r, \theta, \varphi) = N_{n,l} R^{-3/2} j_l(z_{n+1,l} r/R) Y_{l,m}(\theta, \varphi), \quad (18)$$

where $R = d/2$ is the dot radius, $Y_{l,m}(\theta, \varphi)$ is the spherical harmonics, $z_{n+1,l} \neq 0$ is the $(n+1)$ -th non-null zero of the spherical Bessel function $j_l(x)$ ($z_{0,l} \equiv 0$), and

$$N_{n,l} = \left[\int_0^1 j_l^2(z_{n+1,l} u) u^2 du \right]^{-1/2} \quad (19)$$

is the normalization constant. The light beam can be considered as parallel with the Oz axis, due to the spherical symmetry of the dots. Then, the absorption selection rules are $\Delta l = \pm 1$, $\Delta m = \pm 1$.

If we compute now the internal quantum efficiency for the absorption threshold, we find that, as in the case of 2D structures, the wavelength is proportional with the square of the size, while the internal quantum efficiency is size-independent. For a Si-SiO₂ structure with dot diameter $d = 5$ nm, we have found that the threshold wavelength is $\lambda_{thr} \approx 19.7$ nm and the corresponding internal quantum efficiency is $\eta \approx 4.33\%$.

5 Conclusions

We have analyzed the quantum confinement effects in 2D, 1D, and 0D nanometric structures. These effects were modelled by means of the IRQW approximation. The model explains most of the phenomena observed in such structures. Indeed, *almost all* the energies measured in electrical transport and phototransport can be interpreted as due to transitions between QC levels. The fact that different phenomena lead to different energies is related to the selection rules. The differences between the results of the model and the experimental data are produced by the fact that the depth of the quantum well is finite, as well as by the size and shape distribution.

References

1. Brewer M, Utzinger U, Li Y, Atkinson EN, Satterfield W, Auersperg N, Follen M, Bast R (2002) Fluorescence spectroscopy as a biomarker in a cell culture and in a nonhuman primate model for ovarian cancer chemopreventive agents. J Biomed Optics 7:20–26
2. LaVan DA, McGuire T, Langer R (2003) Small-scale systems for in vivo drug delivery. Nat Biotechnol 21:1184–1191

3. Gaburro Z, Oton CJ, Pavesi L (2004) Opposite effects of NO₂ on electrical injection in porous silicon gas sensors. *Appl Phys Lett* 84:4388–4390
4. Li XJ, Zhang YH (2000) Quantum confinement in porous silicon. *Phys Rev B* 61:12605–12607
5. McDonald SA, Cyr PW, Levina L, Sargent EH (2004) Photoconductivity from PbS-nanocrystal/semiconducting polymer composites for solution-processible, quantum-size tunable infrared photodetectors. *Appl Phys Lett* 85:2089–2091
6. Arango AC (2005) A quantum dot heterojunction photodetector. M.Sc. thesis. MIT, Cambridge, MA, USA
7. Walters RJ, Bourianoff GI, Atwater HA (2005) Field-effect electroluminescence in silicon nanocrystals. *Nat Mater* 4:143–146
8. Fert A (2008) Spintronics: fundamentals and recent developments. 22nd General Conference Cond. Matter Division Eur. Phys Soc, Rome, 25–29 August 2008
9. Ihn T, Gustavsson S, Müller T, Schnez S, Güttinger J, Molitor F, Stampfer C, Ensslin K (2008) Electronic transport in quantum dots: from GaAs to grapheme. 22nd Gen. Conf. Cond. Matter Division Eur. Phys. Soc., Rome, 25–29 August 2008
10. Shields, A.: Nano-phonic devices for quantum information technology. 22nd Gen. Conf. Cond. Matter Division Eur. Phys. Soc., Rome, August 25–29 2008
11. Iancu V, Ciurea ML (1998) Quantum confinement model for electrical transport phenomena in fresh and stored photoluminescent porous silicon films. *Solid-State Electron* 42:1893–1896
12. Ciurea ML, Teodorescu VS, Iancu V, Balberg I (2006) Electrical transport in Si–SiO₂ nanocomposite films. *Chem Phys Lett* 423:225–228
13. Harrison P (2005) Quantum wells, wires and dots. Wiley, Chichester
14. Iancu V, Fara L (2007) Modelling of multi-layered quantum well photovoltaic cells. The 17th International Photovoltaic Science and Engineering Conference PVSEC 17. Fukuoka, 3–7 December 2007
15. Ciurea ML, Iancu V, Teodorescu VS, Nistor LC, Blanchin M-G (1999) Microstructural aspects related to carriers transport properties of nanocrystalline porous silicon films. *J Electrochem Soc* 146:2517–2521
16. Ciurea ML, Baltog I, Lazar M, Iancu V, Lazanu S, Pentia E (1998) Electrical behaviour of fresh and stored porous silicon films. *Thin Solid Films* 325:271–277
17. Delerue C, Allan G, Lannoo M (1993) Theoretical aspects of the luminescence of porous silicon. *Phys Rev B* 48:11024–11036
18. Iancu V, Ciurea ML, Stavarache I, Teodorescu VS (2007) Phototransport and photoluminescence in nanocrystalline porous silicon. *J Optoelectron Adv Mater* 9:2638–2643
19. Heitmann J, Müller F, Yi LX, Zacharias M, Kovalev D, Eichhorn F (2004) Excitons in Si nanocrystals: confinement and migration effects. *Phys Rev B* 69:195309–1–7
20. Teodorescu VS, Ciurea ML, Iancu V, Blanchin M-G (2008) Morphology of Si nanocrystallites embedded in SiO₂ matrix. *J Mater Res* 23:2990–2995
21. Iancu V, Mitroi MR, Lepadatu A-M, Ciurea ML Evaluation of the internal quantum efficiency for quantum dot photovoltaic cells. *Nanotechnol.* (submitted, December 2008)

Part II

Techniques and Applications

Air-Fuel Ratio Control of an Internal Combustion Engine Using CRONE Control Extended to LPV Systems

Mathieu Moze, Jocelyn Sabatier, and Alain Oustaloup

Abstract An extension of the CRONE control method to LPV systems is presented in the paper. A LPV controller whose parameters are scheduled on those of the system is determined from small gain theorem applied with a LFT formulation of the system. This approach permits open loop insensitivity to varying parameters, enabling its optimal parameterization for robustness purposes. The method is finally applied to air/fuel ratio control of internal combustion engine. The approach is validated from simulation results.

1 Introduction

Fractional differentiation is now a well known tool for controller synthesis. Several presentations and applications of fractional PID controller [8, 9, 18, 24] and of CRONE controller [13, 22] (CRONE is the French acronym of “Commande Robuste d’Ordre Non Entier” which means Robust Control using Fractionnal Order) demonstrate their efficiency. Fractional differentiation also permits a simple representation of some high order complex integer systems [5]. Consequently, basic properties of fractional systems have been investigated these last ten years. Criteria and theorems are now available in the literature concerning stability [16], observability, and controllability [17] of fractional systems.

Whereas CRONE methodology have been developed for a large variety of plants such as linear non minimum phase plants, linear unstable plants [23], linear plants with low damped modes [22], linear sampled plants, multivariable plants [12], and non linear plants [25], time variant plants have been only studied through consideration of periodic coefficients [26, 27]. General case of variable parameters plants is still to be developed.

M. Moze, J. Sabatier (✉), and A. Oustaloup

Laboratoire de l’Intégration, du Matériau au Système – 351 cours de la Libération – F33405
TALENCE cedex France

e-mail: mathieu.moze@ims-bordeaux.fr; jocelyn.sabatier@ims-bordeaux.fr;
alain.oustaloup@ims-bordeaux.fr

One particularity of CRONE synthesis is that uncertainties are taken into account through convex hulls in Nichols chart, computed at given frequencies. Transfer function and frequency response concepts being not available for time varying systems, a new formulation of CRONE control is required for such an extension. The paper presents an extension of the method presented in [19] to LPV systems case [20].

In the sequel, the notation A^* is used to refer to transpose conjugate of matrix A . For any hermitian matrix B , notation $B > 0$ ($B \geq 0$) means that B is positive (semi positive) definite. For any operator G , $\|G\|_{\mathcal{L}_2}$ denotes its $\mathcal{L}_2$ gain. If G is Linear Time Invariant, its $\mathcal{L}_2$ gain equals the H_∞ norm of the associated transfer matrix $G(s)$ noted $\|G(s)\|_\infty$.

2 CRONE Control Principles

The CRONE control design procedure is a well defined tool that ensures robustness properties to a controlled system using complex fractional differentiation in the definition of the open loop. Three generations of CRONE control have been developed. The most advanced is now presented.

2.1 Template Definition

The third generation of CRONE control is based on the definition of a template which can be represented in the Nichols chart by an any-direction straight line segment around open loop gain crossover frequency ω_u [23]. This template is based on the real part with respect to imaginary unit i of a fractional integrator [13]:

$$\beta_T(s) = \left(\cosh\left(b\frac{\pi}{2}\right) \right)^{-1} \operatorname{Re}_{/i} \left[\left(\frac{\omega_u}{s} \right)^n \right], \quad (1)$$

where $n = a + ib$, $n \in C_i$, and $s = \sigma + j\omega$, $s \in C_j$, C_i and C_j being respectively time-domain and frequency-domain complex planes. In the Nichols chart, the real order a determines the phase placement of the template, and the imaginary order b determines its angle to the vertical, at frequency ω_u imposed by the designer.

2.2 Open Loop Transfer Function Including the Template

As the design of a third generation CRONE controller takes into account specifications at low and high frequencies, the open loop transfer function is

$$\beta(s) = \beta_l(s)\beta_n(s)\beta_h(s), \quad (2)$$

where

$$\beta_n(s) = Y_0 C^{\text{sign}(b)} \left(C_0 \frac{1 + \frac{s}{\omega_h}}{1 + \frac{s}{\omega_b}} \right)^a \text{Re}/i \left(\left(C_0 \frac{1 + \frac{s}{\omega_h}}{1 + \frac{s}{\omega_b}} \right)^{ib} \right)^{-\text{sign}(b)} \quad (3)$$

defines a limited frequency band template. C and C_0 are such that the gain of $\beta_n(s)$ is Y_0 at any desired resonant frequency ω_r .

At low frequency is added a proportional integrator

$$\beta_l(s) = \left(C_b \left(1 + \frac{s}{\omega_b} \right) / \frac{s}{\omega_b} \right)^{n_b}, \quad (4)$$

where C_b is such that ω_r is the gain crossover frequency of $\beta_l(s)$, and $n_b \in \mathbb{N}^+$ enables the steady state error to be nullified.

At high frequency is added a low-pass filter

$$\beta_h(s) = \left(C_h / \left(1 + \frac{s}{\omega_h} \right) \right)^{n_h}, \quad (5)$$

where C_h is such that $|\beta_h(j\omega_r)| = 1$, and $n_h \in \mathbb{N}^+$ enables the constancy or the decrease of the control effort sensitivity function.

2.3 Parameterization of Optimal Open Loop

CRONE control design guaranties the robustness of both stability and performance through the robustness of the maximum Q of the complementary sensitivity function magnitude. An open loop Nichols locus is considered as optimal if the generalized template around ω_r tangents the Q_r Nichols magnitude contour for the nominal state of the plant, Q_r being the desired magnitude peak, and minimizes the variations of Q for the other parametric states.

The open loop Nichols locus optimization thus consists in determining optimal values of a , b , ω_b , ω_h and Y_0 appearing in (3). Figure 1 shows how these parameters affect the Nichols locus.

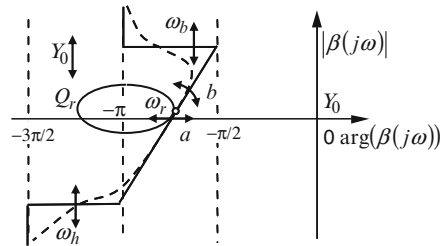


Fig. 1 Effect of a , b , ω_b , ω_h and Y_0 on the asymptotic Nichols locus of $\beta(j\omega)$

In fact, the tangency condition of the Nichols locus of $\beta(s)$ with a Q_r Nichols magnitude contour reduces parameterization to three independent high level parameters [23]: the open loop gain Y_0 at ω_r , and the corner frequencies ω_b and ω_h . The controller tuning is thus reduced to the computation of three parameter optimal values, which is the number of parameters required for standard PID tuning. The fact that only three parameters are needed for controller synthesis is essential from an industrial point of view and of major interest for computational time during optimization.

2.4 Optimization for Optimal Open Loop Determination

Objective function Generally, the optimal template is achieved through minimization of the cost function

$$J = (Q_{\max} - Q_r)^2 + (Q_{\min} - Q_r)^2, \quad (6)$$

or

$$J_{\text{reduced}} = (Q_{\max} - Q_r)^2, \quad (7)$$

where Q_r is the desired sensitivity function magnitude peak, $Q_{\max}$ and $Q_{\min}$ being the extreme values of this peak when the plant is affected by perturbations.

Figure 2 shows how minimisation of criterion (6) positions the template so that the uncertainty domains overlap as little as possible the low stability degree area of the Nichols chart.

Constraints Associated with criterion (6) are constraints on sensitivity function $S(s) = (1)/(1 + \beta(s))$, complementary sensitivity function $T(s) = \beta(s)S(s)$ and control effort function $CS(s) = C(s)S(s)$, through:

$$\sup |CS(j\omega)| < CS_{\max}(\omega) \quad (8)$$

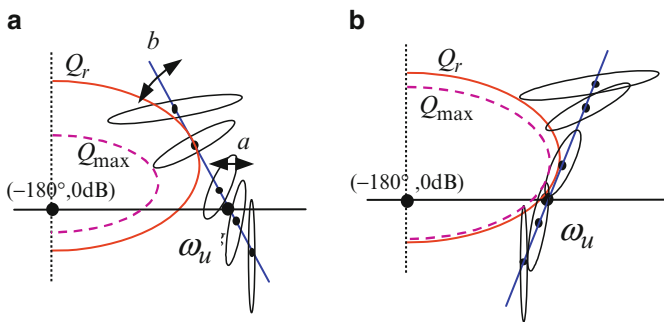


Fig. 2 Open-loop placement by optimal approach (a) unspecified template (b) optimal template

to limit the solicitation level of the plant input,

$$\sup |S(j\omega)| < S_{\max}(\omega) \quad (9)$$

to ensure a satisfactory rejection of plant output disturbances, and

$$\inf |T(j\omega)| > T_{\min}(\omega) \text{ and } \sup |T(j\omega)| > T_{\max}(\omega), \quad (10)$$

to limit the sluggishness of the responses, and to ensure a satisfactory rejection of measurement noise.

2.5 Parametric Synthesis of the Controller

Once the optimal nominal open loop transfer function is determined, fractional controller $C_F(s)$ is defined by its frequency response:

$$C_F(j\omega) = \frac{\beta(j\omega)}{G_0(j\omega)}, \quad (11)$$

where $G_0(j\omega)$ is the frequency response of the nominal plant. The synthesis of the rational controller $C_R(s)$ is then achieved through identification of the ideal frequency response $C_F(j\omega)$ by that of a low order transfer function using any frequency domain system identification technique [23].

3 CRONE Control of LPV Systems

3.1 Principles

Control of LPV systems is often obtained in industry by freezing the time variations of the system around some operating points, performing a synthesis for each system and then interpolating the controller parameters. Such an approach does not guarantee performance or stability outside the operating points, nor even when dynamically passing on them.

The approach proposed in the paper derives from methods presented by Gahinet, Apkarian and Biannic (see [1, 2]) in which the controller is LPV, its parameters being automatically scheduled on those of the system thus ensuring global stability and performance.

The CRONE approach enhances this method as it leads to robust controller that can handle the system perturbations as well as the perturbations of the measured varying parameters.

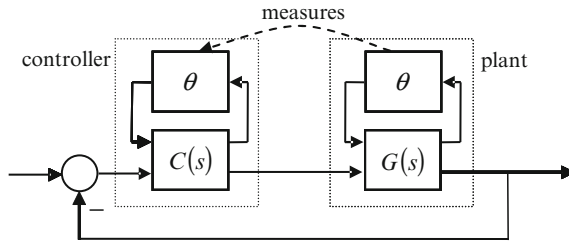


Fig. 3 General interconnection structure for the design of LPV CRONE controller

The general interconnection structure for the LPV CRONE controller design is presented on Fig. 3.

The principle is thus to associate:

- The CRONE approach that consists in dealing with the plant perturbations through an optimal open loop parameterization
- And the LPV controller formulation whose varying parameters are scheduled on the plant varying parameters

The idea is first to calculate the LPV controller from the open loop and the nominal LPV plant, then to verify that the open loop behaviour is robust.

In the sequel, distinction will hence be made between the perturbed varying plant $G(\theta)$ and the nominal plant $G_c(\theta)$ whose perturbations are considered in a nominal state. The latter is used to obtain the controller structure while the former is used for optimization.

3.2 LPV Plant Representation

In the paper, the plant is supposed to admit an LFT description such that:

$$G(\theta) = F_u(G_m(s), \Theta(t)), \quad (12)$$

and

$$G_c(\theta) = F_u(G_m(s), \Theta_0(t)) = F_u(G_{cm}(s), \theta(t)), \quad (13)$$

where s is the Laplace variable, F_u denotes the upper linear fractional transformation defined by

$$F_u\left(\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, K\right) = M_{22} + M_{21}K(I - M_{11}K)^{-1}M_{21}, \quad (14)$$

$$G_m(s) = \left(\begin{array}{cc|c} G_{m\theta\theta}(s) & G_{m\Delta\theta}(s) & G_{mG\theta}(s) \\ G_{m\theta\Delta}(s) & G_{m\Delta\Delta}(s) & G_{mG\Delta}(s) \\ \hline G_{m\theta G}(s) & G_{m\Delta G}(s) & G_{mGG}(s) \end{array} \right), \quad (15)$$

and

$$G_{cm}(s) = \left(\frac{G_{cm\theta\theta}(s)}{G_{cm\theta G}(s)} \middle| \frac{G_{cmG\theta}(s)}{G_{cmGG}(s)} \right). \quad (16)$$

The n_θ varying parameters and the uncertainties are taken into account through

$$\Theta(t) = \left(\theta_m(t)^T, \Delta_m^T \right)^T, \text{ and } \Theta_0(t) = \left(\theta_m(t)^T, \Delta_{m0}^T \right)^T, \quad (17)$$

where

$$\theta_m(t) = \text{diag}(\theta_1(t), \dots, \theta_1(t), \dots, \theta_{n_\theta}(t), \dots, \theta_{n_\theta}(t)), \quad (18)$$

and

$$\Delta_m = \text{diag}(\Delta_1(s), \dots, \Delta_q(s), \delta_1 I, \dots, \delta_r I, \varepsilon_1 I, \dots, \varepsilon_c I), \quad (19)$$

in which $\Delta_i \in C^{k_i \times k_i}$, $\delta_i \in R$ and $\varepsilon_i \in C$ respectively represent the neglected dynamics, the parametric perturbations, and the perturbations that conjointly act on the plant phase and gain. Δ_{m0} is then composed of the nominal values of the perturbations.

3.3 LPV CRONE Controller Structure

LPV controller is directly obtained from the definition of the open loop

$$\beta = G_c(\theta) \circ C(\theta), \quad (20)$$

where $\circ$ stands for the series connection operator:

$$C(\theta) = G_c(\theta)^{-1} \circ \beta. \quad (21)$$

The inverted transfer in (21) can be directly obtained when the direct transfer $G_{cmGG}(s)$ in (16) is invertible. A constraint on open loop high frequency order is added otherwise.

The direct transfer is invertible Combined with (21) and (12), the relation

$$G_c(\theta)^{-1} = F_u(G'_{cm}(s), \theta_m(t)), \quad (22)$$

where

$$G'_{cm}(s) = \begin{pmatrix} G_{cm\theta\theta} - G_{cmG\theta} G_{cmGG}^{-1} G_{cm\theta G} & G_{cmG\theta} G_{cmGG}^{-1} \\ -G_{cmGG}^{-1} G_{cm\theta G} & G_{cmGG}^{-1} \end{pmatrix}, \quad (23)$$

in which any dependence to Laplace variable s is omitted for brevity, then permits the following controller expression:

$$C(\theta) = F_u(C_{cm}(s), \theta_m(t)), \quad (24)$$

where

$$C_{cm}(s) = G'_{cm}(s) \begin{pmatrix} 1 & 0 \\ 0 & \beta(s) \end{pmatrix}. \quad (25)$$

The direct transfer is not invertible This case happens when the plant is strictly proper. Open loop low pass filter is then decomposed into

$$\beta_h(s) = \beta_{h1}(s) \beta_{h2}(s), \quad (26)$$

$\beta_{h1}(s)$ and $\beta_{h2}(s)$ being low pass filters of the form (5) of order n_{h1} and n_{h2} , respectively such that $\beta_{h1}(s)^{-1} G_{cmGG}(s)$ is invertible and $n_{h2} = n_h - n_{h1}$.

LPV controller is then obtained using

$$C(\theta) = (\beta_{h1}^{-1} \circ G_c(\theta))^{-1} \circ \beta_2, \quad (27)$$

where β_2 is the open loop of high frequency order n_{h2} . Relation (24) then holds with

$$C_{cm}(s) = G''_{cm}(s) \begin{pmatrix} 1 & 0 \\ 0 & \beta_2(s) \end{pmatrix}, \quad (28)$$

where

$$G'_{cm}(s) = \begin{pmatrix} G_{cm\theta\theta} - G_{cmG\theta} (\beta_{h1}^{-1} G_{cmGG})^{-1} \beta_{h1}^{-1} G_{cm\theta G} & G_{cmG\theta} (\beta_{h1}^{-1} G_{cmGG})^{-1} \\ -(\beta_{h1}^{-1} G_{cmGG})^{-1} \beta_{h1}^{-1} G_{cm\theta G} & (\beta_{h1}^{-1} G_{cmGG})^{-1} \end{pmatrix}. \quad (29)$$

3.4 Objectives and Constraints

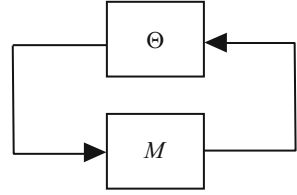
As in LPV case frequency responses of sensitivity functions are not available, minimisation of criterion (7) is performed through minimization of input-output $\mathcal{L}_2$ gain of the closed loop augmented systems that meets the small gain theorem.

Theorem 1. (Small Gain) [11]: The system formed by interconnecting two causal operators M and Θ with finite $\mathcal{L}_2$ gains γ_1 and γ_2 respectively as presented on Fig. 4 has finite $\mathcal{L}_2$ gain given by

$$\gamma = \frac{\gamma_1}{1 - \gamma_1 \gamma_2}. \quad (30)$$

Furthermore, closed loop is stable if $\gamma_1 \gamma_2 < 1$.

Fig. 4 Interconnected system for small gain theorem



This theorem permits to deduce that $\mathcal{L}_2$ gain of SISO system $H = F_u(M_{aug}(s), \Theta)$ where Θ is any causal operator verifying $\|\Theta\|_{\mathcal{L}_2} \leq 1$, verifies $\|H\|_{\mathcal{L}_2} < \gamma$ if

$$\|M'(s)\|_{\infty} < 1, \quad (31)$$

where

$$M'(s) = M_{aug}(s) \cdot \text{diag}(I, \gamma^{-1}), \quad (32)$$

or equivalently if

$$\forall \omega, \quad M'(j\omega)^* M'(j\omega) - I < 0. \quad (33)$$

In order to reduce pessimism associated with small gain theorem, scaling matrices D and G are added. The scaling D takes into account the structure of Θ and verifies $D\Theta = \Theta D$. The scaling G takes into account the nature (real or complex) of the stationary part of Θ . Scalings associated with Θ given by (17) are thus of the form:

$$D = \text{diag}(D_{LPV}, D_{LTI}), \text{ and } G = \text{diag}(0, G_{LTI}), \quad (34)$$

with

$$D_{LTI}(\omega) = \text{diag} \left(\begin{array}{c} d_1(\omega)I, \dots, d_d(\omega)I, \\ D_1(\omega), \dots, D_r(\omega), \\ D_1^c(\omega), \dots, D_c^c(\omega) \end{array} \right), \quad (35)$$

$G_{LTI}(\omega) = \text{diag}(0, G_1(\omega), \dots, G_r(\omega), 0)$, where $d_i \in \mathbb{R}$, $d_i > 0$, $D_i \in \mathbb{C}^{r_i \times r_i}$, $D_i = D_i^* > 0$ and $D_i^c \in \mathbb{C}^{c_i \times c_i}$, $D_i^c = D_i^{c*} > 0$, and where $G_i \in \mathbb{C}^{r_i \times r_i}$, $G_i = G_i^*$, D_{LPV} being of the form $D_{LPV} = \text{diag}(L_1, \dots, L_{n\theta})$, $L_i = L_i^*$.

Relation (33) in this case reduces to the problem:

$$\max_{\omega} \min_{D, G} \alpha < 1$$

$$\forall \omega, \quad M'(j\omega)^* D M'(j\omega) + j(G M'(j\omega) - M'(j\omega)^* G) - \alpha^2 D < 0, \quad (36)$$

where ω is taken in a discrete set $\underline{\omega} = \{\omega_1, \dots, \omega_2\}$ for computational feasibility, as dense as possible.

Figure 5 presents the augmented system used to derive the objective function and the associated constraints of the optimization problem.

Objective The CRONE robustness objective is now obtained by minimizing γ in relation (36), $M_{aug}(j\omega)$ in (32) being replaced by transfer matrix $M_{crit}(j\omega)$ between input $(e_{\theta C}^T, e_{\theta G}^T, e_{\Delta}^T, r)^T$ and output $(s_{\theta C}^T, s_{\theta G}^T, s_{\Delta}^T, z)^T$, as presented on Fig. 5.

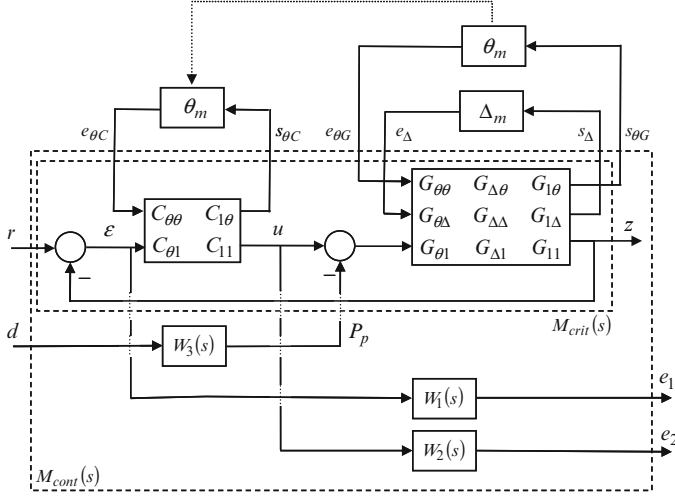


Fig. 5 Functional diagram for constraints and objective definition

Constraints Constraints are similarly taken into account by ensuring bounds on $\mathcal{L}_2$ gain of transfers associated with sensitivity functions presented in Sect. 2.4. This approach leads to verify relation (36) with transfer matrix $M_{const}(j\omega)$ between input $(e_{\theta_C}^T, e_{\theta_G}^T, e_{\Delta}^T, d, r)^T$ and output $(s_{\theta_C}^T, s_{\theta_G}^T, s_{\Delta}^T, e_1, e_2)^T$ as presented on Fig. 5. Filters $W_1(s)$, $W_2(s)$ and $W_3(s)$ are added to frequency constrain the sensitivity functions.

4 Lambda Control of an IC Engine

4.1 Presentation and Motivation

The air/fuel ratio is the ratio of the air mass flow $\dot{m}_a$ and the fuel mass flow $\dot{m}_f$:

$$AFR = \dot{m}_a / \dot{m}_f. \quad (37)$$

In stoichiometric conditions, i.e. when the mixture of air and fuel completely burns resulting in only carbon dioxide and water, the air/fuel ratio value is around $AFR_{st} = 14.5$. This value permits to define lambda:

$$\lambda = AFR / AFR_{st}. \quad (38)$$

Low values of λ defines richness of the mixture, and lead to high engine power typically necessary during ignition phase or during acceleration phase in normal driving conditions. High values of λ lead to a lean mixture which is ideal for low fuel consumption.

Ecological considerations and associated restricting norms enhance the use of mixtures as pure as possible with lambda values around 1. Small deviations (around 0.7%) of λ are then treated by catalytic converters with minimal conversion efficiency of 80% [6]. Such considerations impose precise control of λ by a regulation of the amount of injected fuel. Default in the regulation results in disturbances that must be rejected. Sophisticated lambda sensors such as Universal Exhaust Gas Oxygen (UEGO) sensors permit to measure λ at the exhaust manifold output, before catalytic converter [7].

4.2 Model Description

The amount of fuel to inject for a desired value of lambda at injection throttle λ_{inj} , assuming $\dot{m}_a$ is known, is calculated from the relation

$$\lambda_{inj} = \frac{\dot{m}_a}{\dot{m}_f AFR_{st}}. \quad (39)$$

Various authors [3,4,21,28] propose different models of transfer functions between λ_{inj} and λ_{meas} , the latter being measured by the UEGO at the output manifold.

The model used in this study is based on the system identification around different operating points in terms of velocity and torque. Around each operating point, the transfer appears to be of the form:

$$\frac{\lambda_{meas}(p)}{\lambda_{inj}(p)} = \frac{1}{1 + \tau p} e^{-rp}, \quad (40)$$

this approach thus leading to a LPV model whose varying parameters $\tau(t)$ and $r(t)$ can be determined accordingly to Figs. 6 and 7.

4.3 Delay Approximation

Pade approximation method [15] is based on the approximation of the exponential term appearing in transfer function of a delay. As transfer function is not available

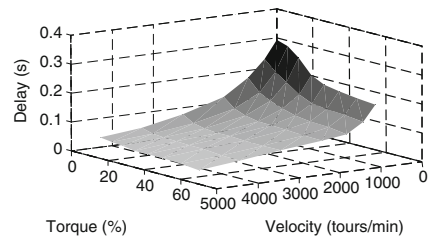
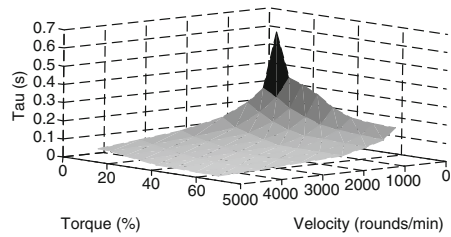


Fig. 6 $r(t)$ versus velocity and torque

Fig. 7 $\tau(t)$ versus velocity and torque



for LPV systems, such a method does not strictly apply for varying delay. However [29] provides justification for replacing delay by system

$$\dot{x}(t) = -\frac{2}{r(t)}x(t) + \frac{4}{r(t)}u(t), y(t) = x(t) - u(t), \quad (41)$$

which is a state-space representation of a first order Pade approximant with judiciously chosen states.

When applied on a transfer function, Pade approximation produces a positive zero. The resulting controller obtained from relation (21) would then have a positive pole. For this reason, the positive zero is included in open loop definition (2) and extracted from the nominal plant used for controller calculation. State-space representation of $G_C(\theta)$ is thus given by:

$$\begin{cases} \dot{x}(t) = -(1/\tau(t)) x(t) + x_r(t), \dot{x}_r(t) = -(2/r(t)) x_r(t) + (4/r(t)) u(t) \\ y(t) = (1/\tau(t)) x(t) \end{cases}, \quad (42)$$

where

$$\begin{aligned} \tau(t) &= 0.3166 + 0.2935 \theta_\tau(t), \text{ and} \\ r(t) &= 0.189 + 9.45 \times 10^{-3} \Delta_r + 0.1417 \theta_r(t), \end{aligned} \quad (43)$$

with $|\theta_\tau(t)| < 1$, $|\theta_r(t)| < 1$, $|\Delta_r| < 1$, these values being obtained from identification data. Uncertainty of 5% is added on nominal value of $r(t)$ through Δ_r to take the Pade approximation into account. This approach results from comparison of both frequency and time responses of the system for fixed delay values and their approximations.

4.4 Results

Low and high open-loop asymptotic behaviours and nominal value of resonant peak are respectively fixed at $n_b = 1$, $n_h = 3$ and $Q_r = 1.5$ dB. Optimisation over discrete sets of open loop parameters lead to optimal values $Y_0 = 8$ dB, $\omega_r/\omega_h = 10^{-2}$ and $\omega_r/\omega_b = 10^2$ for a cross-gain frequency $\omega_u = 0.5$ rad/s. The latter

results in a balance between rapidity and maximum admissible resonant peak, in this case equal to $Q_{\max} = 5.15$, due to the positive zero contained in the open loop definition [10, 14].

Figures 8 to 11 respectively present the Bode magnitude diagrams of sensitivity functions $T(s)$, $S(s)$, $GS(s)$ and $CS(s)$ obtained for nominal values of the parameters and the associated perturbation domains computed with fixed values of the varying parameters. The surimposed templates are those used for computation.

Fig. 8 Magnitude Bode diagram of nominal $T(s)$ and uncertainty domain

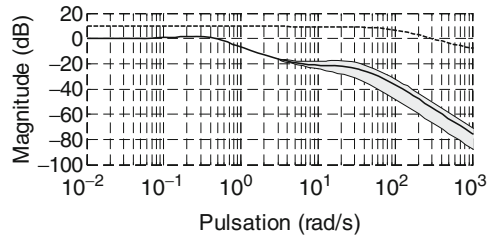


Fig. 9 Magnitude Bode diagram of nominal $S(s)$ and uncertainty domain

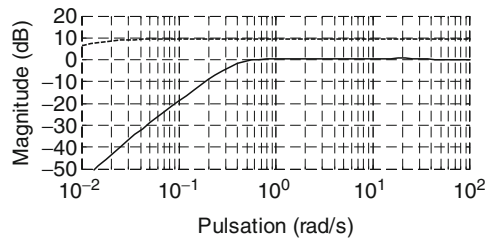


Fig. 10 Magnitude Bode diagram of nominal $GS(s)$ and uncertainty domain

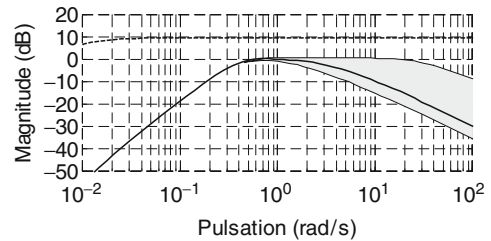
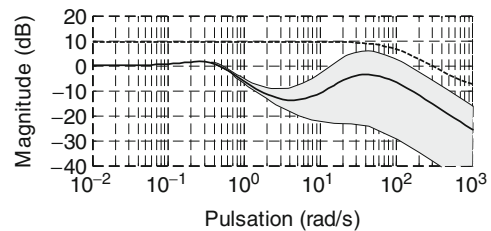


Fig. 11 Magnitude Bode diagram of nominal $CS(s)$ and uncertainty domain



The figures permit to verify the constraints are respected for frozen values, even if this approach only lead to appraise the time responses behaviours for variation of the parameters, only available in simulation or on test bench.

4.5 Simulation

Simulation has been performed for a fixed set of engine phases such as ignition, acceleration or deceleration, thus leading to varying torque and velocity as respectively presented on Figs. 12 and 13.

Figure 14 presents the step responses obtained for different application step times, considering perfect fuel injection. The overshoots is quasi constant, as it varies between 14.5% and 15.5%, independently of torque and velocity variations. Figure 14 thus enhances the insensitivity of the proposed method to parameters variations.

Figure 15 presents response to a step disturbance of magnitude 0.05 kg/s applied at 50 s, which is the highest admissible magnitude. Response is not as insensitive to parameters variations, as those of Fig. 14, but disturbance rejection is done in a very convenient way. This varying parameter sensitivity is due to the fact that the method

Fig. 12 Torque used for simulation

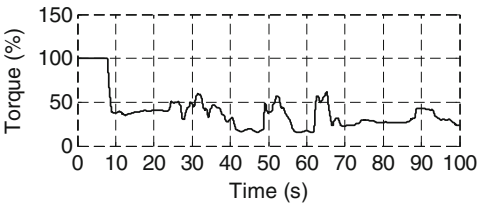


Fig. 13 Velocity used for simulation

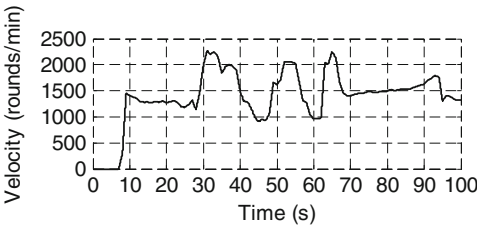


Fig. 14 Step responses for different application step times

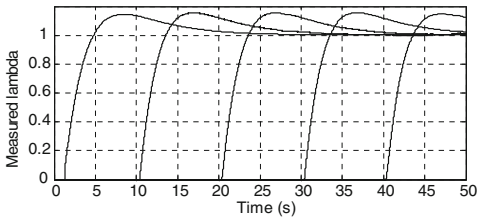
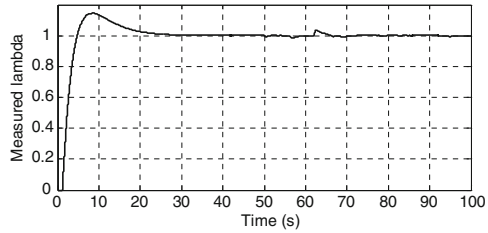


Fig. 15 Response to step disturbance



does not ensure stationary behaviour of close loop to plant input disturbance as shown on Figs. 10 and 11. Template constraints on sensitivity functions $CS(s)$ and $GS(s)$ however satisfactory prevent any excessive behaviours.

5 Conclusion

The method proposed in the paper takes advantage of the CRONE approach that consists in dealing with the plant uncertainties through an optimal open loop parameterization, and proposes to use an LPV controller whose varying parameters are scheduled on the plant varying parameters, that thus desensitise the open loop system. The obtained controller is robust and can handle the system uncertainties as well as the uncertainties of the measured varying parameters.

Method is finally applied for the control of air-fuel ratio of an internal combustion engine modelled with a varying delay which imposes a balance between rapidity and magnitude peak robustness.

Analysis of resulting controller for frozen values of the parameters shows insensitivity of the open loop and of the associated input-output system $T(s)$.

These results have finally been confirmed in simulation for a given set of variations associated with ignition, acceleration, deceleration, and normal driving mode. The first overshoot varies within 14.5% and 15.5% independently of torque and velocity variations.

References

1. Apkarian P, Biannic J-M, Gahinet P (1995) Self-scheduled H_∞ control of missile via linear matrix inequalities. *J Guid Contr Dyn* 18(3):532–538, May–June
2. Apkarian P, Gahinet P (1995) A convex characterization of gain-scheduled H_∞ controller. *IEEE Trans Aut Cont* 40(5):853–864, May
3. Aquino CF (1981) Transient A/F characteristics of the 5 liter central fuel injection engine. SAE Paper No. 810494
4. Balenovic M (2002) Modeling and model-based control of a three-way catalytic converter. Proefschrift, ISBN 90–386–1900–6, Technische Universiteit Eindhoven
5. Battaglia J-L, Cois O, Puissegur L, Oustaloup L (2001) Solving an inverse heat conduction problem using a non-integer identified model. *Int J Heat Mass Tran* 44(14):2671–2680
6. Berggren P, Perkovic A (1996) Cylinder individual lambda feedback control in an SI engine. Report Linköping University, Reg nr LiTH-ISY-EX-1649

7. Bosch R (2004) Automotive handbook. 6th edn. ISBN 1-86058-474-8, Robert Bosch GmbH, October
8. Caponetto R, Fortuna L, Porto D (2004) A new tuning strategy for a non integer order PID controller. First IFAC Workshop on Fractional Derivative and Its Application. FDA 04, Bordeaux, France
9. Chen YQ, Moore KL, Vinagre BM, Podlubny I (2004) Robust PID controller auto tuning with a phase shaper. First IFAC Workshop on Fractional Derivative and Its Application. FDA 04, Bordeaux, France
10. Freudenberg JS, Looze DP (1987) A sensitivity tradeoff for plants with time delay. IEEE Trans Autom Contr AC-32:2, fevrier
11. Khalil HK (2000) Nonlinear systems. 3rd edn. Person Education Int. Corp., New Jersey, USA
12. Lanusse P, Oustaloup A, Mathieu B (2000) Robust control of LTI MIMO plants using two CRONE control design approaches. IFAC symposium on Robust Control Design. Prague, Czech Republic
13. Lanusse P, Oustaloup A, Sabatier J (2005) Step-by-step presentation of a 3rd generation CRONE controller design with an anti-windup system. 5th EUROMECH Nonlinear Dynamics conference, ENOC 05. Eindhoven, The Netherlands
14. Looze DP, Freudenberg JS (1991) Limitations of feedback properties imposed by open-loop right half plane poles. IEEE Trans Autom Contr 36:6, juin
15. Malek-Zavarei M, Jamshidi M (1987) Time delay systems – analysis, optimization and applications, vol. 9. North-Holland Systems and Control series. Elsevier, Amsterdam, The Netherlands
16. Matignon D, Joly (1996) Stability results on fractional differential equations with applications to control processing. In: Computational Engineering in Systems and Application multiconference 2:963–968, IMACS, IEEE-SMC
17. Matignon D, D'Andrea-Novell B (1996) Some results on controllability and observability of finite-dimensional fractional differential systems 2:952–956, IMACS, IEEE-SMC
18. Monje CA, Vinagre BM, Chen YQ, Feliu V, Lanusse P, Sabatier J (2004) Proposals for fractional PID tuning. First IFAC Workshop on Fractional Derivative and Its Application. FDA 04, Bordeaux, France
19. Moze M, Sabatier J, Oustaloup A (2006) Synthesis of third generation CRONE controller using mu-analysis tools. 32nd Annual Conference of the IEEE Industrial Electronics Society (IECON'06). Paris, France, 7–10 November
20. Moze M (2007) Commande CRONE des systèmes Linéaires à Paramètres Variants. Thèse de Doctorat de l'Université Bordeaux 1, Novembre
21. Onder CH, Roduner MR, Simons CA, Geering HP (1998) Wallwetting parameters over the operational region of a sequential fuel injected SI engine. SAE Paper No. 980792
22. Oustaloup A, Mathieu B, Lanusse P (1995) The CRONE control of resonant plants: application to a flexible transmission. In: Eur J Contr 1(2)
23. Oustaloup A, Mathieu B (1999) La commande CRONE du scalaire au multivariable. Hermes Science Publications, Paris
24. Podlubny I (1999) Fractional-Order systems and PID-Controllers. IEEE Trans Autom Contr 44(1):208–214
25. Pommier V, Lanusse P, Sabatier J, Oustaloup A (2001) Input-output linearization and fractional robust control of a non linear plant. European control conference. Porto, Portugal
26. Sabatier J, Garcia Iturricha A, Oustaloup A, Levron F (1998) Third generation CRONE control of continuous linear time periodic systems. In: Proceedings of IFAC Conference on System Structure and Control, CSSC'98. Nantes, France, 8–10 July
27. Sabatier J, Garcia Iturricha A, Oustaloup A (2001) Commande CRONE de systemes lineaires non stationnaires echantillones a coefficients periodiques. Eur J Autom (JESA) 32(1–2): 149–168
28. Turin RCE, Casartelli GB, Geering HP (1994) A new model for fuel supply dynamics in an SI engine. SAE Paper No. 940208
29. Vichnevetsky R (1964) Analog computer simulation of a time-dependant delay using the concept of generaliser transfert function. In: Proceedings of International Association for Analog Computation 6:105–109

Non Integer Order Operators Implementation via Switched Capacitors Technology

Riccardo Caponetto, Giovanni Dongola, Luigi Fortuna, and Antonio Gallo

Abstract In this chapter a Switched Capacitors (SC) implementation of fractional differintegral operator is proposed. Time and frequency domain result tests validate the feasibility and reliability of the SC circuit implementation. A detailed analysis of the influence of the non-idealities is proposed in order to obtain a design ready for Integrated Circuit (IC) implementation. The proposed approach may guarantee a good degree of approximation of fractional differintegral operator and therefore the possibility to realize $PI^\lambda D^\mu$ controller IC based on switched capacitors technology.

1 Circuital Implementation of Fractional Order Integrator

The authors, in [2] and [3] proposed an analog implementation of the non integer order integrator based on Field Programmable Analog Arrays, (FPAAs), able to implement $PI^\lambda D^\mu$ controller, while different practical controller implementations have been suggested in [5] and [1].

In this section a circuital implementation of Oustaloup interpolation [4] of a fractional order integrator is proposed.

The transfer function of Oustaloup interpolation in Eq. 1 can be realized, selecting suitably the values of resistances and capacities of the circuit shown in Fig. 1.

$$I_m(jw) = \prod_{i=1}^N \frac{\left(1 + j \frac{w}{w_i}\right)}{\left(1 + j \frac{w}{w_i}\right)} \quad (1)$$

R. Caponetto (✉), G. Dongola, L. Fortuna and A. Gallo
University of Catania, Engineering Faculty, D.I.E.E.S Viale A. Doria 6, 95125, Catania, Italy
e-mail: riccardo.caponetto@diees.unict.it; gdongola@diees.unict.it; lfortuna@diees.unict.it;
antonio@diees.unict.it

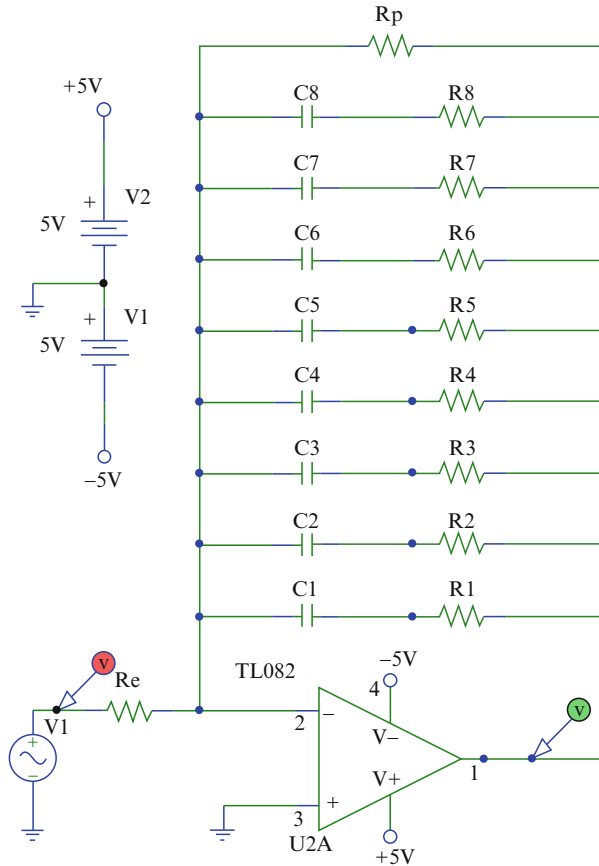


Fig. 1 Circuital implementation of Oustaloup interpolation of a fractional order integrator

In fact the transfer function of the circuit in Fig. 1 is:

$$H(jw) = \frac{R_p}{R_e} \prod_{i=1}^N (1 + jR_i C_i w) * \frac{1}{\prod_{i=1}^N (1 + jR_i C_i w) + R_p \sum_{i=1}^N jC_i w \prod_{k=1, k \neq i}^N (1 + jR_k C_k w)} \quad (2)$$

Thus, matching the Eqs. 1 and 2 a set of equations is obtained, where in the first members are located the known terms joined to the fractional integration order and to the frequency range, according to the Oustaloup interpolation formulas, while in the second members there are the circuit parameters which we must identify. Thus, the following equations are obtained, supposing $\forall i = 1 \dots N$:

$$1 = \frac{R_p}{R_e}, \quad w_i' = \frac{1}{R_i C_i}, \quad \prod_{i=1}^N 1 + j \left(\frac{w}{w_i'} \right) = \prod_{i=1}^N (1 + jR_i C_i w) + R_p \sum_{i=1}^N jC_i w \prod_{k=1, k \neq i}^N (1 + jR_k C_k w) \quad (3)$$

We have an equations system of $2N + 1$ equations and $2N + 2$ parameters (R_p , R_e , R_i and $C_i \forall i = 1 \dots N$). We can fix one and calculate the others. A possible choice, suggested by empirical methods, is to fix $R_p = 10^7 / W_u$, where W_u is a frequency defined by Oustaloup interpolation formulas [3]. From Eq. 3 it is possible to obtain:

$$\sum_{i=1}^N j C_i w \prod_{k=1, k \neq i}^N 1 + j \left(\frac{w}{w_k} \right) = \frac{1}{R_p} \left[\prod_{i=1}^N 1 + j \left(\frac{w}{w_i} \right) - \prod_{i=1}^N 1 + j \left(\frac{w}{w_i} \right) \right] \quad (4)$$

Equation 4 can be written in the following matrix form:

$$\left[j w \prod_{k=1, k \neq 1}^N \left(1 + j \frac{w}{w_k} \right) \dots j w \prod_{k=1, k \neq N}^N \left(1 + j \frac{w}{w_k} \right) \right] \begin{bmatrix} C_1 \\ \vdots \\ C_N \end{bmatrix} = \frac{1}{R_p} \left[\prod_{i=1}^N 1 + j \left(\frac{w}{w_i} \right) - \prod_{i=1}^N 1 + j \left(\frac{w}{w_i} \right) \right] \quad (5)$$

where the terms of the products are placed along the columns. Thus we have a relation of the type $A * X = B \Leftrightarrow X = A^{-1} * B$ from whose it is possible obtain the values of the capacities. Then we obtain the values of the resistances from the values of the capacities via the formula (3), in fact:

$$R_i = \frac{1}{w_i' C_i} \quad \forall i = 1 \dots N \quad (6)$$

2 Switched Capacitors Implementation of Fractional Order Integrator

In order to implement the fractional differintegral operator on chip, it is necessary that each resistance located in the circuit of Fig. 1 is replaced by a switched capacitors configuration.

2.1 Switched Capacitors Introduction

Let the resistance in Fig. 2a, the current is $i = \frac{v_A - v_B}{R} = \frac{v_{AB}}{R}$. But $i = \frac{dQ}{dt} \Rightarrow \frac{dQ}{dt} = \frac{v_{AB}}{R}$. Applying the Laplace transformation it is obtained:

$$i = \frac{Q(s)}{v_{AB}(s)} = \frac{1}{sR} \quad (7)$$

In order to obtain the transfer function of the switched capacitors configuration in Fig. 2b (Euler Behind Configuration) we must solve the equations of the circuit which are finite differential equations because the system is time-discrete.

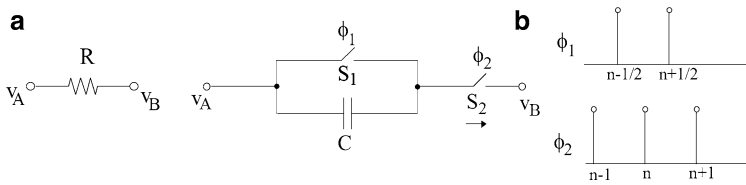


Fig. 2 (a) Resistance, (b) Euler behind configuration

Thus $Q(n) = Q(n-1) + C[v_A(n) - v_B(n)]$ because at the instant n , S_1 is open, S_2 is close and the capacitors is discharge.

Applying the z -transformation it is obtained $Q(z) = z^{-1}Q(z) + C v_{AB}(z)$ or else:

$$H(z) = \frac{Q(z)}{v_{AB}(z)} = \frac{C}{1 - z^{-1}} \quad (8)$$

Using the Euler behind transformation for (7) it is obtained:

$$H(s) \Big|_{s=\frac{1}{T}(1-z^{-1})} = \frac{T}{(1-z^{-1})} = \frac{\frac{T}{R}}{1-z^{-1}} \quad (9)$$

Matching (8) and (9) the relation between the value of the switched capacitors and the value of the resistance is obtained:

$$C = \frac{T}{R} \quad (10)$$

2.2 Switched Capacitors Implementation

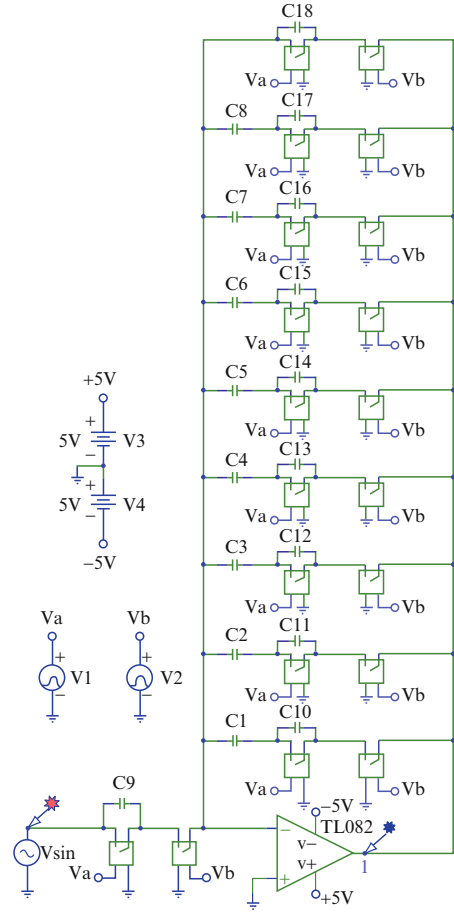
Replacing the resistances of the circuit in Fig. 1 with the switched capacitors, calculated with (10) after selecting the switch time T of the switches, the circuit in Fig. 3 is obtained.

The capacitors used in the switched capacitors technology, in order to obtain a fine accuracy, are realized between two layers of poly-silicon. With these capacitors the minimal dimensions which can be obtained are $20 * 20 \mu\text{m}^2$ with values of capacities about $0.2 - 0.3 \text{ pF}$. The capacities of greater value are realized connecting in parallel the capacities of the smaller value. This procedure allows to have greater tolerances to inaccuracies.

The switches are realized using a parallel configuration of n-MOS and p-MOS in triode configuration in order to avoid the clock-feedthrough and to increase the dynamics of the signals.

Finally, in order to define the values of the parameters of the circuit in Fig. 3, the value R_{on} , which characterizes the transistors in conduction, must be calculated. R_{on} is calculated according the time constant $\tau = R_{on}C$, where τ is the time of

Fig. 3 Switched capacitors implementation of a fractional order integrator



charge of a switch capacitor, as shown in Fig. 4. We must guarantee that the charge of the switched capacitor finishes before the end of time slot, as shown in Fig. 4. Via experimental results it is found the value:

$$R_{on}C \leq \frac{T}{14} \quad (11)$$

3 Results

The identification problem of the parameters of the circuit in Fig. 1, i.e the values of resistances and capacities calculated in (5) and (6), has been resolved developing a procedure in *Matlab*[®] *Environment*, which allows to determine the values of resistances and capacities needed for the Oustaloup approximation of a fixed

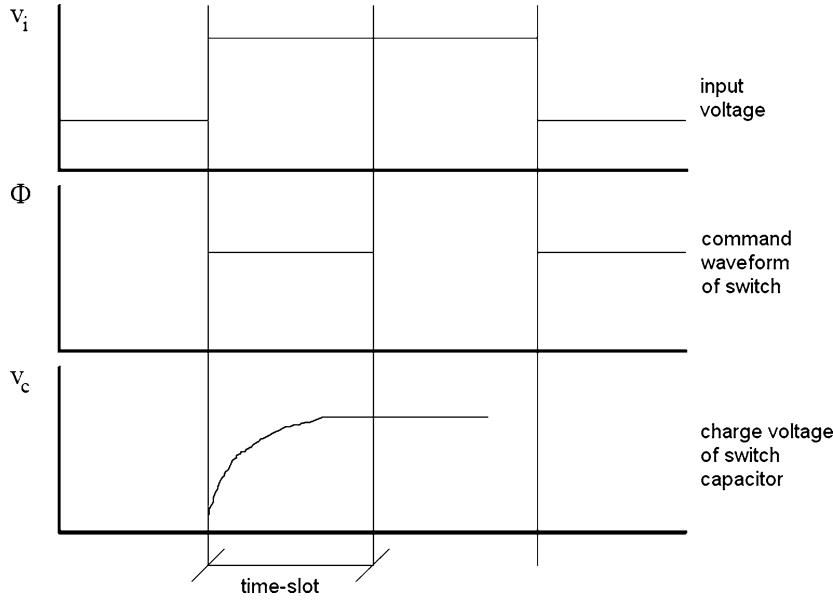


Fig. 4 R_{on} definition

Table 1 Parameters values for a fractional integrator of order 0.5

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
2.825 k Ω	7.85 k Ω	15.29 k Ω	27.936 k Ω	50.294 k Ω	90.939 k Ω	169.71 k Ω	362.97 k Ω
C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
75.12 nF	85.49 nF	138.8 nF	240 nF	422 nF	738 nF	1.25 μ F	1.84 μ F

fractional order of integration on the condition that the range of frequency is fixed. The circuits are developed in *Orcad[®] Environment*.

In Table 1 the values of the resistances and capacities calculated by previous procedure for a fractional integrator of order 0.5 are shown. In Fig. 5 the phase and module bode diagrams are plotted. In Figs. 6 and 7 the output waveform for a sinusoidal input of amplitude 1 V and frequency 100 Hz (5 Hz respectively) are shown. The amplitude of the output signal, about 40 mV (178 mV respectively), is correctly attenuated of a factor $\frac{1}{(2\pi f)^m}$, where f is the frequency of the input signal, and $m = 0.5$ is order of integration.

The phase of the output signal is delayed of $m * 90^\circ = 45^\circ$, i.e., the time delay between the signal output and signal input is 1.25 ms (25 ms respectively), because the time-period of the input signal is 10 ms (200 ms respectively).

In Table 2 the values of the resistances and capacities calculated by previous procedure for a fractional integrator of order 0.2 are shown. In Fig. 8 the phase and module bode diagrams are plotted. In Figs. 9 and 10 the output waveform for a sinusoidal input of amplitude 1 V and frequency 100 Hz (5 Hz respectively) are shown. The amplitude of the output signal, about 280 mV (500 mV respectively), is

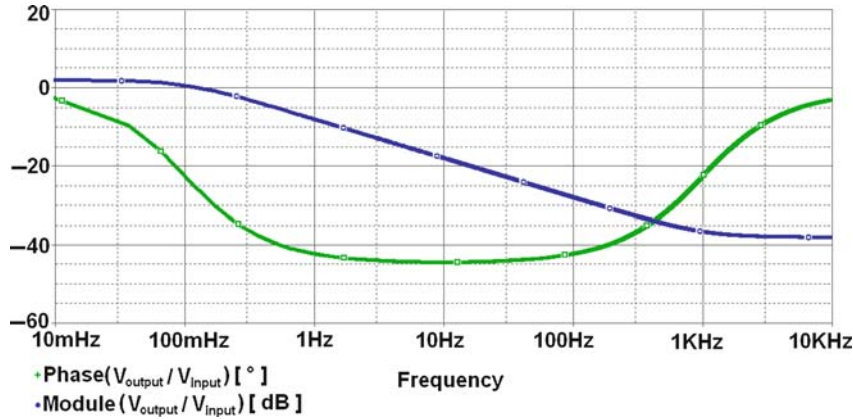


Fig. 5 Bode diagrams of fractional integrator of order 0.5

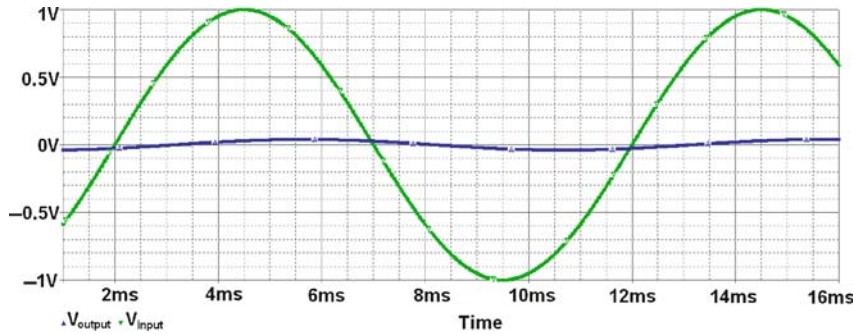


Fig. 6 Waveforms of an integrator of order 0.5 at 100 Hz

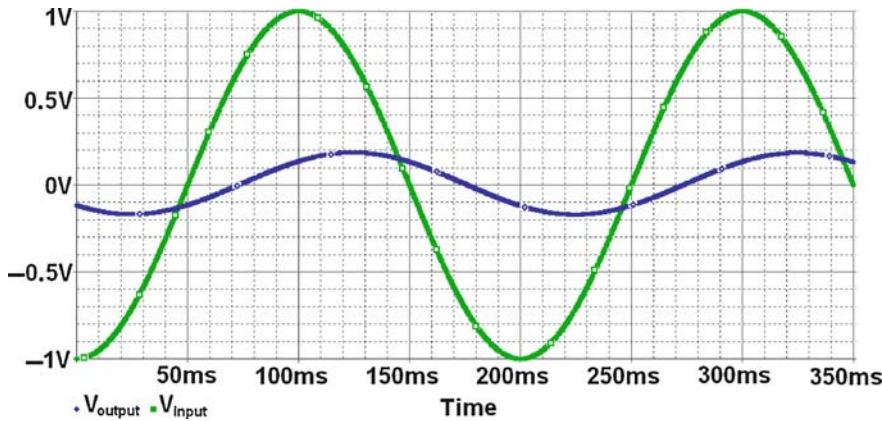


Fig. 7 Waveforms of an integrator of order 0.5 at 5 Hz

Table 2 Parameters values for a fractional integrator of order 0.2

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
108.43 k Ω	155.09 k Ω	201.07 k Ω	255.46 k Ω	323.12 k Ω	409.85 k Ω	527.9 k Ω	727.86 k Ω
C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
2.326 nF	5.143 nF	12.545 nF	31.225 nF	78.066 nF	194.62 nF	477.83 nF	1.096 μ F

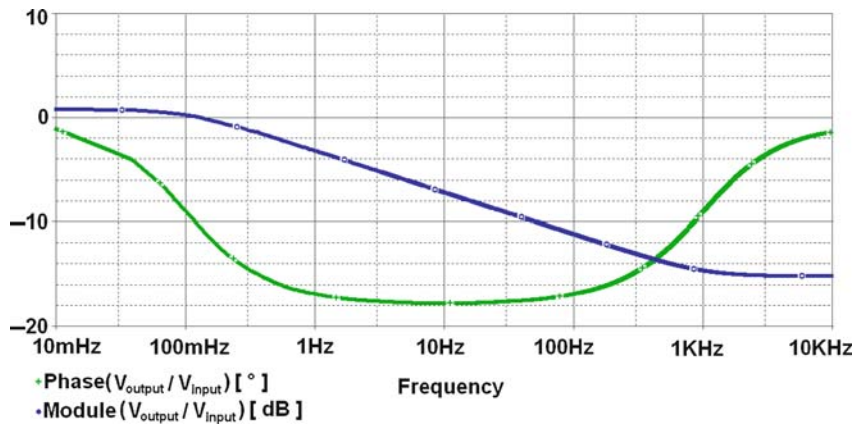


Fig. 8 Bode diagrams of fractional integrator of order 0.2

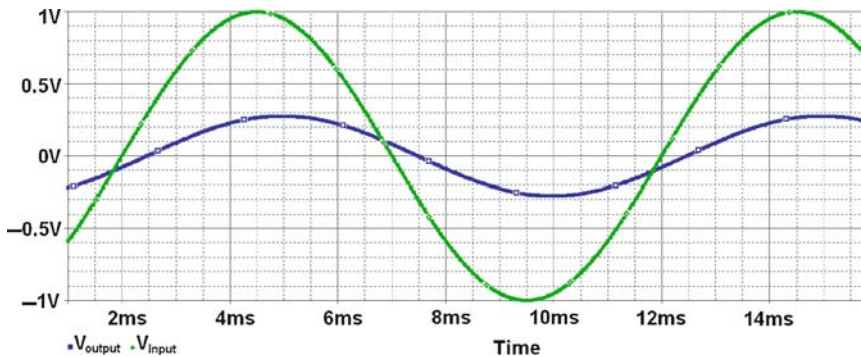


Fig. 9 Waveforms of an integrator of order 0.2 at 100 Hz

correctly attenuated of a factor $\frac{1}{(2\pi f)^m}$, where f is the frequency of the input signal, and $m = 0.2$ is order of integration. The phase of the output signal is delayed of $m * 90^\circ = 18^\circ$, i.e., the time delay between the signal output and signal input is 0.5 ms (10 ms respectively), because the time-period of the input signal is 10 ms (200 ms respectively).

Using the switched capacitors implementation of the first order derivative operator and of the respective fractional integrator is possible to realize a fractional order

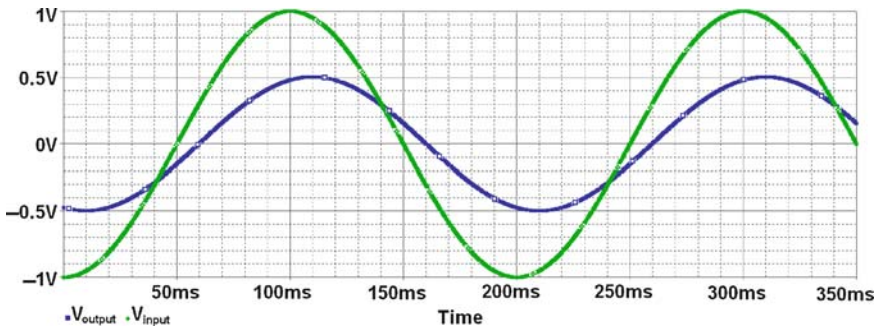


Fig. 10 Waveforms of an integrator of order 0.2 at 5 Hz

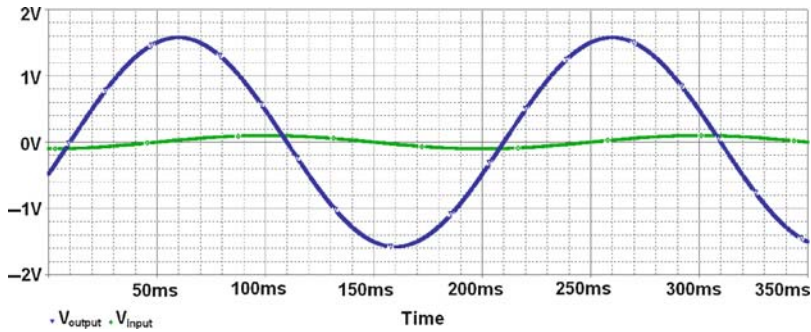


Fig. 11 Waveforms of a derivative operator of order 0.8 at 5 Hz

derivative by switched capacitors approach. The output waveform for a sinusoidal input of amplitude 100 mV and frequency 5 Hz is shown in Fig. 11.

4 Conclusion

In this paper a switched capacitors implementation of a fractional, or more non integer order integrative action has been presented. The results obtained allows us to continue to work on switched capacitors implementation in order to realize on chip implementation of a non integer order $PI^\lambda D^\mu$.

References

1. Bohannan GW (2006) Analog fractional order controller in a temperature control application. IFAC Workshop on Fractional Differentiation and Its Application, Porto
2. Caponetto R, Porto D (2006) Analog implementation of non integer order integrator via field programmable analogic array. IFAC Workshop on Fractional Differentiation and Its Application, Porto

3. Caponetto R, Dongola G (2008) Field programmable analog array implementation of non integer order $PI^\lambda D^\mu$ Controller. *J Comput Nonlinear Dyn* 3:021302
4. Oustaloup A, Levron F, Nanot F, Mathieu B (2000) Frequency-band complex non integer differentiator: characterization and synthesis. *IEEE Trans Circ Syst* 47:25–40
5. Podlubny I, Petras I, Vinagre BM, OLeary P, Dorcak L (2002) Analogue realization of fractional order controller. *Nonlinear Dynam* 29:281–291

Analysis of the Fractional Dynamics of an Ultracapacitor and Its Application to a Buck-Boost Converter

A. Parreño, P. Roncero-Sánchez, X. del Toro García, V. Feliu, and F. Castillo

Abstract This work studies the fractional dynamics of an ultracapacitor which is used in a Buck-Boost power electronic converter as energy storage unit. Ultracapacitors are used in applications such as hybrid vehicles, UPS, power electronic converters connected to the grid, etc, because of their higher energy density. Nevertheless, they exhibit more complex dynamics than standard capacitors due to the diffusion phenomena, which causes a dynamic fractional behaviour in such capacitors.

In this work, the fractional behaviour of the ultracapacitor used in the Buck-Boost converter has been identified by means of an experimental test. Then, the impact of the fractional dynamics of the ultracapacitor on the power electronic converter is analysed. Simulation results are provided in order to compare the ultracapacitor fractional behaviour with an equivalent standard capacitor in terms of voltage dynamics in the Buck-Boost converter.

1 Introduction

Power electronics is nowadays a mature technology and a well established discipline of research [7]. At present, the number of new applications of this technology is considerably increasing due to the growing interest in renewable energy sources [3], the concern about the power quality of the delivered electrical energy, and the increasing number of sophisticated electronic devices, which require high-performance power electronic converters [2]. One of the bottlenecks of this technology is the need to

A. Parreño (✉)

Robotics and Automation Centre, Albacete Science and Technology Park, Paseo de la Innovación 1, 02006 Albacete, Spain
e-mail: alfonso.parreno@pcyta.com

P. Roncero-Sánchez, X. del Toro García, V. Feliu, and F. Castillo

School of Industrial Engineering, University of Castilla-La Mancha, Campus universitario s/n, 13071 Ciudad Real, Spain
e-mail: Pedro.Roncero@uclm.es; Xavier.deltoro@uclm.es; Vicente.Feliu@uclm.es;
Fernando.Castillo@uclm.es

store relatively large amounts of energy in small volumes. This is usually achieved by means of batteries or capacitors, which have strong limitations regarding their energy storage capacity.

A recent alternative is the use of ultracapacitors which are even more efficient than high quality advanced chemistry batteries [1]. At present their main applications are supporting batteries or replacing them in electric vehicles, but there are many other applications such as laptop computers, uninterrupted power supplies (UPS), etc. Ultracapacitors are attractive for all the applications exposed as they have higher energy density than conventional capacitors and higher power density than batteries. They are known since 1969 and they have become very popular for the last decade because of the application in hybrid vehicles as intermediate storage device between actuators and a primary energy source with high power capability.

Ultracapacitors operation implies an important ionic diffusion process between electrodes through the electrolyte. Then fractional behaviour linked with such process may be expected in these devices. In fact, several researches have characterized such behaviour (see [10]), being very noticeable in some cases.

The ultracapacitor fractional impedance has been approximated by different authors: [6] modelled the ultracapacitor with a semi-infinite lossy RC transmission line [10] modelled the ultracapacitor impedance by means of fractional poles and zeros and the transfer function obtained is valid for the whole range frequencies [4] have recently proposed a simplified fractional model.

This work deals with the application of ultracapacitors as input capacitors in Buck-Boost power-electronic converters. This kind of converters is widely used in industry since they achieve an output voltage which can be greater or lower than the input voltage, depending on the value of the control signal, and with opposite polarities. In particular, this work investigates the impact caused by the fractional dynamics of the ultracapacitor on the performance of the power electronic converter.

In order to carry out the above analysis, several tools of fractional calculus developed in the last decades are employed [9]. These tools have been used to model and simulate the dynamics of several processes and applications as electrochemical processes, viscoelasticity or thermal diffusion processes, among others [5, 8].

The paper is organized as follows. Section 2 describes the ultracapacitor employed, its dynamic model, its parametric identification procedure and its model validation. Section 3 presents the basic scheme of the Buck-Boost converter. Section 4 shows the impact of the fractional behaviour on the converter by simulating its dynamics including the fractional order model of the ultracapacitor. This behaviour is compared to the results obtained with an equivalent standard capacitor. Finally the overall conclusions are given.

2 Ultracapacitor Model

The standard model of a capacitor is the well-known impedance expression:

$$Z_{uc}(s) = \frac{1}{Cs} \quad (1)$$

This model has also been used as a simple approximation of the ultracapacitor impedance. However, the diffusion phenomena is present in these devices producing fractional dynamic behaviours. Ref. [13] proposed a more complex impedance model of the form:

$$Z_{uc}(s) = \frac{1}{Cs^\alpha}, \quad 0 < \alpha < 1 \quad (2)$$

A much more accurate model was proposed by [10] which fits the ultracapacitor frequency response in all the frequency range:

$$Z_{uc}(s) = R + \frac{\left(1 + \frac{s}{\omega_0}\right)^\alpha}{Cs^\beta}, \quad 0 < \alpha, \beta < 1 \quad (3)$$

Very recently, a much simpler model has been proposed by [4] which seems to work fine at low and medium frequencies:

$$Z_{uc}(s) = R + \frac{1}{Cs^\alpha} \quad (4)$$

Parameters of all these models (1)-(4) were obtained by fitting these expressions to the impedances obtained experimentally, employing identification procedures in the frequency domain.

Very little effort has been devoted to identify the parameters of these models from fittings in the time domain (experimental time-domain responses), in spite of the potential advantages of this technique, which needs much less time than frequency response identification. In this sense, the work of [4] is cited again, which fitted a discrete fractional state space model to time-domain responses.

The Buck-Boost converter of this work uses as storage element an ultracapacitor with the following characteristic: BestCap Supercapacitors by AVX, capacitance: 22 mF, rated voltage (+20%) 12 V, ESR (miliohm.) at 1 KHz: 350 m Ω , leakage current (max): 5 μ A. As a first step to analyse the whole converter, the dynamics of this ultracapacitor is characterized in this section. The procedure followed to identify and validate the model of the ultracapacitor is based on the temporal response to a step change in the input voltage. Figure 1a shows the scheme of the experimental arrangement and Fig. 1b shows the laboratory setup. The value of the resistor connected in series with the ultracapacitor R_T is 5 Ω .

Figure 2 shows both recorded signals, the applied input voltage and the voltage across the ultracapacitor. It can be observed that the ultracapacitor voltage experiences a discontinuity at the first instant when the input step command is applied. This suggests that this element exhibits some resistive behaviour. Therefore, the initial choice is to fit the model defined by (4) to this data for simplicity. If the results obtained with the fitting are not accurate enough, the more elaborated model defined by (3) could be used instead. From the scheme in Fig. 1a, the following expression can be obtained:

$$U_{uc}(s) = \frac{1 + RCs^\alpha}{1 + (R_T + R)Cs^\alpha} U(s) \quad (5)$$

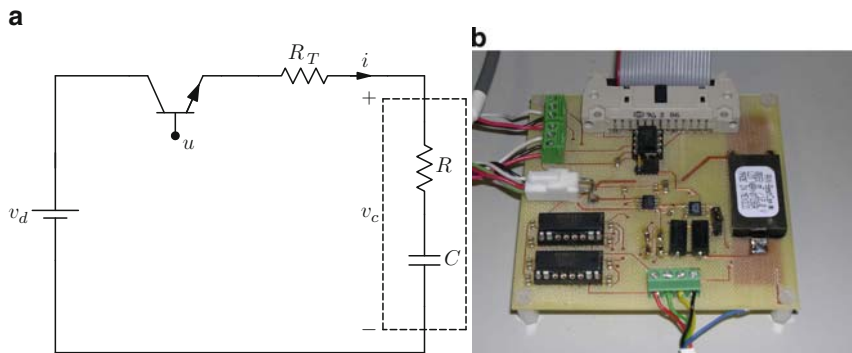


Fig. 1 Laboratory setup (a) Electrical scheme of the experiment. The dashed rectangle shows the ultracapacitor (b) Prototype

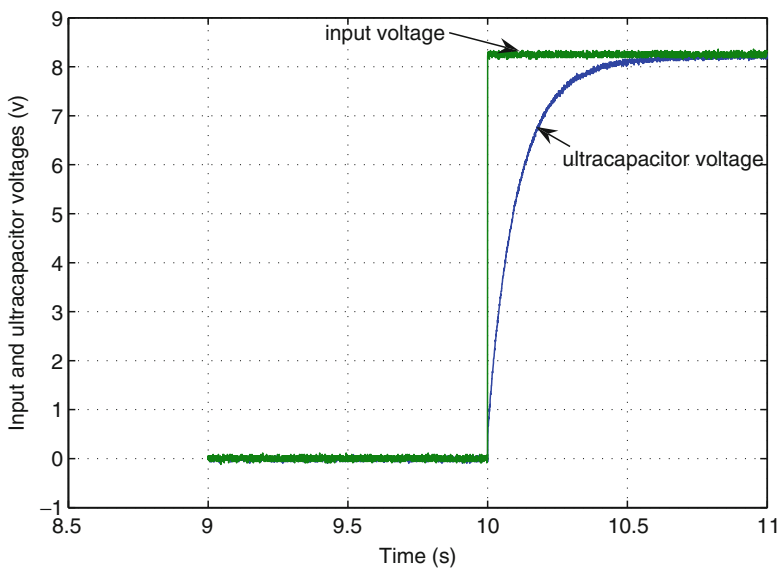


Fig. 2 Voltage response of the ultracapacitor to a step input

where $U(s)$ and $U_{uc}(s)$ are, respectively, the Laplace transforms of the input and ultracapacitor voltages; R , C and α are given by expression (4); and R_T is the resistor connected in series with the ultracapacitor. Denoting $\tau = (R_T + R)C$ and $\gamma = RC$ the above expression yields

$$U_{uc}(s) = X(s) + \gamma s^\alpha X(s), \quad X(s) = \frac{U(s)}{1 + \tau s^\alpha} \quad (6)$$

This can be expressed in the time domain as:

$$u_{uc}(t) = x(t) + \gamma \frac{d^\alpha}{dt^\alpha} x(t), \quad x(t) + \tau \frac{d^\alpha}{dt^\alpha} x(t) = u(t)$$

(7)

The fitting procedure employed optimizes the two parameters α and τ in such way that the following error function is minimized:

$$J_{error} = \sqrt{\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \left(u_{uc}^e(t) - u_{uc}^f(t) \right)^2 dt}$$

(8)

where $u_{uc}^e(t)$ is the experimental data, and $u_{uc}^f(t)$ is the forecasted data obtained from the model. A standard search procedure is used for that. For each pair of values α and τ , $x(t)$ is obtained and the third parameter γ is easily obtained from the left side expression of (7) by using a simple linear fitting procedure. The fractional derivatives were implemented by using the discrete fractional operator obtained from the Grunwald–Letnikov definition of fractional derivatives (e.g. [9]).

The results obtained from this identification process by using model (4) are shown in Table 1. The results of the optimum fitting of (4) as discussed in Table 1 for a standard capacitor are also included for comparison purposes. Figure 3a shows the minimum error (8) obtained for each fractional derivative α in (4), and Fig. 3b

Table 1 Identification process results

	Opt. fit.	Opt. fit. assuming a cap. ($\alpha = 1$)
α	0.97	1
τ	0.1113	0.1109
γ	0.0058	0.0091
Error (J_{error})	0.0323	0.0612

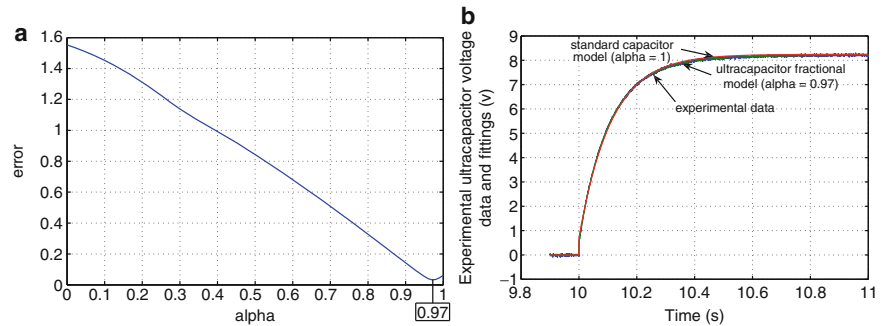


Fig. 3 Minimum error J_{error} and comparison of the ultracapacitor time responses (a) Minimum error (8) in function of α (b) Model fittings to the experimental data of the ultracapacitor time response to a step input

shows the fittings to the experimental data of the fractional model and the standard one. It can be noticed in this Figure that though the “fractionality” of the model is quite small ($\alpha = 0.97$) its effect on the dynamic response is noticeable (in fact the error is reduced to half its value as Table 1 shows). This Figure also shows that the fitting obtained with (4) is excellent, so it is not necessary to look for more complex models. Additionally, the estimated physical parameters are obtained: $C = 22$ mF, $R = 263.6$ m Ω and $R_T = 4.8$ Ω .

3 Buck-Boost Converter

The main application of Buck-Boost converters is in controlled DC power supplies where an output voltage with negative-polarity is required with respect to the input voltage. The output voltage can be either higher or lower than the input voltage [7].

Figure 4a shows the converter which can be divided in two subsystems depending on the switch position. When turning on the switch ($u = 1$), the input voltage v_d is applied across the inductor L . Turning off the switch ($u = 0$), the inductor current flows through the diode transferring its stored energy to the output capacitor C_o and the load resistance R_o .

The control signal u has a steady state duty-ratio D , being the input/output voltage ratio of the converter.

$$\frac{v_o}{v_d} = -\frac{D}{1-D} \quad (9)$$

Applying Kirchoff's laws to the two subsystems and combining the obtained models, the differential equations that describe the converter are (see [11]):

$$L \frac{di(t)}{dt} = (1 - u(t))v_o(t) + u(t)v_d(t) \quad (10)$$

$$C_o \frac{dv_o(t)}{dt} = -(1 - u(t))i(t) - \frac{v_o(t)}{R_o} \quad (11)$$

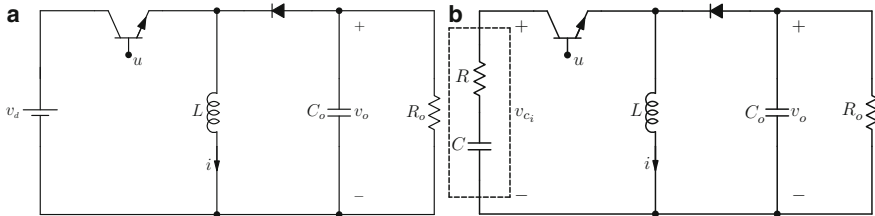


Fig. 4 Buck-Boost power electronic converter (a) Standard Buck-Boost converter (b) Buck-Boost converter supplied by a capacitor. The dashed rectangle shows the capacitor

4 Simulation Results

In this section, the behaviour of an ultracapacitor connected to a Buck-Boost converter is analysed and compared in simulation to the one obtained with a conventional capacitor and with the behaviour of an ultracapacitor with bigger fractionality (taking $\alpha = 0.75$). The DC voltage source v_d which supplies the converter in Fig. 4a is replaced by a capacitor (either the ultracapacitor or the conventional one) charged to an initial voltage v_{c_i} as Fig. 4b shows. The energy stored in the capacitor will be transferred to the load resistance R_o by means of the converter.

The following differential equation of the ultracapacitor has been added to Eqs. 10-11:

$$v_{c_i}(t) - v_{c_i}(0^+) = -\frac{1}{C_i} D^{-\alpha} \{u(t)i(t)\} - Ru(t)i(t) \quad (12)$$

where $v_{c_i}(0^+)$ is the initial voltage of the ultracapacitor, and the input voltage $v_d(t)$ in (10) is replaced with the ultracapacitor voltage $v_{c_i}(t)$.

Three different simulations are carried out in Matlab/Simulink for two different values of α and for the conventional integer behaviour of the capacitor. The values used for the capacitance and the equivalent series resistance are those obtained in Section 2 and they are the same for the three cases. The fractional behaviour is modelled using the fractional control toolbox for Matlab, developed by [12].

The chosen parameters for the converter elements are: $C = 22$ mF, $R = 236.6$ m Ω , $L = 20$ mH, $C_o = 470$ μ F, $R_o = 200$ Ω . The switching frequency of the control signal (u) is 1 kHz with a duty cycle of 50%. The total simulation time is 10 s.

Figure 5a plots the input voltages for the ultracapacitor with $\alpha = 0.97$ and for the conventional capacitor. The results show that the conventional capacitor is discharged faster than the ultracapacitor. However, the difference is not very remarkable because value of α is close to the unity. A second fractional ultracapacitor with higher fractional behaviour than in the previous case ($\alpha = 0.75$) has been simulated.

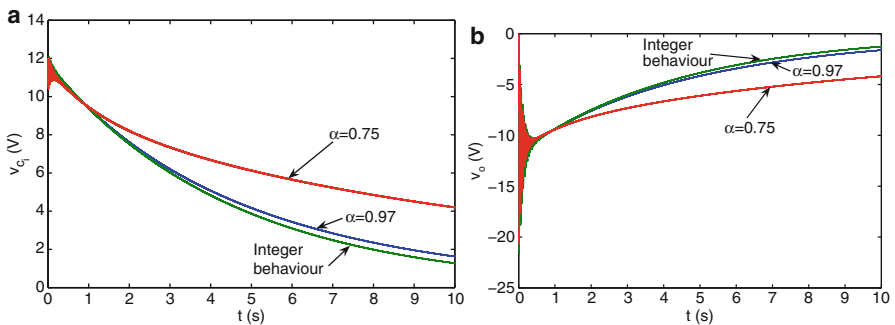


Fig. 5 Time responses of the ultracapacitor voltage and the output voltage, respectively, for different values of α (a) Ultracapacitor voltage (b) Output voltage

As it was expected, the ultracapacitor is discharged much slower than the other capacitors, which is advantageous from the point of view of the energy storage.

Figure 5b show the time response of the converter output voltage. As the capacitors are discharging, the output voltage tends to zero: the worst case is obtained for the standard capacitor as it was expected (the output voltage at the end of the simulation period is $v_o(10) = -1.28$ V), while the output voltage obtained for $\alpha = 0.97$ shows a better time response ($v_o(10) = -1.63$ V). Finally, the best result is obtained for $\alpha = 0.75$ with an output voltage at $t = 10$ s equal to -4.2 V.

5 Conclusions

In this work the dynamic behaviour of a power electronic Buck-Boost converter with an ultracapacitor as storage element has been studied. Ultracapacitors exhibit a remarkable fractional behaviour so the analysis of the mentioned circuit entails some complexity. The new results presented here are as follows: (a) a method to estimate the parameters and the order of the fractional model has been proposed. The method is based on fitting temporal responses, unlike most of the previous approaches that use frequency domain techniques, (b) it has been verified that a simple fractional model is enough to characterize the dynamics of small ultracapacitors as the one employed, and (c) it has been demonstrated that even low levels of “fractionality” may lead to behaviours that noticeably differ from what can be expected using a standard capacitor model. This can be observed both in the ultracapacitor considered as an isolated component, and in the whole converter circuit.

Finally, it should be mentioned that the results here reported are a first step towards designing more efficient control algorithms for these converters, that will take into account the fractional dynamics shown here.

References

1. Atcitty S (2006) Electrochemical capacitor characterization for electric utility applications. PhD thesis, Virginia Polytechnic Institute and State University
2. Bollen MHJ (2000) Understanding Power Quality Problems: Voltage Sags and Interruptions. IEEE Press, New York
3. Burton T, Sharpe D, Jenkins N, Bossanyi E (2001) Wind energy handbook. Wiley, Chichester
4. Dzielinski A, Sierociuk D (2008) Ultracapacitor modelling and control using discrete fractional order state-space model. *Acta Montanistica Slovaca* 13(1):136–145
5. Feliu V, Feliu S (1997) Method of obtaining the time domain response of an equivalent circuit model. *J Electroanal Chem* 435(1–2):1–10
6. Kötzt R, Carlen M (2000) Principles and application of electrochemical capacitors. *Electrochimica Acta* 45:2483–2498
7. Mohan N, Undeland TM, Robbins WP (2003) Power electronics: converters, applications and design, 3rd edn. Wiley, New York

8. Petras I, Vinagre B, Dorcak V, Feliu V (2002) Fractional digital control of a heat solid: Experimental results. In: Proceedings of the International Carpathian Control Conference (ICCC-2002). Malenovice, Czech Republic
9. Podlubny L (1999) Fractional differential equations. Academic, San Diego, USA
10. Quintana J, Ramos A, Nuez I (2006) Identification of the fractional impedance of ultracapacitors. In: Proceedings of the 2nd IFAC, Workshop on fractional differentiation and its applications. Porto, Portugal
11. Sira-Ramírez H, Silva-Ortigoza R (2006) Control design techniques in power electronics devices, 1st edn. Springer, London
12. Valério D (2005) Ninteger v. 2.3 fractional control toolbox for matlab. <http://web.ist.utl.pt/duarte.valerio/ninteger/>
13. Westerlund S, Ekstam L (1994) Capacitor theory. IEEE Trans Dielec Elec In 1(5):826–839

Approximation of a Fractance by a Network of Four Identical RC Cells Arranged in Gamma and a Purely Capacitive Cell

Xavier Moreau, Firas Khemane, Rachid Malti, and Pascal Serrier

Abstract In this paper a storage I -element and a fractance are associated, thus defining a fractional system. In order to achieve the fractance, an approximation by a network of 4 identical RC cells arranged in gamma and a purely capacitive cell is proposed, thus defining a rational system. When compared to each other, the dynamic behaviors of the fractional and the rational systems show an excellent superposition of frequency and time-domain responses. Moreover, the robustness of stability margins obtained with both systems is illustrated versus variations of the I -element.

1 Introduction

The port-based approach (as represented in a bond-graph) has demonstrated the benefits of using an integral causal form of the constitutive relations of storage ports, both for numerical simulation and the modelling process itself. In numerical simulation, integration is preferred to differentiation for obvious reasons like numerical noise and proper handling of initial conditions. For example, with a storage C -element (using bond-graph terminology, it stands for: springs, torsion bars, electrical capacitors, gravity tanks, accumulators, ...) [1], the causal relation between the power variables is given by

$$e_C(t) = \frac{1}{c} \int_0^t f_c(\tau) d\tau + e_C(0), \quad (1)$$

where $f_c(t)$ and $e_C(t)$ are the generalized flow and the generalized effort, $e_C(0)$ being an initial condition (I.C.) on the effort and c a characteristic parameter of the C -element. With a storage I -element (using bond-graph terminology, it stands for:

X. Moreau, F. Khemane (✉), R. Malti, and Pascal Serrier
Bordeaux University – IMS, 351, cours de la Libération, 33405 Talence Cedex, France
e-mail: Xavier.Moreau@ims-bordeaux.fr; Firas.Khemane@ims-bordeaux.fr;
Rachid.Malti@ims-bordeaux.fr; Pascal.Serrier@ims-bordeaux.fr

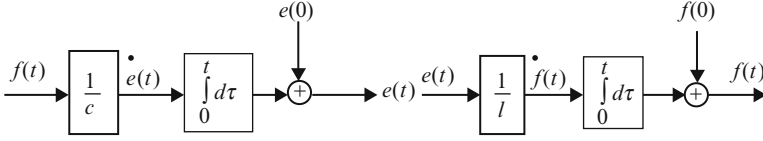


Fig. 1 Block diagrams of the *C*-element and *I*-element

mass in translation, inertia in rotation, electrical or hydraulic self inductance, ...), the causal relation between the power variables is given by

$$f_I(t) = \frac{1}{l} \int_0^t e_I(\tau) d\tau + f_I(0), \quad (2)$$

where $f_I(t)$ and $e_I(t)$ are the generalized flow and the generalized effort, $f_I(0)$ being an initial condition on the flow and l a characteristic parameter of the *I*-element. Figure 1 presents two block diagrams that illustrate the causal relations for the *C*-element and the *I*-element.

For fractional systems, the benefits of using integral causal forms are the same as for rational systems [7].

In this paper, the fractional system studied is composed of a storage *I*-element and a fractance defined by [3]. For the fractance, the generalized effort $e_\lambda(t)$ is proportional to the fractional integral of the generalized flow $f_\lambda(t)$, namely:

$$e_\lambda(t) = \frac{1}{\lambda} \int_0^t \frac{1}{\Gamma(1-m)(t-\tau)^{-m}} f_\lambda(\tau) d\tau + e_\lambda(0), \quad (3)$$

where $e_\lambda(0)$ is a function that takes into account the initial conditions [2, 4, 5] and where $\lambda \in \mathbb{R}^+$ and $m \in [0, 1]$; if $m = 0$ then the fractance is a purely capacitive *C*-element, if $m = 1$ then the fractance is a purely resistive *R*-element.

The objective of this paper is

- Firstly, to compare the dynamical behaviors obtained with the fractance and with an approximation by a network of N identical *RC* cells, in particular when N is small (for example, $N = 4$)
- Then, to highlight the damping robustness versus variation of the *I*-element

After this brief introduction, Sect. 2 presents the modeling of the fractional system studied in this paper. Section 3 focuses on the analysis of the forced motion and Sect. 4 on the free motion. Finally, conclusions are given in Sect. 5.

2 Fractional Dynamic System

In order to be generic, (3) is rewritten as a convolution product:

$$e_\lambda(t) = g(t) * f_\lambda(t) + e_\lambda(0), \quad (4)$$

where $g(t)$ is the impulse response of either the fractance $h(t)$ or its approximation $\tilde{h}(t)$.

This study being generic, no application domain is privileged. However, in order to facilitate the representation, electric diagrams are used.

Figure 2 presents the diagram of the studied fractional system where $e_0(t)$ is a generalized effort generator and $f(t)$ the generalized flow of the I -element. More precisely, Fig. 2a presents the association of the I -element with the fractance and Fig. 2b that of the I -element with the approximation of the fractance.

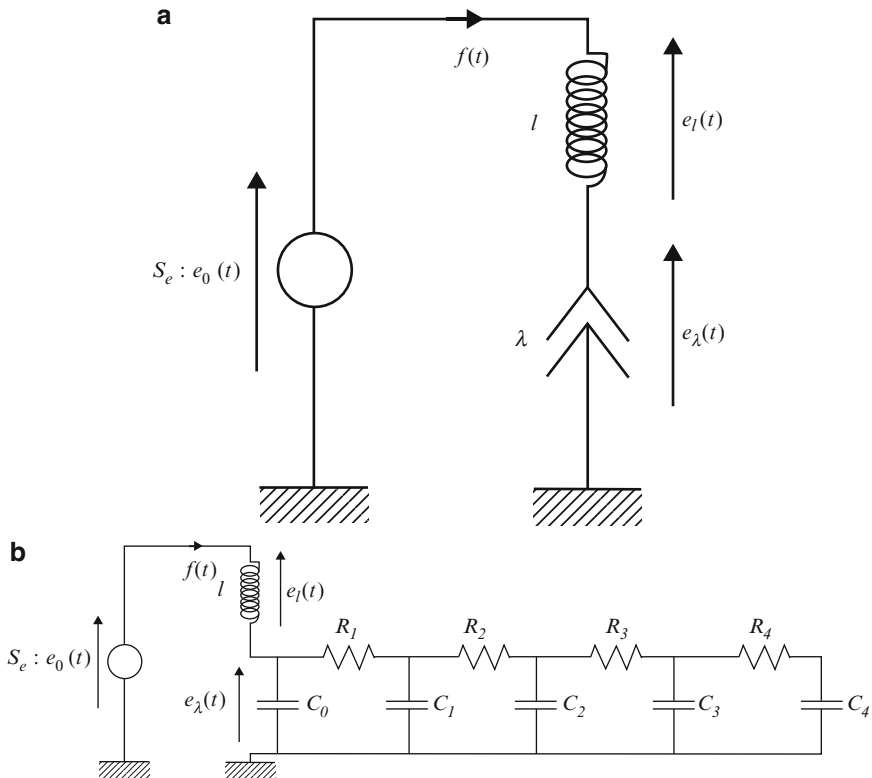


Fig. 2 Electric circuits composed of an I -element with a fractance (a) and an I -element with an approximation of this fractance (b)

The approximation is a cascaded network of four identical RC cells arranged in gamma, and a purely capacitive cell. The role of the capacitance C_0 is essential in the achievement of a fractional integrator with a limited number of cells. Additional details of this network are given in [6].

The I -element and the fractance being in series, the generalized flow $f(t)$ through each element is the same. Hence, the generalized effort $e_0(t)$ is equal to the sum of $e_I(t)$ and $e_\lambda(t)$:

$$e_0(t) = e_I(t) + e_\lambda(t). \quad (5)$$

Finally, the causal relations of the system are:

$$\begin{cases} e_I(t) = e_0(t) - e_\lambda(t), \\ f(t) = \frac{1}{l} \int_0^t e_I(\tau) d\tau + f(0), \\ e_\lambda(t) = g(t) * f(t) + e_\lambda(0). \end{cases} \quad (6)$$

Figure 3 presents a causal diagram established from relations (6) and used for numerical simulations.

Such a diagram presents a closed-loop structure. Relations between open-loop and closed-loop are used for analysis purposes in following paragraph. Moreover, the system being linear, the superposition principle is applied to study forced response ($e_0(t) \neq 0$ and $I.C = 0$), and then free response ($e_0(t) = 0$ and $I.C \neq 0$).

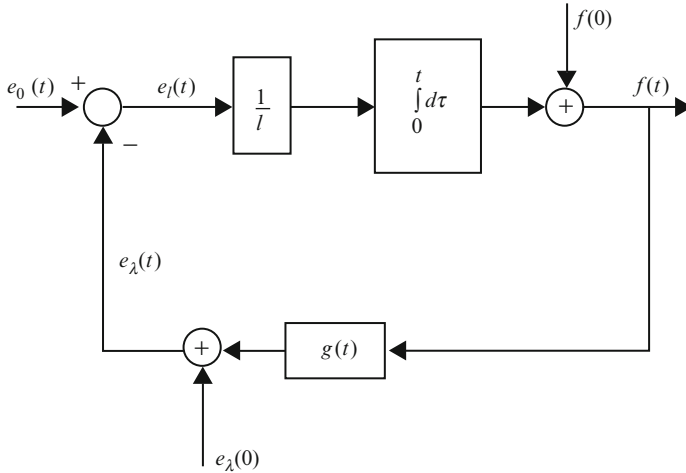


Fig. 3 Causal diagram used for numerical simulations

3 Forced Response

By supposing the initial conditions equal to zero, the Laplace transform of relations (6):

$$\begin{cases} E_I(s) = E_0(s) - E_\lambda(s), \\ F(s) = \frac{1}{l s} E_I(s), \\ E_\lambda(s) = G(s) F(s), \end{cases} \quad (7)$$

allows to establish the functional diagram of Fig. 4 where $\beta(s)$ is the open-loop transfer function given by:

$$\beta(s) = G(s) \frac{1}{l s}, \quad (8)$$

with

$$G(s) = H(s) = \frac{1}{\lambda s^{1-m}} \quad (9)$$

for the fractance, and

$$G(s) = \tilde{H}(s) = \frac{D_0}{s} \left(\frac{1 + \sum_{i=1}^{N=4} b_i (RCs)^i}{1 + \sum_{i=1}^{N=4} a_i (RCs)^i} \right) \quad (10)$$

for its approximation [6].

In the case of the fractance, the expression of $\beta(s)$ is

$$\beta(s) = \frac{b}{s^n}, \quad (11)$$

where $n = 2 - m$ and $b = 1/(l\lambda)$. The closed-loop transfer is given by

$$T(s) = \frac{E_\lambda(s)}{E_0(s)} = \frac{b}{s^n + b}. \quad (12)$$

Figure 5 presents the responses of the fractional system obtained with the fractance (green line) and its approximation (blue dotted line) for the nominal value l_0 of the

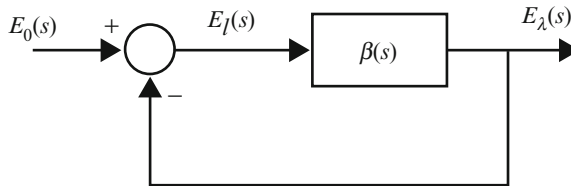


Fig. 4 Functional diagram for analysis

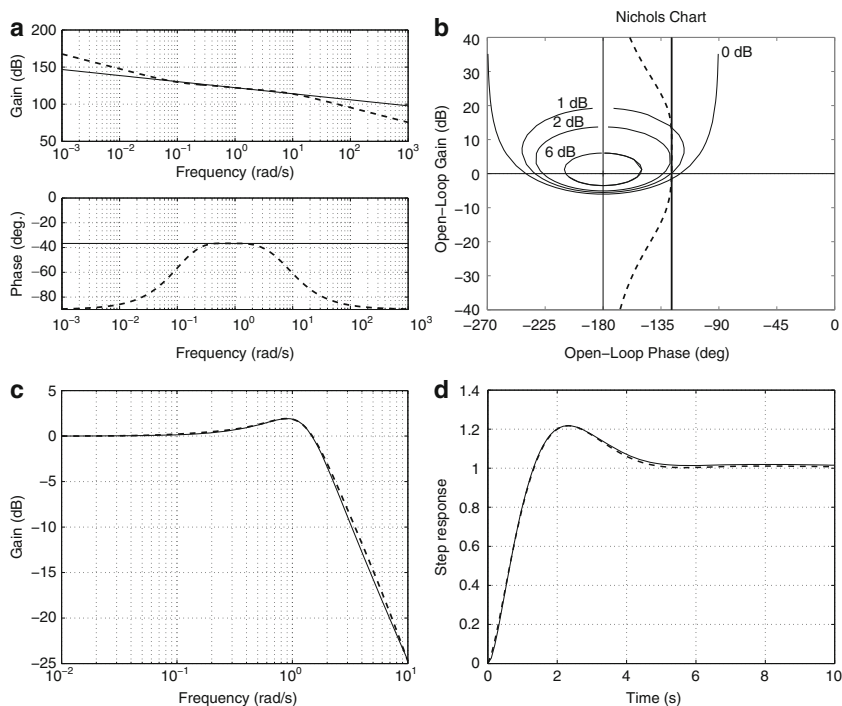


Fig. 5 Comparison between the fractional system obtained with the fractance (green line) and its approximation (blue dotted line): (a) open-loop frequency responses $\beta(s)$ and its approximation; (b) open-loop Nichols chart and its approximation; (c) closed-loop gain diagram and its approximation; (d) closed-loop step response and its approximation

parameter l . More precisely, the frequency responses of the fractional integrator and its approximation are presented in Fig. 5a; the open-loop Nichols loci in Fig. 5b; the gain diagrams of the closed-loop transfer function in Fig. 5c and the step responses of $e_\lambda(t)$ in Fig. 5d.

It is important to note the excellent superposition of the frequency responses (Fig. 5c) and of the step responses (Fig. 5d) of the closed-loop obtained with the fractional integrator and its approximation. This result is very interesting as far as the fractional behavior is synthesized in a single decade (Fig. 5a). In fact, it is fundamental that the open-loop cross-over frequency ω_u belongs to this decade to obtain such a result (Fig. 5b).

Remark

In a general way, the numerical simulation of frequency responses of a fractional system is not a problem. But, the numerical simulation of time responses is more delicate. So, the response $e_\lambda(t)$ to a unit step $u(t)$ (Fig. 5d) is obtained firstly from its Laplace transform

$$E_\lambda(s) = T(s) \frac{1}{s} = \frac{b}{s(s^n + b)} \quad (13)$$

and after a partial fraction expression

$$E_\lambda(s) = \frac{1}{s} - \frac{s^n}{s(s^n + b)}. \quad (14)$$

then by the inverse Laplace transform of (14) is given by

$$e_\lambda(t) = TL^{-1} \left\{ \frac{1}{s} - \frac{s^n}{s(s^n + b)} \right\} = u(t) - E_n[-bt^n], \quad (15)$$

where $E_n[-bt^n]$ is the Mittag-Leffler function [2] defined by

$$E_n[-bt^n] = \sum_{i=0}^{\infty} \frac{(-b)^i t^{in}}{\Gamma(in + 1)}. \quad (16)$$

So, the step response presented in Fig. 5d is obtained by programming the relation (16) where the sum is truncated at $i = 100$. This value is chosen very large in order to obtain a fractional system response very close to the true one.

Considering the excellent approximation capability that has the network of four identical *RC* cells arranged in gamma to reproduce the behavior of the fractance associated with an *I*-element, the next paragraph focuses on the performance obtained with this approximation.

Moreover, the l parameter of the *I*-element is considered as uncertain ($l \in [l_{min}; l_{max}]$).

Figure 6 illustrates the influence of the l variation on the dynamic behavior of the fractional system. More precisely, Fig. 6a presents the open-loop Nichols loci, Fig. 6b the gain diagrams of the closed-loop transfer and Fig. 6c the step responses of $e_\lambda(t)$.

The variation of l leads to an open-loop gain variation that is why the open-loop frequency responses are tangent to the same Nichols magnitude contour (Fig. 6a). So, one can observe the robustness of the resonant peak (Fig. 6b) and the robustness of the first overshoot (Fig. 6c) versus l variation.

4 Free Response

In order to facilitate the analysis of the free response, a state space representation of the fractance approximation is used, namely:

$$\begin{cases} \dot{\underline{x}} = A\underline{x} + Bu, \\ y = C\underline{x} + Du. \end{cases} \quad (17)$$

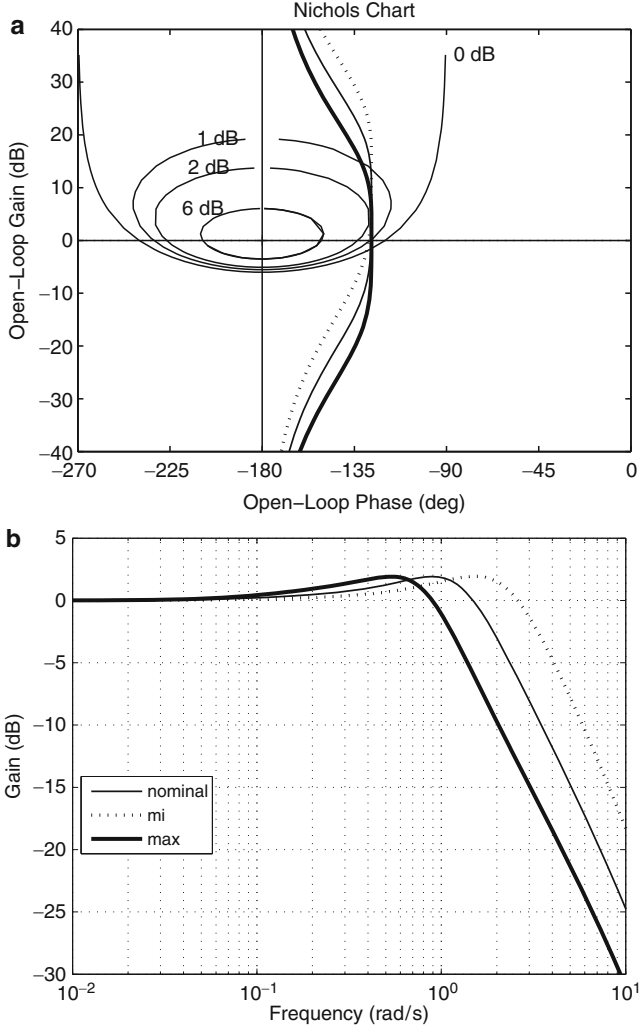


Fig. 6 Illustration of the stability margin robustness versus l variations (l_0 blue, l_{min} green and l_{max} red); open-loop Nichols-loci (a); gain diagrams of the closed-loop transfer function (b) and step responses of generalized effort $e_\lambda(t)$ (c)

where

$$u = f(t), \quad \underline{x} = \begin{pmatrix} x_1 = e_\lambda(t) \\ x_2 = e_{C1}(t) \\ x_3 = e_{C2}(t) \\ x_4 = e_{C3}(t) \\ x_5 = e_{C4}(t) \end{pmatrix} \quad y = e_\lambda(t), \quad (18)$$

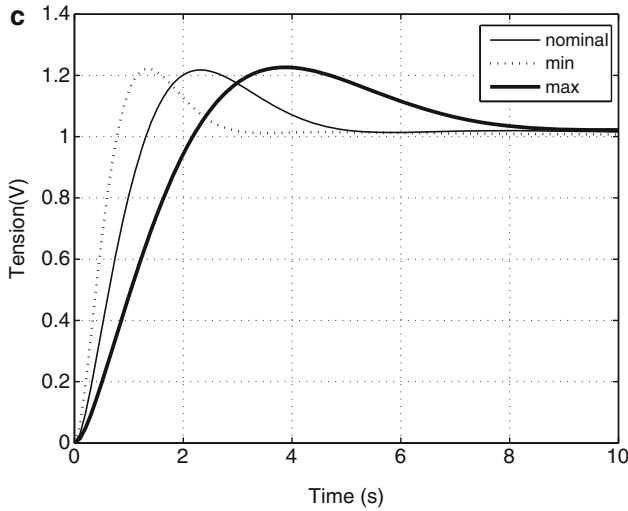


Fig. 6 (continued)

the $e_{C_i}(t)$ being the generalized efforts of elements C_i , and

$$A = \begin{pmatrix} -1/R_1 C_0 & 1/R_1 C_0 & 0 & 0 & 0 \\ 1/R_1 C_1 & -\frac{1}{C_1} \left(\frac{R_1 + R_2}{R_1 R_2} \right) & 1/R_2 C_1 & 0 & 0 \\ 0 & 1/R_2 C_2 & -\frac{1}{C_2} \left(\frac{R_2 + R_3}{R_2 R_3} \right) & 1/R_3 C_2 & 0 \\ 0 & 0 & 1/R_3 C_3 & -\frac{1}{C_3} \left(\frac{R_3 + R_4}{R_3 R_4} \right) & 1/R_4 C_3 \\ 0 & 0 & 0 & 1/R_4 C_4 & -1/R_4 C_4 \end{pmatrix}, \quad (19)$$

$$B = \begin{pmatrix} 1/C_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (C = 1 \ 0 \ 0 \ 0 \ 0), \quad (D = 0). \quad (20)$$

The Laplace transform of relation (17) leads to

$$\begin{matrix} Y &= & (C[sI - A]^{-1}) \underline{x}(0) &+ & (C[sI - A]^{-1} B) U, \\ (1 \times 1) & & (1 \times 5) & & (5 \times 1) & & (1 \times 1) & & (1 \times 1) \end{matrix} \quad (21)$$

where $\underline{x}(0)$ is the initial condition vector associated with the elements C_i . The relation (21) can be rewritten as:

$$Y = \sum_{i=1}^5 (\Psi_i(s) x_i(0)) + \tilde{H}(s) U, \quad (22)$$

where

$$\Psi_i(s) = (C[sI - A]^{-1}) \text{ and } \tilde{H}(s) = (C[sI - A]^{-1}B), \quad (23)$$

Finally, back to the time-domain by inverse Laplace transform leads to

$$e_\lambda(t) = \sum_{i=1}^5 (TL^{-1}\{\Psi_i(s) x_i(0)\}) + \tilde{H}(t) * f(t), \quad (24)$$

which has the following form:

$$e_\lambda(t) = e_\lambda(0) + \tilde{e}_\lambda(t), \quad (25)$$

where

$$e_\lambda(0) = \sum_{i=1}^5 (TL^{-1}\{\Psi_i(s) x_i(0)\}) \quad (26)$$

and

$$\tilde{e}_\lambda(t) = \tilde{H}(t) * f(t). \quad (27)$$

So, the functional diagram of Fig. 3 is completed in accordance with previous developments as illustrated Fig. 7.

For the study of free response $e_0(t) = 0$, the initial conditions are

$$f(0) = 0 \text{ and } e_{Ci}(0) = 1 \quad \forall i \in [0; 4]. \quad (28)$$

In this case and for the nominal value l_0 of the l parameter, Fig. 8a presents the contributions of each term of the sum $e_\lambda(0)$ defined by the relation (26). Figure 8b shows the plot of $e_\lambda(0)$ (green) and the plots of $\tilde{e}_\lambda(t)$ (red) and $e_\lambda(t)$ (blue).

It is important to note that the sum $e_\lambda(0)$ of all contributions $e_{Ci}(0)$ is equal to unity.

Finally, Fig. 9 presents free response of $e_\lambda(t)$ for the same initial conditions and for the values l_0 , l_{min} and l_{max} of l , showing thus the robustness of the stability margin versus l variations.

In a first step, the interest of using the fractance approximation for taking into account initial conditions is to observe (in this particular case) that $e_\lambda(0)$ is equal to a constant (unity if all the $e_{Ci}(0) = 1$ and if $f(0) = 0$). In a second step, it is possible to affirm (always in this particular case) that $e_\lambda(0)$ is a unit constant and to replace the impulse response $\tilde{h}(t)$ of the approximation by that of the fractance, namely $h(t)$. Thus, with $e_0(t) = 0$, $f(0) = 0$ and $e_\lambda(0) = 1$, the relation (6) is rewritten as

$$\begin{cases} e_l(t) = e_0(t) - e_\lambda(t), \\ f(t) = \frac{1}{T} \int_0^t e_l(\tau) d\tau, \\ e_\lambda(t) = h(t) * f(t) + e_\lambda(0). \end{cases} \quad (29)$$

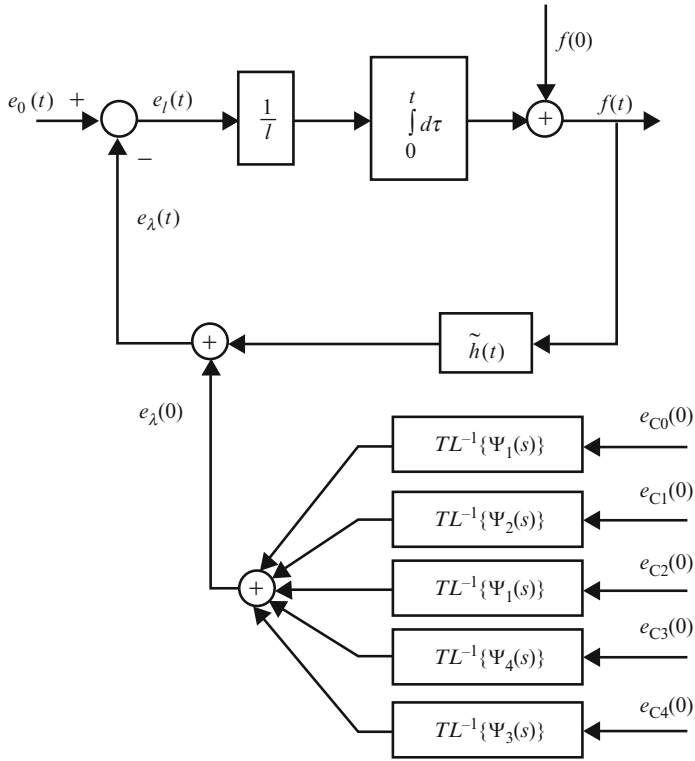


Fig. 7 Functional diagram for simulation by taking into account the initial conditions $e_{Ci}(0)$ associated with elements C_i

where

$$h(t) = \frac{t^{-m}}{\lambda \Gamma(1-m)} \quad (30)$$

The Laplace transform of relation (29), namely:

$$\begin{cases} E_l(s) = -E_\lambda(s), \\ F(s) = \frac{1}{l s} E_l(s), E_\lambda(s) = \frac{1}{\lambda s^{1-m}} F(s) + \frac{e_\lambda(0)}{s}, \end{cases} \quad (31)$$

leads to an expression of the form

$$E_\lambda(s) = -\frac{1}{\lambda s^{1-m}} - \frac{1}{l s} E_\lambda(s) + \frac{e_\lambda(0)}{s}, \quad (32)$$

which can be reduced to

$$E_\lambda(s) = \frac{s^n}{s(s^n + b)} e_\lambda(0), \quad (33)$$

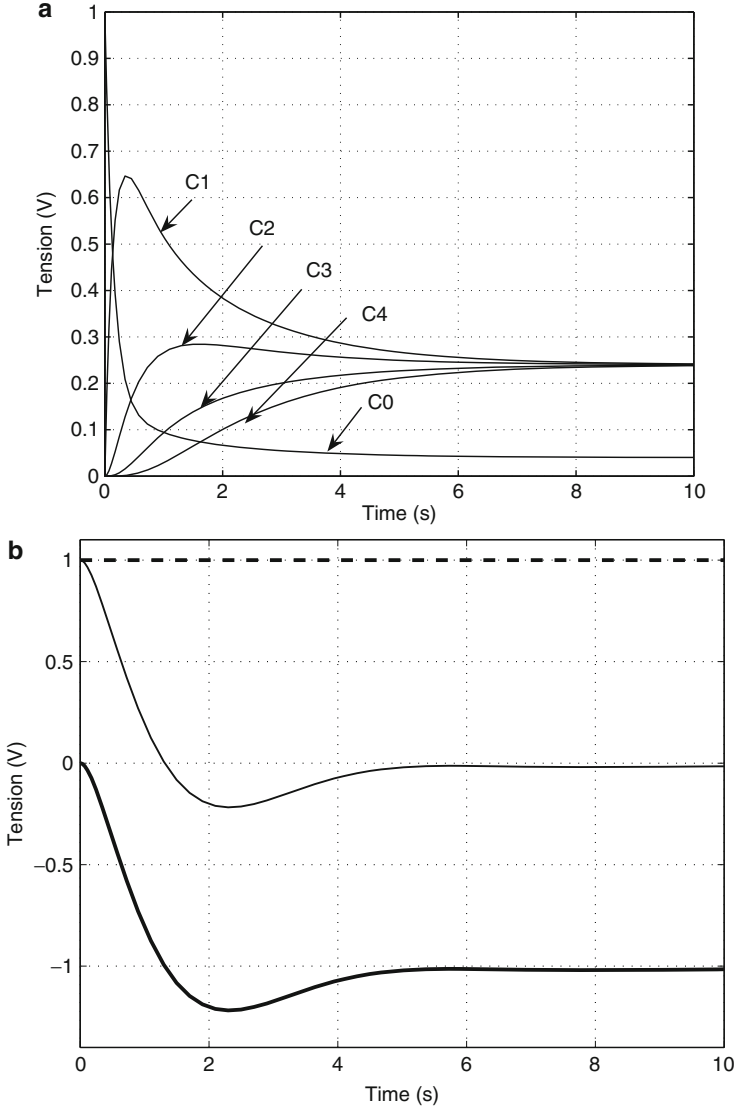


Fig. 8 (a) Contribution of each term of the sum $e_\lambda(0)$ defined by the relation (26). (b) plots of $e_\lambda(0)$ (green), $\tilde{e}_\lambda(t)$ (red) and $e_\lambda(t)$ (blue)

always by putting $n = 2 - m$ and $b = 1/(l\lambda)$. Knowing that $e_\lambda(0) = 1$, the inverse Laplace transform of (33) leads to:

$$e_\lambda(t) = TL^{-1}\left\{\frac{s^n}{s(s^n + b)}\right\} = E_n[-bt^n], \quad (34)$$

where $E_n[-bt^n]$ is the Mittag-Leffler function defined by the relation (16). Figure 10 presents, for the nominal value l_0 , the free motion of $e_\lambda(t)$ obtained with

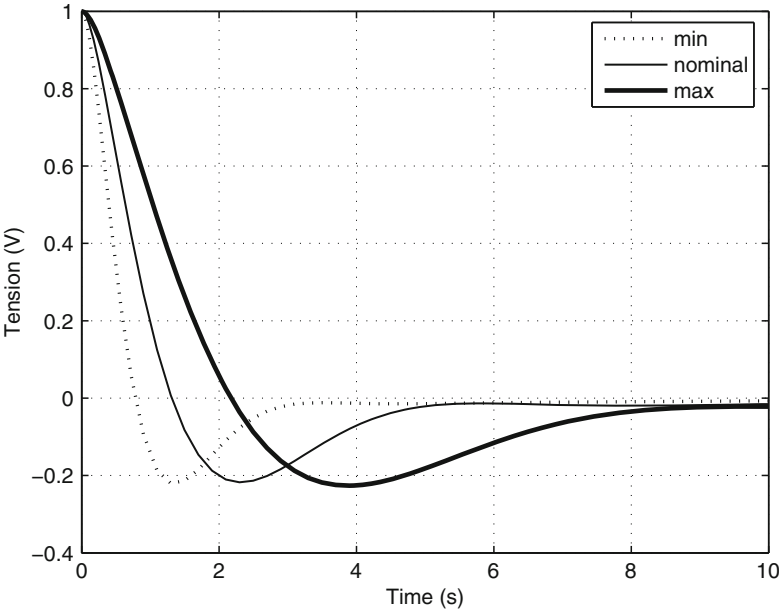


Fig. 9 Free response of $e_\lambda(t)$ for the same initial conditions and for the values l_0 , l_{min} and l_{max} of l

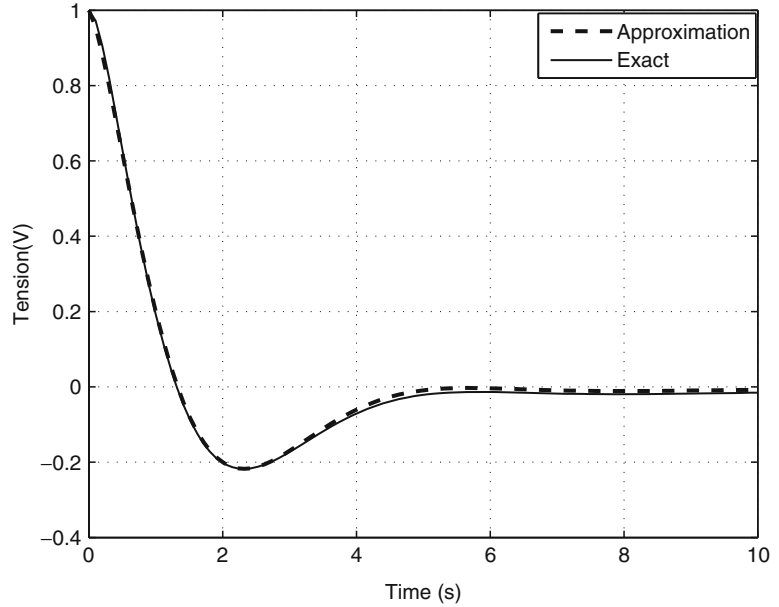


Fig. 10 Free response of $e_\lambda(t)$ obtained with the approximation (blue dotted line) and with the Mittag–Leffler function (green line) truncated at $i = 100$

the approximation (blue dotted line) and with the Mittag-Leffler function (green line) truncated at $i = 100$. One can observe the excellent superposition of the two plots.

5 Conclusion

In this paper, a fractional system and its approximation have been studied. The comparison of the dynamic behaviors obtained with both systems shows the excellent capability that has the network of four identical RC cells arranged in gamma to reproduce the behavior of the fractance. Moreover, the robustness of stability margin obtained with both systems is illustrated versus the I -element variations. The perspective in the continuity of this study, is to take into account uncertainties and/or non-linearities of the R and C elements according to the application context.

References

1. Dauphin-Tanguy G (2000) Les bond-graphs. Edition Hermès, Paris
2. Hartley TT, Lorenzo CF (2002) Dynamics and control of initialized fractional-order systems. *J Nonlinear Dynam* 29:201–233
3. Le Méhauté A, Nigmatullin R et L (1998) Nivanen: Flèche du temps et géométrie fractale. Edition Hermès, Paris
4. Lorenzo CF, Hartley TT (2007) Initialization of fractional differential equations: background and theory. In: Proceedings of the ASME 2007, DETC2007-34810. Las Vegas, Nevada, USA, 4–7 September
5. Lorenzo CF, Hartley TT (2007) Initialization of fractional differential equations: theory and application. Proceedings of the ASME 2007, DETC2007-34814. Las Vegas, Nevada, USA, 4–7 September
6. Moreau X, Serrier P, Oustaloup A (2008) From fractional systems to localised parameter systems: synthesis and analysis. Special Issue on APII-JESA, N 6-7-8, volume 42, September
7. Trigeassou JC, Poinot T, Lin J, Oustaloup A, Levron F (1999) Modeling and identification of a non integer order system. In: Proceedings of ECC'99, European Control Conference. Karlsruhe, Germany

Part III

Mathematical Tools

On Deterministic Fractional Models

Margarita Rivero, Juan J. Trujillo, and M. Pilar Velasco

Abstract This chapter is dedicated to presenting some aspects of the so-called Ordinary and/or Partial Fractional Differential Equations. During last 20 years the main underground reason that explain the interest of the applied researchers in the fractional models have been the known close link that exists between such kind of models and the so-called “Jump” stochastic models, such as the CTRW (Continuous Time Random Walk). During the second half of the twentieth century (until the 1990s), the CTRW method was practically the main tool available to describe subdiffusive and/or superdiffusive phenomena associated with Complex Systems for the researches that work in applied fields. The fractional operators are non-local, while the ordinary derivative is a local operator, and on the other hand, the dynamics of many anomalous processes depend of certain memory of its own dynamics. Therefore, the fractional models linear and/or non-linear look as a good alternative to the ordinary models. Note that fractional operators also provide an alternative method to the classical models including dilate terms. The main of this chapter is dedicated to do a first approach to show how introduce Fractional Models only under a deterministic basement. We will considerate Fractional Dynamics Systems with application to study anomalous growing of populations, and on the other hand, the Fractional Diffusive Equation.

M. Rivero

Universidad de La Laguna, Departamento de Matemática Fundamental, 38271 La Laguna, Tenerife, Spain

e-mail: mrivero@ull.es

J.J. Trujillo (✉)

Universidad de La Laguna, Departamento de Análisis Matemático, 38271 La Laguna, Tenerife, Spain

e-mail: jtrujill@ullmat.es

M.P. Velasco

Universidad Complutense de Madrid, Departamento de Matemática Aplicada, 28040 Madrid, Spain

e-mail: mvcebian@mat.ucm.es

1 Introduction

Fractional Calculus deals with the study of so-called fractional order integral and derivative operators over real or complex domains, and their applications. The definitions of the fractional derivative (which generalizes the ordinary differential operator) are diverse. Questions such as ‘What is understood by the Fractional Derivative?’, ‘What does the $1/3$ or $\sqrt{2}$ derivative of a function mean?’, and ‘How can these Fractional Operators be applied, losing as they do so many fundamental properties with respect to ordinary derivatives?’ have been the center of attention of many important scientists over the last two centuries. A rigorous, encyclopedic study of Fractional Calculus was written by Samko, Kilbas and Marichev in 1987 (the English edition was published in 1993 [11]).

Fractional order differential equations, that is, those involving real or complex order derivatives, have assumed an important role in modeling the anomalous dynamics of numerous processes related to complex systems in the most diverse areas of Science and Engineering.

Despite the attention given to fractional differential equations by many authors, only a few steps have been taken toward what may be called a clear and coherent theory of said equations that supports the use of this tool in the applied sciences in a manner analogous to the ordinary case.

The theoretical interest in fractional differential equations as a mathematical challenge can be traced back to 1918, when O’Shaughnessy [9] gave an explicit solution to the differential equation $y^{(\alpha)} = y/x$, after he himself had suggested the problem. In 1919, Post [10] proposed a completely different solution. First, note that the problem was not rigorously defined, since there is no mention of what fractional derivative is being used in the proposed differential equation. This explains why both authors found such different solutions, and why neither was wrong (see Kilbas-Trujillo [4]–[5] and Bonilla-Kilbas-Trujillo [1]).

A fractional derivative is nothing more than an operator which generalizes the ordinary derivative, such that if the derivative is represented by the operator D^α then, when $\alpha = 1$, it lets us get back to the ordinary differential operator D . As one would expect, there are many ways to set up a fractional derivative, and it is not unusual to see nowadays at least six different definitions (see Samko-Kilbas-Marichev [11]).

Note that since approximately 1990, there has been a spectacular increase in the use of fractional models to simulate the dynamics of many different anomalous processes, especially those involving ultraslow diffusion.

As one would expect, since a fractional derivative is a generalization of an ordinary derivative, it is going to lose many of its basic properties; for example, it loses its geometric or physical interpretation, the index law is only valid when working on very specific function spaces, the derivative of the product of two functions is difficult to obtain, and the chain rule is not straightforward to apply. It is natural to ask, then, what properties fractional derivatives have that make them so suitable for modeling certain Complex Systems. **We think the answer lies in the property exhibited by many of the aforementioned systems of “non-local**

dynamics”, that is, the processes’ dynamics have a certain degree of memory and fractional operators are non-local, while the ordinary derivative is clearly a local derivative.

The stochastic interpretation of both the fundamental solution to the ordinary diffusion equation and Brownian motion has been known since the early twentieth century. It was done by associating two random variables to a particle at each position, one to predict the probability of the length of the particle’s jump, and the other to predict the particle’s wait time before its next move. For the ordinary case, these probabilities are Gaussian and, along with the Central Limit Theorem, serve to completely explain the dynamics involved.

In 1949, Gnedenko and Kolmogorov [2] introduced a generalization of the Central Limit Theorem, extending the validity of said theorem to a random variable which was in turn the sum of random non-Gaussian variables (specifically of the so-called Levy stable distribution type). In 1965 Montroll and Weiss [8] applied this generalization of stochastic theory in introducing the CTRW (Continuous Time Random Walk) method for describing subdiffusive anomalous phenomena associated with the mechanical behavior of certain materials. This method is simply a generalization of the above mentioned method for normal diffusion processes.

The CTRW method lets us describe subdiffusive phenomena very well when $0 < \alpha < 1$, and superdiffusive phenomena when $\alpha > 1$. Thus, this method has been the tool of choice used by applied scientists to characterize and describe those types of anomalous diffusive processes from the middle of the twentieth century until today. The use of Laplace and Fourier integral transforms lets us prove that, for a subdiffusive process, the density function of the random variable for the average particle displacement, X , satisfies the following fractional diffusion equation:

$$D^\alpha U = k \frac{\partial^2 U}{\partial x^2}. \quad (1)$$

We recommend reading the excellent paper by Metzler and Kafer [7], which, according to the JCR, was the most cited article in 2003 in the area of Physics, and in which the reader can find not only a more accurate description of the above explanation, but also a clear explanation of the role played, from an applied point of view, by many other linear or non-linear diffusive fractional models, such as the Fokker-Planck fractional equation.

Without going into further detail, let us point out that this connection with stochastic models is not the only way to use fractional derivatives to model the dynamics of anomalous phenomena, even despite not being able to use a geometric of physical description of said fractional operators, which is the most common modeling approach for the ordinary case.

We hereby propose a few examples of the use of fractional models in the different branches of Science, such as:

1. In **Materials theory**
2. In **Transport theory or fluid of contaminant flow phenomena through heterogeneous porous media**

3. In Physics theory, electromagnetic theory, wave propagation, thermodynamics or in mechanics
4. In Signal theory and chaos theory and/or fractals
5. In Geology and astrophysics
6. In Control theory, biology and other sciences connected with the live, economics or chemistry, and so on

The reader can find many more references in the bibliographies of the above-cited papers, in particular those of Metzler and Klafter [7] and in the book by Kilbas, Srivastava and Trujillo [11].

The main objective of this work is break the limitations of the mentioned stochastic Methods and present a realistic first approach to show how we can introduce Fractional Models only under a deterministic basement. We will considerate Fractional Dynamics Systems with application to study anomalous growing of populations, and on the other hand, the Fractional Diffusive Equation. There for we are opening new perspectives for the application of the fractional models to simulate the anomalous dynamics of Complex Processes, with a scientific justification of such models.

2 Fractional Integral and Differential Operators

Next we describe various properties of some fractional operators of interest (see, for instance, Samko, Kilbas, Marichev [11]; McBride [6]; Sneddon [12]; Kilbas, Srivastava, Trujillo [3]), as a prelude to explain, with simples examples and without resorting to stochastic techniques, why certain fractional operators are well suited to simulating subdiffusive phenomena, but are inadequate for describing superdiffusive processes. Additionally, this will also allow us to construct a fractional operator suitable for simulating both phenomena and give answer to several interesting open questions.

Let $\alpha > 0$, $[a, b] \subset \mathbb{R}$, $n = -[-\alpha] \in \mathbb{R}$, where the notation $[\bullet]$ represent the integer part of the argument, and f is a measurable Lebesgue function, for example, $f \in L_1(a, b)$. Then the well known Riemann–Liouville and the Liouville integral operators of order α are defined by

1. Riemann–Liouville

$$(I_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \quad (x > a) \quad (2)$$

$$(D_{a+}^{\alpha} f)(x) = D^n (I_{a+}^{n-\alpha} f)(x) \quad (x > a) \quad (3)$$

$$(I_{b-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt \quad (x < b) \quad (4)$$

$$(D_{b-}^{\alpha} f)(x) = (-D)^n (I_{b-}^{n-\alpha} f)(x), \quad (x < b) \quad (5)$$

where D the regular derivative.

2. Liouville (or Weyl). If $x \in \mathbb{R}$

$$(I_+^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-t)^{\alpha-1} f(t) dt \quad (6)$$

$$(D_+^\alpha f)(x) = D^n (I_+^{n-\alpha} f)(x) \quad (7)$$

$$(I_-^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t) dt \quad (8)$$

$$(D_-^\alpha f)(x) = (-D)^n (I_-^{n-\alpha} f)(x), \quad (9)$$

Another derivative whose use is widespread in fractional modeling is the **Caputo Fractional Derivative**:

$$({}^C D_{a+}^\alpha f)(x) = (I_{a+}^{n-\alpha} D^n f)(x) \quad (x > a) \quad (10)$$

This derivative is closely related with the Riemann–Liouville derivative. For example, if $n-1 < \alpha < n$:

$$({}^C D_{a+}^\alpha f)(x) = (D_{a+}^\alpha f)(x) + \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{\Gamma(1+j-\alpha)} (x-a)^{j-\alpha} \quad (11)$$

The following properties are important:

$$({}^C D_{a+}^\alpha 1) = 0 \quad (12)$$

$$(D_{a+}^\alpha 1) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \quad (13)$$

The following generalization of the Liouville operator will be of interest later (we must assume that $g(x) > 0$ if $x > 0$, g' exist in the corresponding interval):

$$(I_{+;g}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x \frac{g'(t)f(t)}{(g(x)-g(t))^{1-\alpha}} dt \quad (x > a) \quad (14)$$

$$(D_{+;g}^\alpha f)(x) = D_g^n (I_{+;g}^{n-\alpha} f)(x) \quad (x > a) \quad (15)$$

$$(I_{-;g}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{g'(t)f(t)}{(g(t)-g(x))^{1-\alpha}} dt \quad (x < b) \quad (16)$$

$$(D_{-;g}^\alpha f)(x) = (-1)^n D_g^n (I_{-;g}^{n-\alpha} f)(x), \quad (x < b) \quad (17)$$

where it is assumed that $g'(x) \neq 0$ if $x > 0$.

The corresponding generalized Riemann–Liouville operators are defined as follow:

$$(I_{a+;g}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)f(t)}{(g(x)-g(t))^{1-\alpha}} dt \quad (x > a), \quad (18)$$

$$(D_{a+;g}^\alpha f)(x) = D_g^n (I_{a+;g}^{n-\alpha} f)(x) \quad (x > a), \quad (19)$$

$$(I_{b-;g}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t)f(t)}{(g(t) - g(x))^{1-\alpha}} dt \quad (x < b), \quad (20)$$

$$(D_{b-;g}^\alpha f)(x) = (-1)^n D_g^n (I_{b-;g}^{n-\alpha} f)(x), \quad (x < b), \quad (21)$$

where $D_g^n = \left(\frac{1}{g'(x)} D \right)^n$.

Under the above conditions for the $g(x)$ function, we introduce the following g-convolution definitions:

$$(K(t) *_g^1 f(t))(x) = \int_0^\infty K(g(x) - g(t))f(t)dt. \quad (22)$$

$$(K(t) *_g^2 f(t))(x) = \int_a^x K(g(x) - g(t))f(t)dt, \quad (a \in \mathbb{R}). \quad (23)$$

$$(K(t) *_g^3 f(t))(x) = \int_x^b K(g(x) - g(t))f(t)dt, \quad (b \in \mathbb{R}). \quad (24)$$

$$(K(t) *_g^4 f(t))(x) = \int_a^b K(g(x) - g(t))f(t)dt, \quad (a, b \in \mathbb{R}). \quad (25)$$

Such kind of convolutions include in the kernel certain memory or non-local effect. **Therefore, the fractional models are a good and natural alternative to the known classical non-local equations called “delay differential equations”.**

3 Time-Fractional Derivative: An Application of Fractional Differential Operators in Modeling Subdiffusive and Superdiffusive Problems

First we now define certain functions which generalize the ordinary exponential function:

$$e_\alpha^{kx} = x^{\alpha-1} E_{\alpha,\alpha}(kx^\alpha), \quad (26)$$

where

$$E_{\alpha,\beta}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\alpha j + \beta)}, \quad (27)$$

is the generalized Mittag–Leffler function. We are also interested in the following exponential function:

$$E_\alpha(kx^\alpha) \quad \text{with} \quad E_\alpha(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\alpha j + 1)}. \quad (28)$$

These functions have the following properties, among others:

The behavior of $y(x) = e_{\alpha}^{-k(x-a)}$ ($k > 0$) is as follows; If $0 < \alpha < 1$ and if $\alpha > 1$, respectively (Figs. 1 and 2):

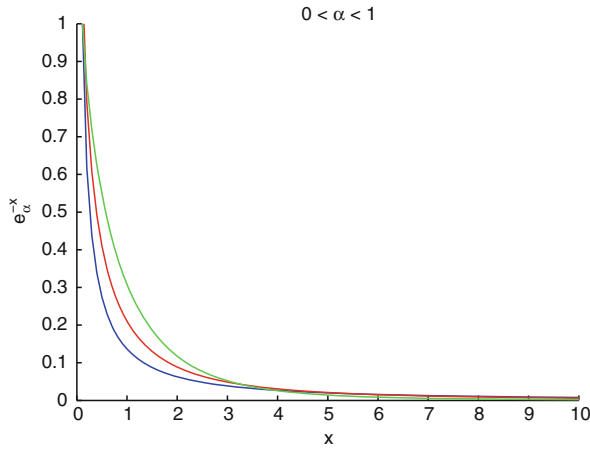


Fig. 1 Graphics of $E_{\alpha}(-x)$ with $\alpha = 0.5$ (blue), $\alpha = 0.7$ (red) and $\alpha = 0.9$ (green). For colors, see online version

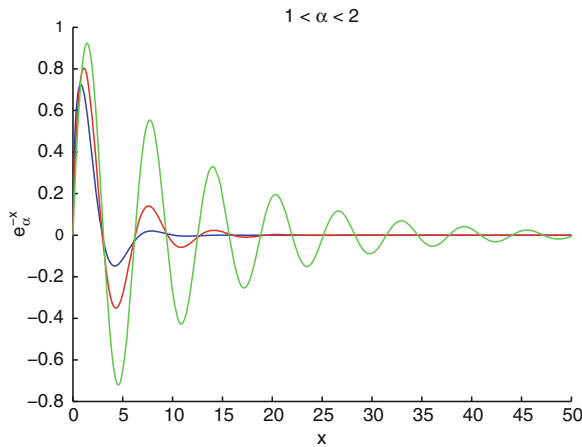


Fig. 2 Graphics of $E_{\alpha}(-x)$ with $\alpha = 1.5$ (blue), $\alpha = 1.7$ (red) and $\alpha = 1.9$ (green). For colors, see online version

Moreover, $(D_{a+}^{\alpha} y(t))(x) = k y(x)$

On the other hand, the behavior of $y(x) = E_{\alpha}(-kx)$ ($k > 0$) is as follows, if $0 < \alpha < 1$ and if $\alpha > 1$, respectively (Figs. 3 and 4):

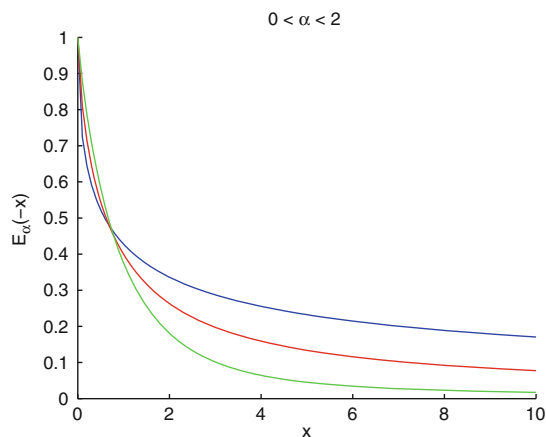


Fig. 3 Graphics of $E_\alpha(-x)$ with $\alpha = 0.5$ (blue), $\alpha = 0.7$ (red) and $\alpha = 0.9$ (green). For colors, see online version

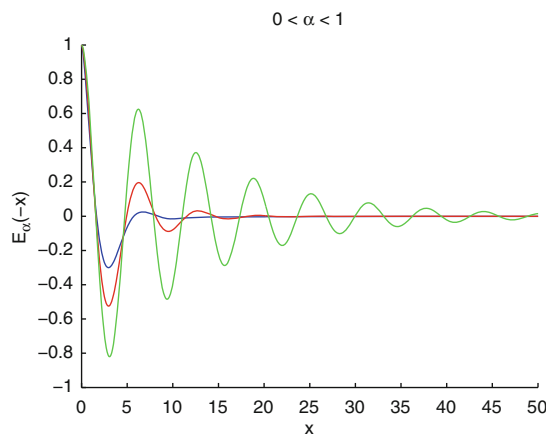


Fig. 4 Graphics of $E_\alpha(-x)$ with $\alpha = 1.5$ (blue), $\alpha = 1.7$ (red) and $\alpha = 1.9$ (green). For colors, see online version

Moreover, $({}^c D_{a+}^\alpha y(t))(x) = ky(x)$

On the other hand, $(D_{-;g}^\alpha e^{-k g(t,\alpha)})(x) = ke^{-k g(t,\alpha)}$.

If $g(t, \alpha) = t^\alpha$ or $g(t, \alpha) = e^{t^\alpha}$, then naturally $e^{-k g(t,\alpha)}$ will behave like an ordinary negative exponent e^{-kt} , decaying more slowly when $0 < \alpha < 1$, and considerably faster than e^{-kt} as the values of α increase above 1.

Now, let us briefly recall the method of separation of variables to formally solve the unidimensional problem associated with, for example, heat diffusion in an isolated bar of length 1 and whose ends are kept at 0 (this is the same equation which regulates normal diffusion processes). Assume a diffusivity constant of ρ^2 :

$$\frac{\partial U(x, t)}{\partial t} = \rho^2 \frac{\partial^2 U(x, t)}{\partial x^2}; \quad (\rho > 0; \quad t > 0; \quad 0 < x < l) \quad (29)$$

$$U(0, t) = U(l, t) = 0; \quad (30)$$

$$U(x, 0) = f(x); \quad f \text{ a suitable function.} \quad (31)$$

In searching for solutions of the type

$$U(x, t) = X(x)T(t), \quad (32)$$

we have that the following must hold for $X(x)$ and $T(t)$:

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{\rho^2 T(t)} = -\lambda^2, \quad (33)$$

Therefore:

$$X''(x) + \lambda^2 X(x) = 0; \quad (X(0) = X(l) = 0), \quad (34)$$

and

$$T'(t) + (\lambda\rho)^2 T(t) = 0. \quad (35)$$

So, eigenfunctions $X_n(x)$ to (34), and the corresponding solutions to (35) for each eigenvalues $\lambda_n = a = \frac{n\pi}{l}$ ($n \in \mathbb{N}$), being, respectively:

$$X_n(x) = \sin(ax) \quad \text{and} \quad T_n(t) = e^{-(a\rho)^2 t} \quad (36)$$

Therefore the solution to problems (29)–(30)–(31), as a series function, turns out to be (by imposing the initial condition and using nothing more than the Fourier expansion of $f(x)$):

$$U(x, t) = \sum_{n=0}^{\infty} c_n X_n(x) T_n(t), \quad (37)$$

with

$$c_n = \frac{2}{l} \int_0^l \sin(ax) f(x) dx; \quad (n \in \mathbb{N}) \quad (38)$$

In conclusion, from (37) to (38), the solution $U(x, t)$ clearly varies in time according to

$$T_n(t) = e^{-(a\rho)^2 t}; \quad \left(a = \frac{n\pi}{l}; \quad n \in \mathbb{N}\right) \quad (39)$$

and so, **it decays in time like any other ordinary negative exponent**. We must then ask ourselves:

What would we have to do if we wanted the solution to the diffusion problem to proceed in time at an extremely slower rate? What would its analytical differential model be like?

We could undoubtedly adopt many strategies, but the most direct one is the following: replace the part of the solution (37) that affects the solution's time variance, in other words, the diffusion process, with another that not only fulfils this function but which also depends on an additional parameter that lets us control the speed

of the diffusion process. This is not difficult to do since it would suffice to substitute the negative exponent (36) with another function that plays the same role but which decays much more slowly, according to the values of the aforementioned parameter. This new function may be either of the two exponents previously defined: $e_\alpha^{k(t-b)}$ ($b < 0$) or $E_\alpha(kt^\alpha)$, with $0 < \alpha < 1$. The decay rates of these functions become increasingly slower as the parameter α approaches 0, that is:

$$T_n(t) = e_\alpha^{-(a\rho)^2(t-b)} \left(b \leq 0; a = \frac{n\pi}{l}; n \in \mathbb{N} \right) \quad (40)$$

or

$$T_n(t) = E_\alpha(-(a\rho)^2 t) \left(a = \frac{n\pi}{l}; n \in \mathbb{N} \right) \quad (41)$$

The problem now lies in trying to find an operator that satisfies relation (35) for each of the solutions (40) and (41). For (40) the operator is the Riemann–Liouville fractional derivative D_{b+}^α , while for (41), it is the Caputo fractional derivative ${}^c D_{0+}^\alpha$, which follows by just keeping in mind the properties already mentioned for these operators in Sect. 3. This leads to the following corresponding fractional models for solutions (40) and (41), respectively:

$$D_{b+,t}^\alpha U(x,t) = \rho^2 \frac{\partial^2 U}{\partial x^2} \quad (42)$$

and

$${}^c D_{0+,t}^\alpha U(x,t) = \rho^2 \frac{\partial^2 U}{\partial x^2}, \quad (43)$$

with the conditions stated in (30)–(31).

It is important to note that both models represent subdiffusive phenomena; however, neither of them can model superdiffusive phenomena, the reason being that both the fractional exponent associated with the Riemann–Liouville and with the Caputo fractional derivative stop behaving as a negative exponent when $\alpha > 1$, as in the ordinary case. So if we wish to propose a superdiffusive model, we shall have to find another generalization for the exponent and another fractional operator to play the role of the corresponding derivatives. If we consider the aforementioned properties of the Liouville generalized operator

$$(D_{-,g}^\alpha e^{-kg(t,\alpha)})(x) = ke^{-kg(x,\alpha)}, \quad (44)$$

with respect to the ordinary exponent $e^{-kg(t,\alpha)}$, for example whether $g(t,\alpha)$ is t^α or e^{t^α} , it is obvious that the following model results in subdiffusive behavior for $0 < \alpha < 1$, while for $\alpha > 1$ it results in superdiffusive behavior:

$$(-D_{-,g,t}^\alpha U)(x,t) = \rho^2 \frac{\partial^2 U}{\partial x^2}, \quad (45)$$

with the conditions stated in (30)–(31).

Note that the previous models are easily generalized to three dimensions if in the second term we substitute the second order spatial derivative with the corresponding Laplacian.

It is important to remark that here we have use, in some sense, a inverse method on the operator to explain this kind of time-fractional diffusion problems.

Lastly we point out that, although fractional differential equations have captured the attention of many authors, only a few steps have been taken toward devising a clear and coherent theory for these equations, analogous to the ordinary case, so as to justify the use of this tool for Fractional Modeling. There is still much work to be done.

4 Fractional Dynamics Systems

Here we must point out that there exist many example of applications of Fractional linear and non-linear Partial differential equations to model the dynamics of anomalous processes, but no too many mathematical studies and/or applications of fractional dynamics systems. That have a very simple justification, the main reason is that do not exist a nice connection between the fractional differential equation involving non-sequential fractional derivatives and the corresponding fractional system of fractional differential equations, join to the physical interpretation of the fractional derivatives is unknown.

Such difficult can be solved by using the sequential fractional derivatives, and we present here some examples on the behavior of such dynamics systems around the critical points:

4.1 Linear Case Using the Caputo Fractional Derivative

We considerate the following elementary system, for $0 < \alpha < 1$:

$$\begin{aligned} {}^C D_{0+}^{\alpha} x &= ax + by \\ {}^C D_{0+}^{\alpha} y &= cx + dy \end{aligned} \quad (46)$$

where a, b, c, d are real constants, with the condition $ad - bc \neq 0$, and ${}^C D_{0+}^{\alpha}$ is the Caputo fractional derivative. The explicit solutions of this system in terms of the Mittag-Leffler functions $E_{\alpha}(rt^{\alpha})$, which generalize the ordinary exponential, are well known. The general solution of (46) is (see the mentioned book by Kilbas–Srivastava–Trujillo (2006)):

$$\begin{aligned} x &= A_1 E_{\alpha}(r_1 t^{\alpha}) + A_2 E_{\alpha}(r_2 t^{\alpha}) \\ y &= B_1 E_{\alpha}(r_1 t^{\alpha}) + B_2 E_{\alpha}(r_2 t^{\alpha}) \end{aligned} \quad (47)$$

where A_1 , A_2 , B_1 and B_2 are real constants, and r_1 and r_2 are two different real solutions of the characteristic equation

$$r^2 - (a + d)r + (ad - bc) = 0 \quad (48)$$

and so on.

First we remark that here the only one critical point of the system is $(0, 0)$, but for another system with a critical point, for instance (x_0, y_0) it can be reduce such system to an analogous as the above one, using the following change:

$$x(t) = x_0 + u(t) \quad (49)$$

$$y(t) = y_0 + v(t). \quad (50)$$

It is obvious that phase plane of the system, in general is not the same to the fractional phase plane of such system, but in this case the critical point are the same to such of the corresponding ordinary system (the justification is very easy).

Therefore, we ask us about the phase plane of the system (46), and about what changes we can find between the trajectories of such system around the critical points comparing with the corresponding ordinary system, how we can applied them. See master Ph.D. work of Velasco, UCM, 2008.

4.2 Quasi-Linear Case Using the Caputo Fractional Derivative

Also, there are not problem to extend the known results of the ordinary case when the system is nonlinear, but quasi-Linear.

Let we consider a non-linear homogeneous system of fractional differential equations, with the point $(0, 0)$ as a isolated critical point:

$$\begin{aligned} {}^C D_{0+}^\alpha x &= F(x, y) \\ {}^C D_{0+}^\alpha y &= G(x, y) \end{aligned} \quad (51)$$

where in a neighborhood of $(0, 0)$, the functions F and G have the form

$$\begin{aligned} F(x, y) &= ax + by + F_1(x, y) \\ G(x, y) &= cx + dy + G_1(x, y) \end{aligned} \quad (52)$$

where a , b , c and d are real constants with $ad - bc \neq 0$ and F_1 , G_1 are continuous functions with continuous partial derivatives and $F_1(0, 0) = 0$, $G_1(0, 0) = 0$, and verify

$$\frac{F_1(x, y)}{r} \rightarrow 0 \quad \frac{G_1(x, y)}{r} \rightarrow 0 \quad \text{when} \quad r \rightarrow 0 \quad (53)$$

where $r = (x^2 + y^2)^{1/2}$. The it is easy to proof that the behavior of the trajectories of the non-linear system (51) is the same to the corresponding linear system (46).

4.3 Conclusion

The behavior of the trajectories around the critical point $(0, 0)$ of the system (46), is as follows:

- (a) When the two roots of the characteristic equation (48) are real, or complex, but with real part no equal to 0, then the trajectories around the critical point is very same that in the ordinary case, but its evolution to the critical point are so much slow that in the ordinary case, under the control of the parameter α .
Here we have considerate special case where the order $0 < \alpha < 1$, but as the generalized exponential $E_\alpha(\lambda t^\alpha)$, if $\lambda > 1$ have the same behavior of the ordinary exponential but it grow so much fast as α is greater than 1, it is obvious that we can obtain same results that in the above around the critical point of the system, but its evolution of the trajectories to the critical point will be so much fast that in the ordinary case, under the control of the parameter α .
- (b) When the roots of the characteristic Eq. 48 are complex, with the real part equal to 0, then the behavior of the trajectories around the critical point are very different that in the ordinary case: they are not Centers, but they are spiral with the special characteristic that they keep inside to the center corresponding to the associate ordinary system in the same critical point, tend to the critical point, and arrange from the mentioned center, if such point is stable. In another case the behavior of the trajectories around the critical point are spiral that keep outside to the center corresponding to the associate ordinary system in the same critical point.
- (c) Here we have considerate the case where the order of the fractional derivative in each equation of the system is $0 < \alpha < 1$, but there are not problem to generalized such case to systems where the order of the derivatives that involved in each equation of the system be different, for example, $0 < \alpha < 1$ and $\beta > 1$.
- (d) We also can considerate the more general n -dimensional case without any problem.

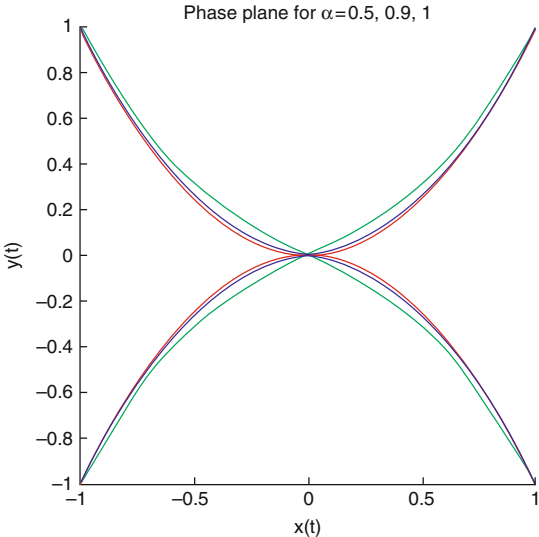
4.4 Examples

In the following graphics of the phase plane around the critical point we will use the following colors: green for $\alpha = 0.5$, blue for $\alpha = 0.9$, and red for $\alpha = 1$ (ordinary case).¹

Example 1. The phase plane of a system with real roots different, but with same sign:

$$\begin{aligned} {}^C D_{0+}^\alpha x &= -x \\ {}^C D_{0+}^\alpha y &= -2y \end{aligned}$$

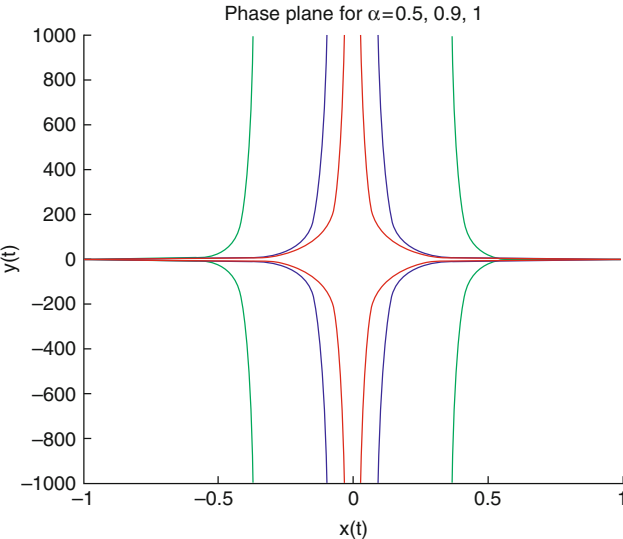
¹ For colors, see online version.



Here the evolution of the trajectories are very slowly in the fractional case.

Example 2. The phase plane of a system with real roots different, but with different sign:

$$\begin{aligned} {}^C D_{0+}^\alpha x &= -x \\ {}^C D_{0+}^\alpha y &= 2y \end{aligned}$$



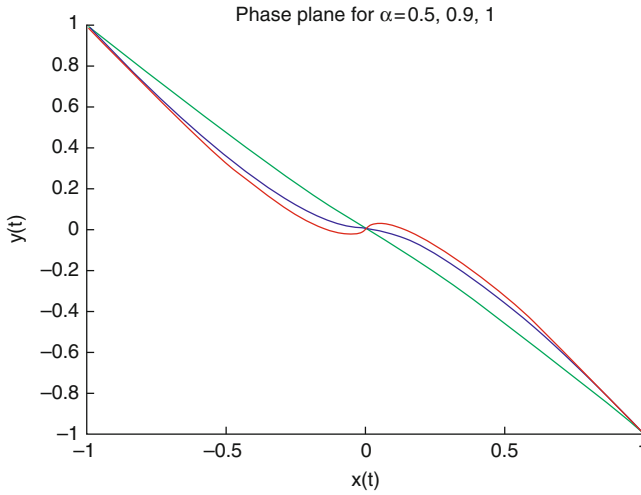
Here the evolution of the trajectories are very slowly in the fractional case, and tend to ∞ .

Example 3. The phase plane of a system with equal real roots:

$$\begin{aligned} {}^C D_{0+}^\alpha x &= -x \\ {}^C D_{0+}^\alpha y &= 2y \end{aligned}$$

which have the following general solution:

$$\begin{aligned} x &= A E_\alpha(-2t^\alpha) \\ y &= B E_\alpha(-2t^\alpha) + A \frac{t^\alpha}{\alpha} E_{\alpha,\alpha}(-2t^\alpha) \end{aligned}$$



Here the evolution of the trajectories are very slowly in the fractional case, and tend to $(0, 0)$.

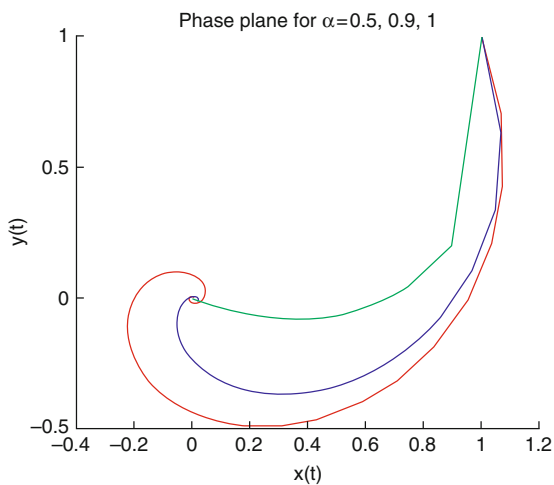
Example 4. The phase plane of a system with complex roots, with real part different to 0:

$$\begin{aligned} {}^C D_{0+}^\alpha x &= -x + 2y \\ {}^C D_{0+}^\alpha y &= -2x - y \end{aligned}$$

which have the following general solution:

$$\begin{aligned} x &= A \Re(E_\alpha((-1 + 2i)t^\alpha)) + B \Im(E_\alpha((-1 + 2i)t^\alpha)) \\ y &= B \Re(E_\alpha((-1 + 2i)t^\alpha)) - A \Im(E_\alpha((-1 + 2i)t^\alpha)) \end{aligned}$$

Here the evolution of the spiral trajectories are very slowly in the fractional cases, and tend to $(0, 0)$.

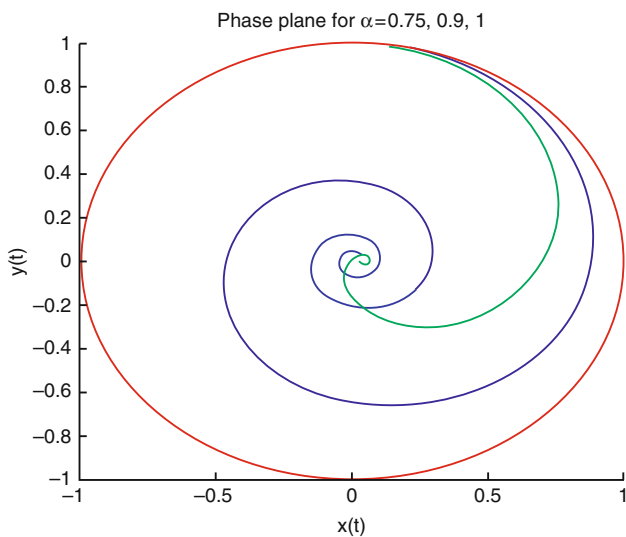


Example 5. The phase plane of a system with complex roots and real part equal to 0:

$$\begin{aligned} {}^C D_{0+}^\alpha x &= y \\ {}^C D_{0+}^\alpha y &= -x \end{aligned}$$

which have the following general solution:

$$\begin{aligned} x &= A \Im(E_\alpha(it^\alpha)) \\ y &= A \Re(E_\alpha(it^\alpha)) \end{aligned}$$



This is a interesting case. The evolution of the spiral trajectories are very slowly in the fractional cases, and tend to $(0, 0)$, arranging from the center corresponding to the ordinary case.

We must remark here that this study can be extending, without problem, in several ways, first with the number of equations a different variables, also by the use of different order of derivation $\alpha_1 < \alpha_2 < \dots < \alpha_n$ and/or $\alpha_n = r_n \alpha_1$, where r_n is a integer. With this last condition we establish that ratio of growing of each variable can be different. On the other hand, if $0 < \alpha_i < 1$ ($i = 1, 2, \dots, n$) we have the control of a more slowly moving of each one variable, but if any of the α_i is great than 1, that permit to us to control a faster evolution of the corresponding variable, so on.

4.5 Applications

Here we ask to us, if we can give natural deterministic hypothesis in some known problems so we could applied the above study of fractional systems?. The answer is, of course, YES.

We will illustrate this answer with some known examples on competition of populations:

The following example correspond to the classical model of **the competition of two populations with densities x and y**

$$\begin{aligned}\frac{dx}{dt} &= x(k_1 - \sigma_1 x - \gamma_1 y) \\ \frac{dy}{dt} &= y(k_2 - \sigma_2 y - \gamma_2 x)\end{aligned}\tag{54}$$

where $k_1, k_2, \sigma_1, \sigma_2, \alpha_1$ y α_2 real positive constants.

In this model we suppose two hypothesis:

1. Both populations $x(t)$ and $y(t)$ growing in a proportional way to $x(t)$ and $y(t)$, respectability, that is, in the classical notation $\frac{dx}{dt} = k_1 x(t)$ and $\frac{dy}{dt} = k_2 y$.
2. When one of the population grow too much, then the other decrees and stop the growing of the populations that have been growing quickly, and viceversa. This hypothesis justify the non-linear terms of the model (54).

The first hypothesis can be writing in the less usual form $x(t) = C_1 \exp k_1 t$ and $y(t) = C_2 \exp k_2 t$. Then, such hypothesis can be read in a very different form “the evolution of the populations in the time follow to an classical exponential”.

When one understand such hypothesis, it is obvious deduce that the growing of many populations can be as a exponential but no necessarily as the classical one. For instance the population $x(t)$ could growing as the following generalized exponential $C_1 E_\alpha k_1 t^\alpha$ and $y(t) = C_2 E_\beta k_2 t^\beta$, where C_1, C_2, k_1, k_2 are real constants, and $0 < \alpha, \beta \leq 1$.

Now, we could replace in the classical model (54), the first hypothesis, for instance, for the following $x(t) = C_1 E_\alpha(k_1 t^\alpha)$ and $y(t) = C_2 E_\alpha(k_2 t^\alpha)$, but we keep the second hypothesis. Therefore, the such hypothesis can be writing by the following $({}^C D_{0+}^\alpha x)(t) = k_1 x(t)$ and $({}^C D_{0+}^\alpha y)(t) = k_2 y(t)$, and we arrive to the following generalized fractional model:

$$\begin{aligned} {}^C D_{0+}^\alpha x &= x(k_1 - \sigma_1 x - \gamma_1 y) \\ {}^C D_{0+}^\alpha y &= y(k_2 - \sigma_2 y - \gamma_2 x) \end{aligned} \quad (55)$$

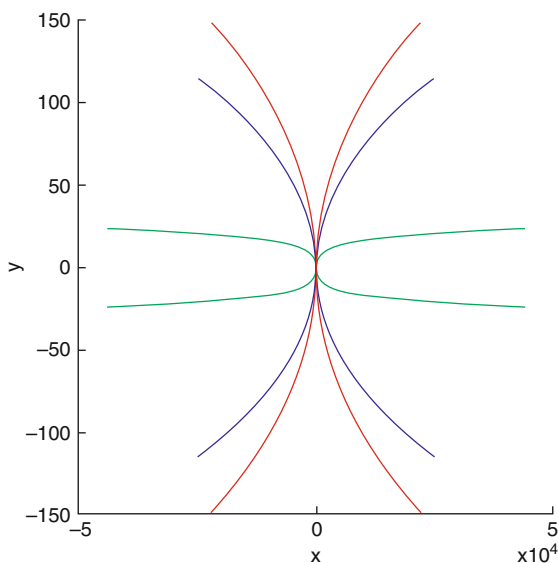
where k_1, k_2, σ_1 are real constants, $\sigma_2, \gamma_1, \gamma_2$ real positive constants, and $0 < \alpha \leq 1$.

Example 6. This is a particular example which is illusive of the mentioned deflexion. We will use it to applied the study of its critical points

$$\begin{aligned} {}^C D_{0+}^\alpha x &= x(1 - x - y) \\ {}^C D_{0+}^\alpha y &= y\left(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x\right) \end{aligned} \quad (56)$$

This quasi-linear system have four critical points:

1. $\mathbf{x}_0 = \mathbf{0}$ and $\mathbf{y}_0 = \mathbf{0}$



The associated linear system is the following:

$$\begin{aligned} {}^C D_{0+}^\alpha x &= x \\ {}^C D_{0+}^\alpha y &= \frac{1}{2}y \end{aligned}$$

which have the following general solution:

$$\begin{aligned}x &= AE_\alpha(t^\alpha) \\ y &= BE_\alpha\left(\frac{1}{2}t^\alpha\right)\end{aligned}$$

and the phase plane around the critical point is the following (*blue line for $\alpha = 0.9$, green line for $\alpha = 0.5$, and red line for $\alpha = 1$*).²

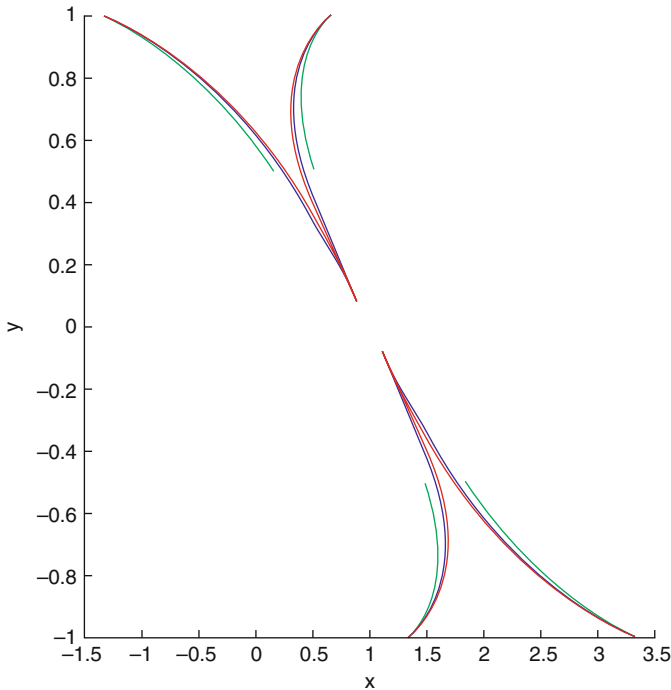
The trajectories go away slowly from the critical point $(0, 0)$, and it is a node.

2. $\mathbf{x}_0 = \mathbf{1}$ and $\mathbf{y}_0 = \mathbf{0}$

After to replace the variables $x = 1 + u$ and $y = 0 + v$ in (56), we have the following associated linear system and its phase plane around the critical point are the following

$$\begin{aligned}{}^CD_{0+}^\alpha u &= -u - v \\ {}^CD_{0+}^\alpha v &= -\frac{1}{4}v\end{aligned}$$

and the phase plane around the critical point is the following (*blue line for $\alpha = 0.9$, green line for $\alpha = 0.5$, and red line for $\alpha = 1$*).



Therefore the critical point $(1, 0)$ is a slowly stable node.

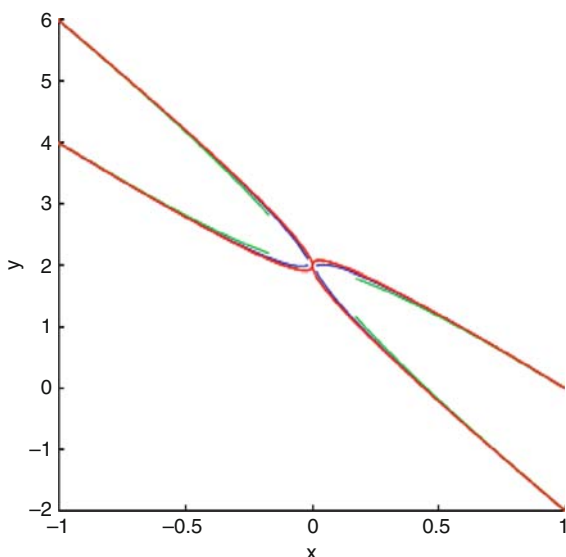
² For colors, see online version.

3. $\mathbf{x}_0 = \mathbf{0}$ and $\mathbf{y}_0 = 2$

After to replace the variables $x = 0 + u$ and $y = 2 + v$, we have the following associated linear system

$$\begin{aligned} {}^C D_{0+}^\alpha u &= -u \\ {}^C D_{0+}^\alpha v &= -\frac{1}{2}v - \frac{3}{2}u \end{aligned}$$

and the phase plane around the critical point is the following (blue line for $\alpha = 0.9$, green line for $\alpha = 0.5$, and red line for $\alpha = 1$).



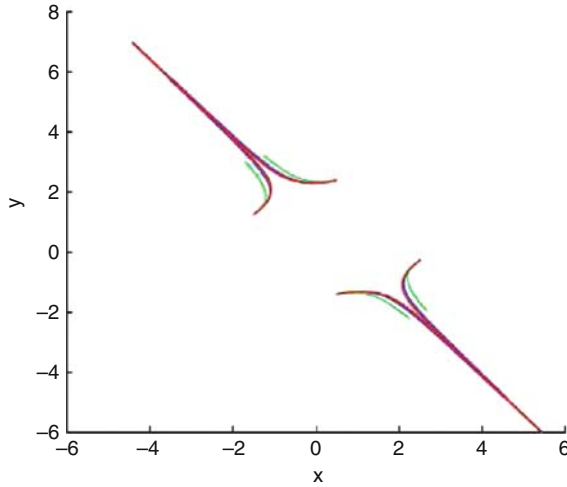
Therefore the critical point $(0, 2)$ is a fast improper stable node.

4. $\mathbf{x}_0 = \frac{1}{2}$ and $\mathbf{y}_0 = \frac{1}{2}$

After to replace the variables $x = \frac{1}{2} + u$ and $y = \frac{1}{2} + v$, we have the following associated linear system

$$\begin{aligned} {}^C D_{0+}^\alpha u &= -\frac{1}{2}u - \frac{1}{2}v \\ {}^C D_{0+}^\alpha v &= -\frac{3}{8}u - \frac{1}{8}v \end{aligned}$$

and the phase plane around the critical point is the following (blue line for $\alpha = 0.9$, green line for $\alpha = 0.5$, and red line for $\alpha = 1$).



Therefore the critical point $(\frac{1}{2}, \frac{1}{2})$ is a slowly instable chair.

Example 7. In this example we considerate a generalization of **the known model Predator–Prey of Lotka–Volterra**:

$$\begin{aligned}\frac{dx}{dt} &= x(k_1 - \gamma_1 y) \\ \frac{dy}{dt} &= y(-k_2 + \gamma_2 x)\end{aligned}\quad (57)$$

We will study the following particular case

$$\begin{aligned}{}^C D_{0+}^\alpha x &= x(1 - y) \\ {}^C D_{0+}^\alpha y &= y(-1 + x)\end{aligned}\quad (58)$$

This system have only two critical point:

I. $(0, 0)$ The associated linear system to (58) is the following:

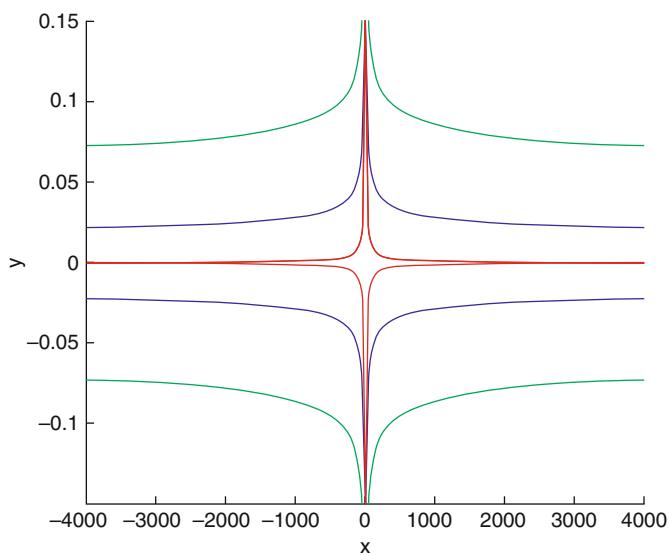
$$\begin{aligned}{}^C D_{0+}^\alpha x &= x \\ {}^C D_{0+}^\alpha y &= -y\end{aligned}$$

and the phase plane around the critical point is the following (blue line for $\alpha = 0.9$, green line for $\alpha = 0.5$, and red line for $\alpha = 1$).

Therefore the critical point $(0, 0)$ is a slowly instable chair point.

I. $(1, 1)$

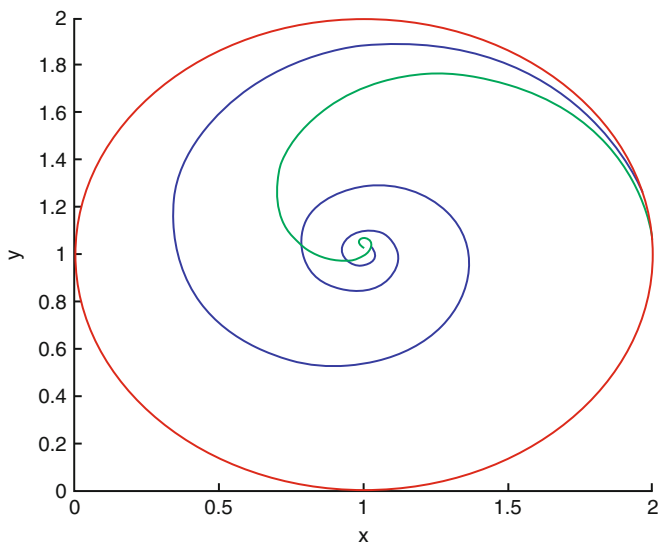
After to replace the variables $x = 1 + u$ $y = 1 + v$, we have the following associated linear system



The associated linear system to (58) is the following:

$$\begin{aligned} {}^C D_{0+}^{\alpha} u &= -v \\ {}^C D_{0+}^{\alpha} v &= u \end{aligned}$$

and the phase plane around the critical point is the following (blue line for $\alpha = 0.9$, green line for $\alpha = 0.5$, and red line for $\alpha = 1$).



Therefore the critical point $(1, 1)$ is a slowly stable spiral center point.

5 Spatial–Fractional Derivative: Anomalous Diffusion

With a generalization of the arguments used above one could generalize linear or non-linear ordinary partial differential model to fractional Model. Only we must considerate the no-locality or the memory involving in the dynamics of some anomalous processes. For instance, we will try to generalize the classical hypothesis that produce the one dimensional linear diffusive equation, same arguments can be applied to more general nonlinear models involving partial differential equations.

As it is well known the model in (29)–(31)

$$\frac{\partial U(x, t)}{\partial t} = \rho^2 \frac{\partial^2 U(x, t)}{\partial x^2}; \quad (\rho > 0; \quad t > 0; \quad 0 < x < l) \quad (59)$$

$$U(0, t) = U(l, t) = 0; \quad U(x, 0) = f(x). \quad (60)$$

Therefore, our objective is to change the classical hypothesis that support the above model to justify the corresponding time-fractional subdiffusive one. As it is well known such hypothesis are the following

5.1 Spatial–Fractional Diffusion Equation

First of all we remember the deterministic physics laws for the classical linear diffusion equation:

- (a) **The Fourier Law of Heat Conduction:** The heat flows through the area on the hottest to the coldest. The amount of heat that flows is directly proportional to the section that runs space. Mathematically, if we denote $q(x, t)$ to heat flow and $\kappa(x) > 0$ a thermal conductivity, we can write such law as follows:

$$q(x, t) = \kappa(x) \frac{\partial U}{\partial x} \quad (61)$$

- (b) **Law of Energy Conservation:** The rate of exchange of energy in a finite section of a bar is equal to the total amount of heat flowing in that section of the bar, that is:

$$\frac{\partial}{\partial t} \int_x^{x+\Delta x} c(z)U(z, t)dz = -q(x + \Delta x) + q(x, t) \quad (62)$$

where $c(x)$ is the heat conductivity of the material.

Remark 1. We will considerate here that the heat flows from left to right.

If it is assumed that $c(z)U(z, t)$ is a suitable regular function, then we can interchange the integral and the time-derivative in the expression (62). Now, we

can divide both side of the resulting expression by expression Δx . Then we take $\Delta x \rightarrow 0$ and apply the known theorem of the mean valuer. Therefore:

$$c(x) \frac{\partial U}{\partial t} = -\frac{\partial q}{\partial x} \quad (63)$$

If now we considerate (61), held that

$$c(x) \frac{\partial U}{\partial t} = \frac{\partial}{\partial x} \left(\kappa(x) \frac{\partial U}{\partial x} \right) \quad (0 < x < l) \quad (64)$$

A particular case is when the bar is homogenies, that is $c(x)$ and $\kappa(x)$ are constant. In this case we can obtain the well known expression of the following ordinary diffusion equation:

$$\frac{\partial U}{\partial t} = \gamma \frac{\partial^2 U}{\partial x^2} \quad (0 < x < l) \quad (65)$$

with $\gamma = \frac{\kappa}{c}$.

Here we point out that the hypothesis (61) can be writing in the following equivalent integral form, which is possible to see as a simple convolution:

$$U(x, t) = \int_x^l \frac{q(z, t)}{\kappa(z)} dz + \varphi(t) = 1 *_l \left(\frac{q(x, t)}{\kappa(x)} \right) + u_2(t) \quad (66)$$

where $U(l, t) = u_2(t)$.

With the objective to simplify the example, we will assume that $u_2(t) = 0$, $c(x) = c > 0$, and $\kappa(x) = \kappa > 0$. Therefore:

$$U(x, t) = \frac{1}{\kappa} \int_x^l q(z, t) dz = 1 *_l \frac{q(x, t)}{\kappa} \quad (0 < x < l) \quad (67)$$

Now we are in condition to generalized, in a natural way, the Law of Fourier introducing a memory factor in (67), Instead of replacing a suitable kernel in 1. For example we could choose $K(x) = x^{\alpha-1}$ $0 < \alpha \leq 1$. Then:

$$(x, t) = K(x) * \frac{q(x, t)}{\kappa} = \frac{1}{\kappa} \int_x^l (z-x)^{\alpha-1} q(z, t) dz = \frac{\Gamma(\alpha)}{\kappa} \left(I_{l-,x}^\alpha q \right) (x, t), \quad (68)$$

which is equivalent to

$$q(x, t) = \frac{\kappa}{\Gamma(\alpha)} \left(D_{l-,x}^\alpha U \right) (x, t) \quad (69)$$

Then if we combine (69) and (63), it held the following spatial–fractional diffusion equation:

$$\left(\frac{\partial U}{\partial t}\right)(x, t) = \gamma \left(D_{l-,x}^{\beta} U\right)(x, t) \quad (0 < x < l, \quad t > 0) \quad (70)$$

where $1 < \beta = \alpha + 1 \leq 2$, $\gamma = \frac{\kappa}{c \Gamma(\alpha)} > 0$ and with the initial condition $U(x, 0) = f(x)$.

Remark 2. We must remark that If we work with the Riemman–Liouville derivative D_{0+}^{α} , we can not use the boundary condition $U(0, t) = 0$. But this problem has a easy solution. It is enough to solve the problem to place the bar of the example o the x -axis from $t_0 > 0$ to $l + t_0$ instead from 0 to l .

Remark 3. It is obvious that there are not problem to follow a same way to deduce more general models as the spatial–fractional diffusion or advection, linear or no-linear, equations. It is also obvious that here we had not necessity to use stochastic arguments.

6 Fractional Differential Equations and Chaos

As we have mentioned above a sequential fractional derivative of order $n\alpha$ is given as following:

$$\begin{cases} D^{\alpha} = D^{\alpha}, & (0 < \alpha \leq 1) \\ D^{k\alpha} = D^{\alpha} D^{(k-1)\alpha}, & (k = 2, 3, \dots) \end{cases} \quad (71)$$

where D^{α} is a fractional derivative. We note here the following property which will be of great use in some applications, in particular to illustrate the fractional Chaos:

$$(D^n y)(t) = (\mathcal{D}_{a+}^{p\alpha} y)(t) \quad (\forall n, p \in \mathbb{N}; \alpha = \frac{n}{p}; p > n; t > a). \quad (72)$$

This is an immediate remembering that, if $y \in C([a, b])$, then

$$(I_{a+}^{\beta} y)(a+) = 0 \quad (\forall \beta \in \mathbb{R}^+). \quad (73)$$

Therefore, we can concluded that any ordinary differential equation can be expressed as a sequential fractional differential equation, and of course, as a system of fractional differential equations of order $\alpha = \frac{1}{p}$.

As an example, we present the following simple ordinary differential equation

$$x'(t) - \lambda^2 x(t) = 0 \quad (\lambda > 0), \quad (74)$$

which has the general solution $x(t) = C e^{\lambda t}$, with one freedom degree.

On the other hand, as $x'(t) = (D_{0+}^{1/2} D_{0+}^{1/2} x)(t)$ for $t > 0$, such equation can be expressed as the following:

$$(\mathcal{D}_{0+}^{2\alpha} x)(t) - a^2 x(t) = 0 \quad (\alpha = 1/2). \quad (75)$$

By using the corresponding existence theorem, its general solution is given by

$$x(t) = C_1 e_a^{at} + C_2 e_a^{-at}, \quad (76)$$

with two freedom degree. Now we can check that both solutions are the same:

Since $x(0) < \infty$, we have $C_2 = -C_1$. Therefore,

$$x(t) = C_1 \sum_{j=1}^{\infty} \frac{[1 - (-1)^j] a^j t^{j\alpha + \alpha - 1}}{\Gamma[(j+1)\alpha]}; \quad (77)$$

Since $[1 - (-1)^j] = 0$ for $j = 2k$ ($k \in \mathbb{N}$), the solution (77) takes the form

$$x(t) = 2C_1 a \sum_{j=1}^{\infty} \frac{a^{2j} t^j}{j!} = C e^{a^2 t}, \quad (78)$$

which is the general solution to (74), as expected.

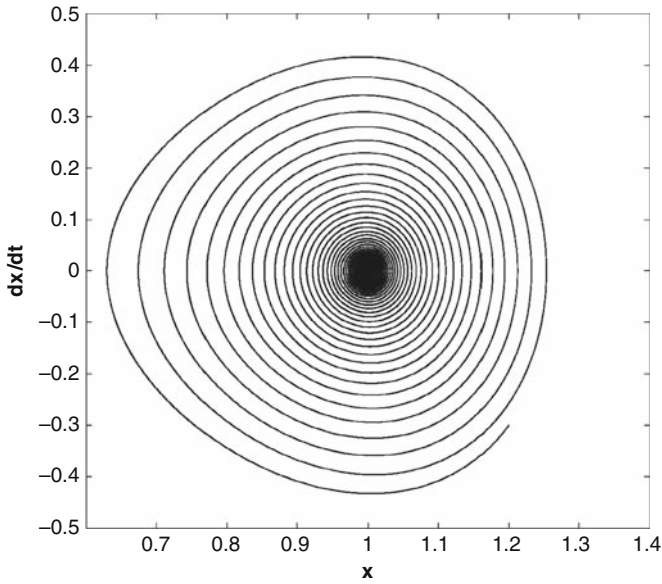
It is known the importance to find chaotic attractors associated to differential models. From the above reflection we can conclude that all the chaotic behavior connected with ordinary differential equations can be considerate as particular cases of the fractional chaotic behavior connected with fractional differential equations. Then the conclusion, is obvious in this more general case we have, at less one new freedom degree (the order of derivative) so is clear that we can get so many more interesting chaotic cases from fractional models.

Here we present a simple example taking from the interesting series of papers published during last years by Zaslavsky et al. see for instance [13]:

A trajectories in the Phase plane of the fractional oscillator equation

$$({}^C D_{0+}^{\alpha} X)(t) - X(t) + X^3(t) = 0$$

is the following:

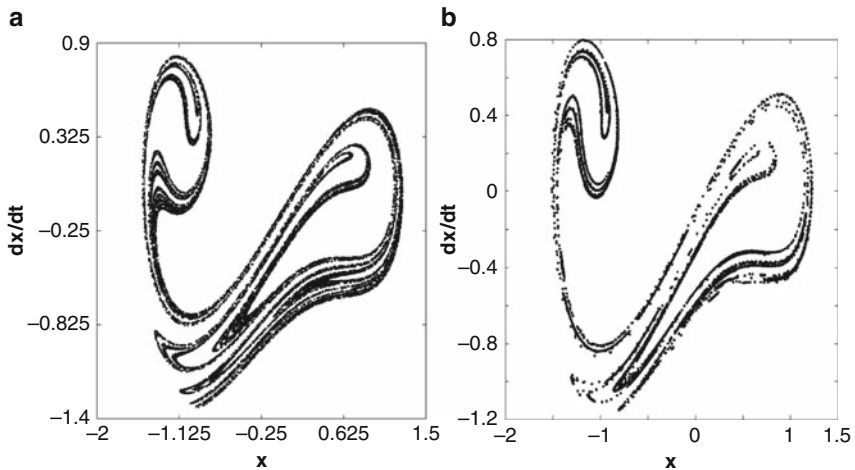


A chaotic attractor in the phase plane of the following ordinary with dissipative term and fractional oscillator equations dissipative term without, and with a external oscillatory force:

(a) $X''(t) + 0.1172X'(t) - X(t) + X^3(t) = 0.3 \sin(t)$

(b) $(^c D_{0+}^\alpha X)(t) - X(t) + X^3(t) = 0.3 \sin(t)$

with $\alpha = 1.9$ and $t_{max} = 16\pi$ (2,000 trajectories):



Acknowledgements The authors express their gratitude to MININN of Spain Government (MTM2004-00327), and to the Scholarship FPU of M. Velasco, call order 19843/2007 of 25 of October.

References

1. Bonilla B, Kilbas AA, Trujillo JJ (2003) *Cálculo Fraccionario y Ecuaciones Diferenciales Fraccionarias*. Madrid, Uned
2. Gnedenko BV, Kolmogorov AN (1949; 1954) *Limit distributions for sums of independent random variables*. Addison-Wesley, Massachusetts
3. Kilbas AA, Srivastava HM, Trujillo JJ (2006) *Theory and applications of fractional differential equations*. Elsevier, Amsterdam
4. Kilbas AA, Trujillo JJ (2001) Differential equations of fractional order: methods, results and problems I. *Appl Anal* 78(1–2):157–192
5. Kilbas AA, Trujillo JJ (2002) Differential equations of fractional order: methods, results and problems II. *Appl Anal* 81(2):435–493
6. McBride AC (1979) *Fractional calculus and integral transforms of generalized functions*. Ed. Pitman, London, Adv Publ Program
7. Metzler R, Klafter J (2000) The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys Rep* 339(1):1–77
8. Montroll EW, Weiss GH (1965) Random walk on lattices II. *J Math Phys* 6:167–181
9. O'Shaughnessy L (1918) Problem # 433. *Amer Math Month* 25:172–173
10. Post EL (1919) Discussion of the solution of $D^{1/2}y=y/x$, problem # 433. *Amer Math Month* 26:37–39
11. Samko SG, Kilbas AA, Marichev OI (1993) *Fractional integrals and derivatives: theory and applications*. Gordon and Breach Science Publishers, Switzerland
12. Sneddon, I. N. (1966) *Mixed boundary value problems in potential theory*. Amsterdam, North-Holland Publ.
13. Zaslavsky GM, Stanislavsky AA, Edelman M (2006) Chaotic and pseudochaotic attractors of perturbed fractional oscillator. *Chaos* 16:013102

A New Approach for Stability Analysis of Linear Discrete-Time Fractional-Order Systems

Said Guermah, Said Djennoune, and Maamar Bettayeb

Abstract In this paper, an approach using a new formalism is proposed to analyze the stability of linear discrete-time fractional-order systems. Asymptotic stability of such systems is examined. Practical asymptotic stability is introduced and illustrated by a numerical example.

1 Introduction

In the few past years, modern control theory has received a tremendous amount of contributions in the field of fractional-order systems, based on the concept of non-integer derivatives [2, 26, 30, 33]. One feature of importance is that they exhibit hereditary properties and long memory transients. This concept of non-integer derivative and integral is increasingly used to model the behavior of real systems in various fields of science and engineering [6, 19, 20]. Some fundamental developments of the fractional calculus theory are given in [17, 25, 27, 31].

The state-space representation, in continuous time, has been exploited in the analysis of system performances [7, 16, 21, 29]. In fact, the solution of the state-space equation has been derived by using the Mittag–Leffler function [24]. The stability of the fractional-order system has been investigated [22]. A condition based on the argument principle has been established to guarantee the asymptotic stability of the fractional-order system. Further, the controllability and the observability properties have been defined and some algebraic criteria of these two properties have been derived [1, 3, 23].

S. Guermah (✉) and S. Djennoune

Laboratoire de Conception et Conduite des Systèmes de Production (L2CSP), Université Mouloud Mammeri de Tizi-Ouzou – B.P. 17 RP, 15000 Tizi-Ouzou, Algeria
e-mail: saidguermah@yahoo.fr; s.djennoune@yahoo.fr

M. Bettayeb

Electrical & Computer Engineering Department, University of Sharjah, P.O. Box 27272, Sharjah, United Arab Emirates
e-mail: maamar@sharjah.ac.ae

Linear discrete-time fractional-order systems modeled by a state space representation have been introduced in [9–11, 32]. These contributions are devoted respectively to a stability condition, to the design of an observer, Kalman filter design and finally to an adaptive feedback control for discrete fractional-order systems. In [13], some new results concerning the controllability and observability properties of linear discrete-time fractional-order systems have been derived. The concept of Controllability Realization Index (*CRI*) already introduced for linear discrete-time systems with time-delay in state [28] is extended to the fractional order system with, in addition, the concept of Observability Realization Index *ORI* [4]. A state space analysis of linear fractional order systems is exposed in [14], in which is given a procedure for obtaining a continuous-time state-space fractional-order model of a system initially modeled by a continuous-time transfer function. Besides, this latter work gives for the derived discrete-time fractional-order state-space model an analysis of the controllability and the observability properties, as well as its input–output behaviour.

The objective of the present work is to give a further investigation in the analysis of fractional-order discrete-time linear systems. New results concerning asymptotic stability are developed. Besides, we introduce the concept of practical stability, then some mathematical conditions to check this property are established. This paper is organized as follows: In Sect. 2, we expose the fractional-order discrete-time model as defined in [9] and we introduce extra notations that reveal a new form, making it possible to take into account the past behavior of the system and to analyze the structural properties. Section 3 addresses the asymptotic stability and practical stability properties. A numerical example is treated in Sect. 4 to illustrate the theoretical results [15].

2 Linear Discrete-Time Fractional-Order Systems

There are different definitions of the fractional derivative, [17, 25, 31]. The Grünwald-Letnikov definition, which is the discrete approximation of the fractional order derivative, is used here. The Grünwald–Letnikov fractional order derivative of a given function $f(t)$ is given by

$${}_a^G D_t^\alpha f(t) = \lim_{h \rightarrow 0} \frac{{}_a \Delta_h^\alpha f(t)}{h^\alpha} \quad (1)$$

where the real number α denotes the order of the derivative, a is the initial time and h is a sampling time. The difference operator Δ is given by:

$${}_a \Delta_h^\alpha f(t) = \sum_{j=0}^{\lceil \frac{t-a}{h} \rceil} (-1)^j \binom{\alpha}{j} f(t - jh) \quad (2)$$

The binomial term can be obtained by the relation:

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0 \\ \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & \text{for } j > 0 \end{cases} \quad (3)$$

and $\lfloor \cdot \rfloor$ takes the integer part.

Now, let us consider the traditional discrete-time state-space model of integer order, i. e. when α is equal to unity:

$$x(k+1) = Ax(k) + Bu(k); x(0) = x_0 \quad (4a)$$

$$y(k) = Cx(k) + Du(k) \quad (4b)$$

Where $u(k) \in \mathbf{R}^p$ and $y(k) \in \mathbf{R}^q$ are respectively the input and the output vectors, $x(k) = [x_1(k) \ x_2(k) \ \dots \ x_n(k)] \in \mathbf{R}^n$ is the state vector. Its initial value is denoted $x_0 = x(0)$ and can be set equal to zero without loss of generality. (A, B, C, D) are the conventional state space matrices with appropriate dimensions.

The first-order difference for $x(k+1)$ can be defined as

$$\Delta^1 x(k+1) = x(k+1) - x(k)$$

Therefore, using Eq. 4a we deduce that:

$$\begin{aligned} \Delta^1 x(k+1) &= Ax(k) + Bu(k) - x(k) \\ &= A_d x(k) + Bu(k) \end{aligned}$$

where $A_d = A - I_n$ and I_n is the n -dimensional identity matrix.

The generalization of this integer-order difference to a non integer-order (or fractional-order) difference has been addressed in [9] where the discrete fractional-order difference operator with the initial time taken equal to zero is defined as follows:

$$\Delta^\alpha x(k) = \frac{1}{h^\alpha} \sum_{j=0}^k (-1)^j \binom{\alpha}{j} x(k-j) \quad (5)$$

In the sequel, the sampling time h is taken equal to 1. These results conducted to conceive the linear discrete-time fractional-order state-space model, using the equations:

$$\Delta^\alpha x(k+1) = A_d x(k) + Bu(k); x(0) = x_0 \quad (6)$$

In this model, the differentiation order α is taken the same for all the state variables $x_i(k)$, $i = 1, \dots, n$. This is referred to as commensurate order. Besides, from Eqs. 5 and 6 we have:

$$x(k+1) = A_d x(k) - \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x(k-j+1) + Bu(k) \quad (7)$$

The discrete-time fractional order system is represented by the following state space model:

$$x(k+1) = \sum_{j=0}^k A_j x(k-j) + Bu(k); x(0) = x_0 \quad (8a)$$

$$y(k) = Cx(k) + Du(k) \quad (8b)$$

where $A_0 = A_d - c_1 I_n$ and $A_j = -c_{j+1} I_n$ for $j \geq 2$ with $c_j = (-1)^j \binom{\alpha}{j}$, $j = 1, 2, 3, \dots$. This description can be extended to the case of non-commensurate fractional-order systems modeled in [9, 10] by introducing the following vector difference operator:

$$\Delta^{\mathcal{R}} x(k+1) = A_d x(k) + Bu(k)$$

$$x(k+1) = \Delta^{\mathcal{R}} x(k+1) + \sum_{j=1}^{k+1} A_j x(k-j+1)$$

where:

$$\Delta^{\mathcal{R}} x(k+1) = \begin{bmatrix} \Delta^{\alpha_1} x_1(k+1) \\ \vdots \\ \Delta^{\alpha_n} x_n(k+1) \end{bmatrix}$$

Then, in the case of non commensurate-order, the systems is described by Eqs. 8a and 8b where the matrices A_j , $j = 0, 1, 2, \dots$ take the following expressions:

$$A_0 = A_d - \text{diag}\left\{-\binom{\alpha_i}{1}, i = 1, \dots, n\right\}$$

and

$$A_j = \text{diag}\left\{-(-1)^{j+1} \binom{\alpha_i}{j+1}, i = 1, \dots, n\right\}$$

for $j = 1, 2, 3, \dots$

The model described by (8) can be classified as a discrete-time system with time delay in state. Whereas the models addressed in [5, 28] consider a finite constant number of steps of time-delays, System (8) has a varying number of steps of time-delays, equal to k , i.e. increasing along with time. Let us define matrices G_k such that:

$$G_k = \begin{cases} I_n & \text{for } k = 0; \\ \sum_{j=0}^{k-1} A_j G_{k-1-j} & \text{for } k \geq 1 \end{cases} \quad (9)$$

Theorem 1 ([13]). *The solution of Equation (8a) is given by*

$$x(k) = G_k x_0 + \sum_{j=0}^{k-1} G_{k-1-j} B u(j) \quad (10)$$

We deduce that the corresponding transition matrix can be defined as

$$\Phi(k, j) = G_{k-j}, \quad \Phi(0, 0) = G_0 = I_n$$

Obviously, this transition matrix does not enjoy the semi group property as for the integer order case. In fact:

$$\Phi(k_2, k_0) \neq \Phi(k_2, k_1) \Phi(k_1, k_0); \quad \forall k_2 > k_1 > k_0 \geq 0$$

3 Asymptotic Stability

In this section we study the asymptotic stability of the following unforced system:

$$x(k+1) = \sum_{j=0}^k A_j x(k-j), \quad x(0) = x_0 \quad (11)$$

Definition 1. System (11) is asymptotically stable if for each $k \geq 1$ and any initial condition x_0 , the following equality is verified:

$$\lim_{k \rightarrow \infty} \|x(k)\| = 0 \quad (12)$$

We use the 2-norm of the state vector $x(k)$:

$$\|x(k)\| = \sqrt{\sum_{i=1}^n x_i^2(k)}$$

where $x_i(k)$ are the components of $x(k)$. The solution of Eq. 9 for given initial conditions x_0 is

$$\|x(k)\| = G_k x_0; \quad k \geq 1; \quad G_0 = I_n \quad (13)$$

It follows that System (11) is asymptotically stable if and only if

$$\|G_k\| \leq 1; \quad k \geq 1 \quad (14)$$

The 2-norm of the transition matrix is

$$\|G_k\| = \lambda_{\max}^{\frac{1}{2}}(G_k G_k^T) = \sigma_{\max}(G_k)$$

where $\lambda_{\max}$ and $\sigma_{\max}$ refer to respectively the maximal eigenvalue and the maximal singular value of G_k . Let us define the backward shift operator $\mathcal{S}$ as follows [8, 18]: we consider an infinite sequence of samples of vector x , starting from $k = 0$ to infinity, with null values for $k < 0$, assuming the system to be causal.

$$\mathbf{x} = \{\dots, 0, 0, x(0), x(1), x(2), \dots, x(N), \dots\}$$

$\mathcal{S}$ acts on $\mathbf{x}$ as follows:

$$\begin{aligned} \mathcal{S}\mathbf{x} &= \mathcal{S}\{\dots, 0, 0, x(0), x(1), \boxed{x(2)}, \dots, x(k), x(k+1), \dots\} \\ &= \{\dots, 0, 0, 0, x(0), \boxed{x(1)}, x(2), \dots, x(k-1), \dots\} \end{aligned}$$

We then define the column sequence

$$\tilde{\mathbf{x}} = \begin{bmatrix} \vdots \\ 0 \\ x(0) \\ x(1) \\ \vdots \\ x(k) \\ \vdots \end{bmatrix}$$

Similarly, we can use this representation to rewrite Eqs. 8a and 8b in the equivalent form:

$$\tilde{\mathbf{x}} = \tilde{\mathbf{S}}\tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\tilde{\mathbf{u}} \quad (15a)$$

$$\tilde{\mathbf{y}} = \tilde{\mathbf{C}}\tilde{\mathbf{x}} + \tilde{\mathbf{D}}\tilde{\mathbf{u}} \quad (15b)$$

In this representation the expressions of the different components are

$$\tilde{\mathbf{u}} = \begin{bmatrix} \vdots \\ 0 \\ u(0) \\ u(1) \\ \vdots \\ u(k) \\ \vdots \end{bmatrix}, \quad \tilde{\mathbf{y}} = \begin{bmatrix} \vdots \\ 0 \\ y(0) \\ y(1) \\ \vdots \\ y(k) \\ \vdots \end{bmatrix}, \quad \tilde{\mathbf{S}} = \begin{bmatrix} 0 & 0 & 0 & \dots\dots\dots \\ I_n & 0 & \dots\dots\dots \\ 0 & I_n & \dots\dots\dots \\ 0 & 0 & I_n & \dots\dots\dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

and

$$\tilde{\mathbf{A}} = \begin{bmatrix} A_0 & 0 & 0 & 0 & \dots \\ A_1 & A_0 & 0 & 0 & \dots \\ A_2 & A_1 & A_0 & 0 & \dots \\ A_3 & A_2 & A_1 & A_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} B & 0 & 0 & 0 & \dots \\ 0 & B & 0 & 0 & \dots \\ 0 & 0 & B & 0 & \dots \\ 0 & 0 & 0 & B & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$\tilde{\mathbf{C}} = \begin{bmatrix} C & 0 & 0 & 0 & \dots \\ 0 & C & 0 & 0 & \dots \\ 0 & 0 & C & 0 & \dots \\ 0 & 0 & 0 & C & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}; \quad \tilde{\mathbf{D}} = \begin{bmatrix} D & 0 & 0 & 0 & \dots \\ 0 & D & 0 & 0 & \dots \\ 0 & 0 & D & 0 & \dots \\ 0 & 0 & 0 & D & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Theorem 2. Let us put $\mathcal{A}_s = \tilde{\mathbf{S}}\tilde{\mathbf{A}}$, then the system

$$\tilde{\mathbf{x}} = \tilde{\mathbf{S}}\tilde{\mathbf{A}}\tilde{\mathbf{x}} = \mathcal{A}_s\tilde{\mathbf{x}} \quad (16)$$

is asymptotically stable if and only if $\rho(\mathcal{A}_s) \leq 1$, where ρ is the spectral radius of the operator $\mathcal{A}_s$, defined as

$$\rho(\mathcal{A}_s) = \lim_{i \rightarrow \infty} \|\mathcal{A}_s^i\|_*^{\frac{1}{i}} \quad (17)$$

with the norm definition:

$$\|\mathcal{A}_s^i\|_* = \sup_{[I,J]} \|\mathcal{A}_{s[I,J]}^i\|$$

and in which $\mathcal{A}_{s[I,J]}^i$ denotes the $[I, J]th$ block matrix of $\mathcal{A}_s^i$.

The proof of this theorem is as follows:

Proof. 1. Necessity

We can verify that

$$\mathcal{A}_s^i = \begin{bmatrix} G_1 & 0 & 0 & \dots & 0 \\ G_2 & 0 & 0 & \dots & 0 \\ G_3 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ G_i & 0 & 0 & \dots & 0 \\ \times & A_0^{i+1} & 0 & \dots & 0 \\ \times & \times & A_0^{i+2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \times & \times & \times & \times & \times & \times \end{bmatrix} \quad (18)$$

In this matrix the symbol $\times$ stands for non-null quantities depending on the A_j . It follows then that:

$$\lim_{i \rightarrow \infty} \|\mathcal{A}_s^i\|_* = \sup_{k \geq 1} \|G_k\|$$

The assumption that System (16) is asymptotically stable is equivalent to:

$$\|G_k\| \leq 1, \quad k \geq 1$$

Therefore we have:

$$\rho(\mathcal{A}_s) = \lim_{i \rightarrow \infty} \|\mathcal{A}_s^i\|_*^{\frac{1}{i}} = \lim_{i \rightarrow \infty} (\sup_{k \geq 1} \|G_k\|)^{\frac{1}{i}} \leq (1)^{\frac{1}{i}} \leq 1$$

2. Sufficiency

Let $\rho(\mathcal{A}_s) \leq 1$. Then : $\rho(\mathcal{A}_s) = \lim_{i \rightarrow \infty} \|\mathcal{A}_s^i\|_*^{\frac{1}{i}} \leq 1$

Since

$$\lim_{i \rightarrow \infty} \|\mathcal{A}_s^i\|_*^{\frac{1}{i}} = \lim_{i \rightarrow \infty} (\sup_{k \geq 1} \|G_k\|)^{\frac{1}{i}}$$

Then

$$\sup_{k \geq 1} \|G_k\| \leq 1$$

It follows that $\|G_k\| \leq 1$, for all $k \geq 1$, i.e., System (16) is asymptotically stable. $\square$

In practice, a finite time observation of the system is desirable. We think it is worth considering the concept of practical stability [10, 12]. We define it as follows:

Definition 2. System (11) is practically stable in a finite time horizon $L > 0$ if for each $1 \leq k \leq L$ and any initial condition x_0 , the following inequality is verified:

$$\|x(k)\| \leq M \|x_0\| \quad (19)$$

where M is a strictly positive finite given number.

From Solution (13), System (11) is practically stable if and only if $\|G_k\| \leq M$ for $1 \leq k \leq L$.

Using (18), we can write

$$\mathcal{A}_s^L = \begin{bmatrix} G_1 & 0 & 0 & \dots & 0 \\ G_2 & 0 & 0 & \dots & 0 \\ G_3 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ G_L & 0 & 0 & \dots & 0 \\ \times A_0^{L+1} & 0 & \dots & 0 \\ \times \times A_0^{L+2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \times & \times & \times & \times & \times \end{bmatrix}$$

Define the block matrix:

$$\mathcal{A}_s^L = \begin{bmatrix} G_1 & 0 & 0 & \dots & 0 \\ G_2 & 0 & 0 & \dots & 0 \\ G_3 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_L & 0 & 0 & \dots & 0 \end{bmatrix}$$

From the above, we can state that System (11) is practically stable if and only if $\|\mathcal{A}_s^L\|_* \leq M$

4 Numerical Example

Let us consider the following unforced discrete-time fractional-order linear system characterized by:

$$A_0 = \begin{bmatrix} -0.9 & 0 \\ 0 & -0.6 \end{bmatrix}; \quad \alpha_1 = 0.2; \quad \alpha_2 = 0.7$$

We have fixed horizon $L = 25$; the initial conditions for the two state variables x_1 and x_2 are taken equal to -1 and $+1$, respectively. We compute the norm $\|\mathcal{A}_s^{25}\|_*$ and we find that its computed value is equal to 0.7 . Then the bound M can be taken equal to 0.7 . The simulations yield the trajectories of the two state variables shown on Figs. 1 and 2. We observe that both state variables verify the practical stability condition, i.e.,

$$\|x(k)\| \leq 0.7 \|x_0\|$$

along the whole horizon L .

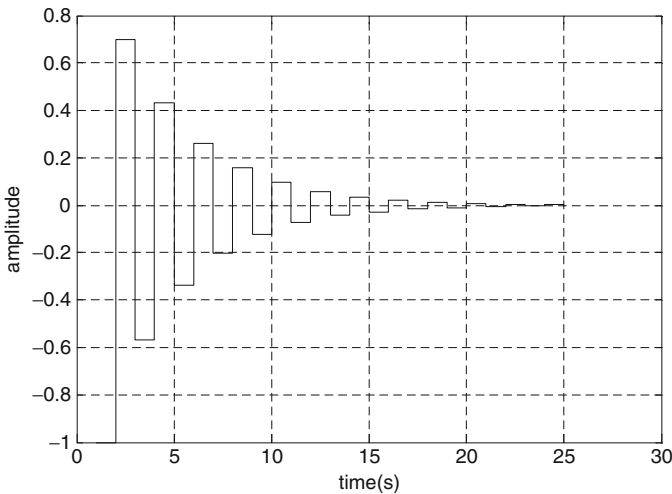


Fig. 1 Trajectory of x_1

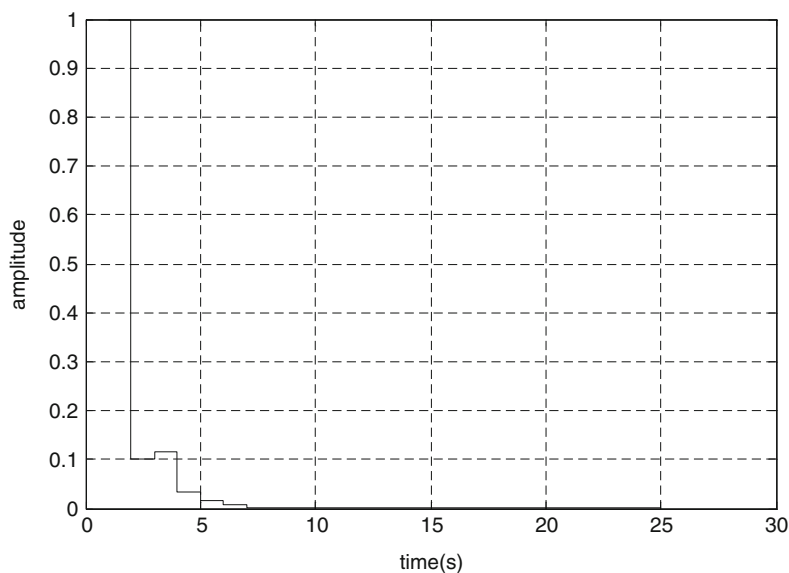


Fig. 2 Trajectory of x_2

5 Conclusion

In this paper some new results concerning the analysis of asymptotic stability and practical stability of discrete-time fractional-order systems have been established. For this purpose, a new description of such systems has been introduced. The preliminary results developed here can be useful for further investigation on stabilization and practical stabilization.

References

1. Antsaklis PJ, Michel AN (1997) Linear Systems. McGraw-Hill, New-York
2. Axtell M, Bise EM (1990) Fractional calculus applications in control systems. In: Proceedings of the IEE 1990 national aerospace and electronics conference, New York, pp 536–566
3. Bettayeb M, Djennoune S (2006) A note on the controllability and observability of fractional dynamical systems. In: Proceedings of the 2nd IFAC workshop on fractional differentiation and its applications, Porto
4. Bettayeb M, Djennoune S, Guermah S (2008) Structural properties of linear discrete-time fractional-order systems. 17th IFAC world congress, Seoul, South Korea, 6–11 July
5. Debeljkovic D Lj, Aleksendric M, Yi-Yong N, Zhang QL (2002) Lyapunov and non Lyapunov stability of linear discrete time delay Systems. Facta Universitatis Series: Mechanical Engineering, pp 1147–1160
6. Debnath L (2003) Recent applications of fractional calculus to science and engineering. IJMMS, Hindawi Publishing 54:3413–3442

7. Dorčák L, Petras I, Kostial I (2000) Modeling and analysis of fractional-order regulated systems in the state space. In: Proceedings of ICCCT'2000. High Tatras, Slovak Republic, pp 185–188
8. Dullerud G, Lall S (1999) A new approach for analysis and synthesis of time-varying systems. *IEEE Trans Automat Contr* 44:1486–1497
9. Dzieliński A, Sierociuk D (2005) Adaptive feedback control of fractional order discrete-time state-space systems. In: Proceedings of the 2005 international conference on computational intelligence for modelling, control and automation, and international conference on intelligent agents, Web Technologies and Internet Commerce, CIMCA-IAWTIC'05
10. Dzieliński A, Sierociuk D (2006) Stability of discrete fractional order state-space systems. In: Proceedings of the 2nd IFAC workshop on fractional differentiation and its applications, Porto
11. Dzieliński A, Sierociuk D (2006) Observers for discrete fractional order systems. In: Proceedings of the 2nd IFAC workshop on fractional differentiation and its applications, Porto
12. Garcia G, Messaoud H, Maraoui S (2005) Practical stabilization of linear time-varying systems. *Sixième Conférence Internationale des Sciences et des Techniques de l'Automatique-STA*, Sousse, Tunisia, 19–21 December
13. Guermah S, Djennoune S, Bettayeb M (2008) Controllability and observability of linear discrete-time fractional-order systems. *Int J App Math Comp Sci (AMCS)* 18(2):213–222
14. Guermah S, Djennoune S, Bettayeb M (2008) State space analysis of linear fractional order systems. *Journal Européen des Systèmes Automatisés* 42(6-7-8):825–838
15. Guermah S, Djennoune S, Bettayeb M (2008) Asymptotic stability and practical stability of linear discrete-time fractional-order systems. 3rd IFAC workshop on fractional differentiation and its applications, Ankara, Turkey, 05–07 November
16. Hotzel R, Fliess M (1998) On linear systems with fractional derivation: introduction theory and examples. *Math Comput Simulat* 4:385–395
17. Kilbas AA, Srivastava HM, Trujillo JJ (2006) Theory and application of fractional differential equations. In: Jan van Mill (ed) *North Holland Mathematics Studies*. Elsevier, UK
18. Lall S, Beck C (2003) Error-bounds for balanced model-reduction of linear time-varying systems. *IEEE Trans Automat Contr* 48(6):946–956
19. Magin RL (2006) *Fractional calculus in bioengineering*. Begell House Publishers, Connecticut
20. Manabe S (1961) The non-integer integral and its applications to control systems. *ETJ Japan* 6:83–87
21. Matignon D (1994) Représentation en variables d'état de modèles de guides d'ondes avec dérivation fractionnaire. Ph.D. Thesis. Université Paris-Sud, Orsay, France
22. Matignon D (1996) Stability results on fractional differential equations with applications to control processing. In: *IAMCS, IEEE SMC proceedings conference*. Lille, France
23. Matignon D, d'Andréa-Novel B (1996) Some results on controllability and observability of finite-dimensional fractional differential systems. In: *IAMCS, IEEE SMC proceedings conference*, Lille, France, pp 952–956
24. Mittag-Leffler G (1904) Sur la Représentation Analytique d'une Branche Uniforme d'une Fonction Monogène. *Acta Mathematica* 19:101–181
25. Oldham KB, Spanier J (1974) *The Fractional Calculus*. Academic, New York
26. Oustaloup A (1983) *Systèmes Asservis Linéaires d'Ordre Fractionnaire*. Masson, Paris
27. Oustaloup A (1995) *La Dérivation Non Entière: Théorie, Synthèse et Applications*. Hermes Editions, Paris
28. Pen Y, Guangming X, Long W (2003) Controllability of linear discrete-time systems with time-delay in state. <http://dean.pku.edu.cn/blsky/1999tzlwj/4.pdf>
29. Raynaud HF, Zergainoh A (2000) State-space representation for fractional-order controllers. *Automatica* 36:1017–1021
30. Sabatier J, Cois O, Oustaloup A (2002) *Commande de Systèmes Non Entiers par Placement de Pôles*. Deuxième Conférence Internationale Francophone d'Automatique, CIFA. Nantes, France
31. Samko SG, Kilbas AA, Marichev OI (1993) *Fractional integrals and derivatives: theory and applications*. Gordon and Breach Science Publisher, Amsterdam

32. Sierociuk D, Dzieliński A (2006) Fractional Kalman filter algorithm for the states, parameters and order of fractional system estimation. *Int J Appl Math Comput Sci* 16:129–140
33. Vinagre B, Monje C, Caldero AJ (2002) Fractional order systems and fractional order actions. Tutorial workshop # 2: Fractional calculus applications in automatic control and robotics, 41st IEEE CDC, Las Vegas, USA

Stability of Fractional-Delay Systems: A Practical Approach

Farshad Merrikh-Bayat

Abstract Sometimes, in dealing with fractional-order transfer functions, the exact location of the poles and zeros on the first Riemann sheet is needed. For example, in order to examine the stability of fractional-delay systems, the location of the poles on the first Riemann sheet should be known since the stability is related to the poles that are located on the right half-plane of the first Riemann sheet. The difficulty is due to the fact that most of the practical multi-valued transfer functions consist of large (possibly infinite) number of poles and zeros which makes the problem of determining their location (and consequently, the stability analysis) a challenging task. In this paper, an effective numerical algorithm for determining the location of poles and zeros on the first Riemann sheet is presented. The proposed method is based on the Rouché's theorem and can be applied to all multi-valued transfer functions defined on a Riemann surface with finite number of Riemann sheets where the origin is a branch point. This covers all practical (finite-dimensional) fractional-order transfer functions and also the so-called fractional-delay systems. An example is presented to confirm the effectiveness of the proposed algorithm.

1 Introduction

In a generic sense, the *fractional-order* systems are identified by non-integer powers of the Laplace variable s . In the field of linear finite-dimensional fractional-order systems, the corresponding transfer functions commonly appear in the general form of

$$H(s) = \frac{b_0 s^{m/\nu} + b_1 s^{(m-1)/\nu} + \dots + b_m}{s^{n/\nu} + a_1 s^{(n-1)/\nu} + \dots + a_n}. \quad (1)$$

F. Merrikh-Bayat (✉)

Zanjan University, Tabriz road km 5, University Boulevard, Zanjan, Iran, P.O. Box: 313
e-mail: farshadbayat@yahoo.com

Such transfer functions are of interest especially for identification purposes. For example, Podlubny [9] proposed the fractional-order transfer function

$$H(s) = \frac{1}{0.7943s^{2.5708} + 5.2385s^{0.8372} + 1.5560}, \quad (2)$$

for a heating furnace. See also [10] for another examples of this type.

In the field of linear infinite-dimensional systems, the corresponding transfer functions commonly appear as a ratio of two polynomials in $\sqrt{s}$ in combination with the fractional delay term $\exp(-\sqrt{s})$. For example,

$$H(s) = \frac{\tanh(\sqrt{s})}{\sqrt{s}}, \quad (3)$$

appears in a boundary controlled and observed diffusion process in a bounded domain [3]. Other examples of the similar type can be found in [3]. In this note we will deal with the fractional-delay systems described by the transfer function:

$$H(s) = \frac{\sum_{m=0}^M \sum_{p=0}^{P_m} a_{m,p} s^{p/v} \exp(b_{m,p} s^{m/v})}{\sum_{n=0}^N \sum_{q=0}^{Q_n} c_{n,q} s^{q/v} \exp(d_{n,q} s^{n/v})}, \quad (4)$$

where $v \in \mathbb{N}$, $b_{m,p}, d_{n,q} \in \mathbb{R}^+$, and $a_{m,p}, c_{n,q} \in \mathbb{R}$. The presence of a fractional-delay transfer function like (4) in a feedback control system leads to a characteristic equation with infinity many isolated roots. Such a characteristic equation is often called *fractional-delay equation*.

In this brief, the studies are mainly focused on a certain class of fractional-order transfer functions the domain of definition of which is a Riemann surface with limited number of Riemann sheets where the origin is a branch point. The transfer functions (1), (2), (3) and (4) lie in this category. Note that this is not a considerable loss of generality as it covers almost all practical cases, e.g. those described by heat and wave equations [3]. The important feature of the systems under consideration is that they are of very high (possibly infinite) dimension. Note also that, in fact, it is the very multi-valued nature of the systems like (2) that gives the richness to model.

The aim of this brief is to present a method to find the roots of a fractional-delay equation on the first Riemann sheet. Before introducing the main problem, it should be noted that in some applications the exact location of the poles and zeros of a given multi-valued transfer function on the first Riemann sheet is required. For example, [4] proposed a controller design algorithm for non-minimum phase (integer-order) systems which can be extended to fractional-order systems in the general form of (1). In order to do such an extension we must know the exact location of the unstable zeros of (1) on the first Riemann sheet. As another example, [1] studied the generalized damping equation

$$(D^2 + aD^q + b)x(t) = f(t); \quad q \in (0, 2), \quad (5)$$

where $x(t)$ denotes the displacement at time t of a mass with respect to the equilibrium position, $f(t)$ is the external force per mass; $a, b \in \mathbb{R}$ are the damping and the stiffness constants per mass and D denotes the derivative with respect to time. As it is mentioned in [1], in order to calculate the impulse response of (5) one should determine the location of the roots of $s^2 + as^q + b = 0$ on the first Riemann sheet. It is also known that the stability of fractional-order systems under consideration is related to the poles on the right half-plane of the first Riemann sheet [6] (it is shown in [6] that, in general, the numerator of a given fractional-order transfer function is also important for stability analysis). In fact, in order to deal with the stability of fractional-delay systems we should know about the location of the poles on the first Riemann sheet. But, it is a well-known fact that there is no simple and generally applicable analytical approach to find the roots of a given fractional-delay equation on the first Riemann sheet and hence the stability analysis of such systems is, in general, a tricky task.

In this paper, a numerical algorithm for determining the exact location of the roots of a given high-dimensional fractional-order equation on the first Riemann sheet is presented. The proposed method is based on Rouché's theorem and directly determines the location of the roots of a given fractional-order equation on the first Riemann sheet. This method can also be used for bounded-input bounded-output (BIBO) stability testing of the fractional-delay systems. Unlike some other works (such as [5]) that are based on Cauchy's theorem, the proposed method is easily applied and leads to very reliable results.

In what follows, first the proposed algorithm is presented in Sect. 2 and then a numerical example is studied in Sect. 3.

2 The Proposed Numerical Algorithm

Before introducing the main contribution of this work, some preliminaries and definitions are provided. For the systems under consideration, the first Riemann sheet is characterized as

$$\{s : 0 < |s|, -\pi \leq \arg s < \pi\},$$

and the symbol $\mathcal{P}$ denotes all points on the first Riemann sheet that do not lie on the branch cut at $\mathbb{R}^-$, i.e.,

$$\mathcal{P} \triangleq \{s : 0 < |s|, -\pi < \arg s < \pi\}.$$

We continue this section with a brief review of an explanation of the Rouché's theorem. Consider the (single-valued) function $f : \mathbb{C} \rightarrow \mathbb{C}$ which has (finite) zeros of orders $m_1, \dots, m_k$ respectively at $\alpha_1, \dots, \alpha_k$ and (finite) poles of orders $p_1, \dots, p_n$ respectively at $\beta_1, \dots, \beta_n$. Such a function can be expressed as

$$f(s) = g(s) \frac{(s - \alpha_1)^{m_1} \dots (s - \alpha_k)^{m_k}}{(s - \beta_1)^{p_1} \dots (s - \beta_n)^{p_n}}, \quad (6)$$

where $g(s)$ has neither pole nor zero. By taking the natural logarithm from both sides of (6) we obtain

$$\ln f(s) = \ln g(s) + m_1 \ln(s - \alpha_1) + \cdots + m_k \ln(s - \alpha_k) - p_1 \ln(s - \beta_1) - \cdots - p_n \ln(s - \beta_n). \quad (7)$$

Derivation with respect to s yields

$$\frac{f'(s)}{f(s)} = \frac{g'(s)}{g(s)} + \frac{m_1}{s - \alpha_1} + \cdots + \frac{m_k}{s - \alpha_k} - \frac{p_1}{s - \beta_1} - \cdots - \frac{p_n}{s - \beta_n}. \quad (8)$$

Now let γ be a simple, closed, counterclockwise contour such that $f(s)$ has neither pole nor zero on it. Then it is concluded from the residue theorem that

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(s)}{f(s)} ds = \sum_{j=1}^k m_j - \sum_{j=1}^n p_j = M - P, \quad (9)$$

according to the fact that $g(s)$ is analytic on and inside γ . Equation 9 is an explanation of the Rouché's theorem. If $f(s)$ has no pole inside γ then it is concluded from (9) that

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(s)}{f(s)} ds = M. \quad (10)$$

Equation 10 can be used to find the number of the roots of $f(s) = 0$ in the given contour γ .

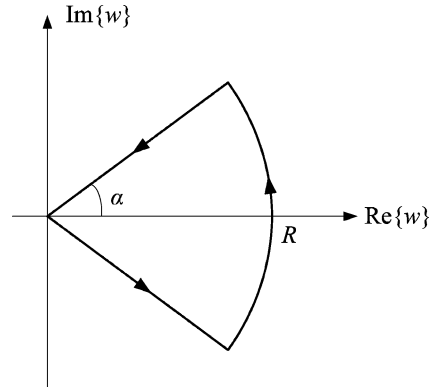
In order to extend the above result to the systems under consideration, let $\Delta(s)$ be a multi-valued characteristic function which can be transformed to the single-valued function $\tilde{\Delta}(w)$ with the change of variable $w = s^{1/v}$. Then the number of the roots of $\Delta(s) = 0$ in $\mathcal{P}$ (denoted by M in the rest of this note) is obtained as

$$M = \frac{1}{2\pi i} \oint_{\gamma} \frac{\tilde{\Delta}'(w)}{\tilde{\Delta}(w)} dw, \quad (11)$$

where γ is the contour shown in Fig. 1 with $\alpha = \pi/v$ and $R \rightarrow \infty$. Note that the sector $|\arg w| < \pi/v$ in the complex w plane corresponds to $\mathcal{P}$ in the s Riemann surface. Since most software packages can only deal with the real integrals, we write the complex integral in (11) as a combination of three real integrals as follows:

$$M(R, \alpha) = \frac{1}{2\pi i} \left(\int_{r=0}^R \frac{\tilde{\Delta}'(re^{-i\alpha})}{\tilde{\Delta}(re^{-i\alpha})} e^{-i\alpha} dr + \int_{\theta=-\alpha}^{\alpha} \frac{\tilde{\Delta}'(Re^{i\theta})}{\tilde{\Delta}(Re^{i\theta})} Re^{i\theta} d\theta + \int_{r=R}^0 \frac{\tilde{\Delta}'(re^{i\alpha})}{\tilde{\Delta}(re^{i\alpha})} e^{i\alpha} dr \right), \quad (12)$$

Fig. 1 The contour of integration in w plane



which after some algebra yields

$$M(R, \alpha) = \frac{1}{\pi} \left(R \int_0^\alpha \Re \left\{ \frac{\tilde{\Delta}'(Re^{i\theta})}{\tilde{\Delta}(Re^{i\theta})} e^{i\theta} \right\} d\theta - \int_0^R \Im \left\{ \frac{\tilde{\Delta}'(re^{i\alpha})}{\tilde{\Delta}(re^{i\alpha})} e^{i\alpha} \right\} dr \right). \quad (13)$$

Now, in order to find the modulus of the roots of $\tilde{\Delta}(w) = 0$ in the sector $|\arg w| < \pi/\nu$, first we let $\alpha = \pi/\nu$ and then calculate $M(R, \pi/\nu)$ for several values of R using (13). Then, the distance of the roots from origin can be investigated by plotting $M(R, \pi/\nu)$ versus R . For instance, let R_1 and R_2 ($> R_1$) be two successive values of R such that

$$M(R_2) \approx M(R_1) + k, \quad (14)$$

for some $k \in \mathbb{N}$ (this means that there is an integer jump in the plot of M versus R). In this case, (14) implies that there are k roots in the region defined by $R_1 < |w| < R_2$ and $|\arg w| < \pi/\nu$. In the proposed method, the values of R_1 and R_2 must be chosen such close that (14) holds for $k = 1$ (or $k = 2$, if $k = 1$ is not possible, i.e. if there is a pair of complex conjugate roots in the contour). In brief, in order to determine the modulus of the roots of $\Delta(w) = 0$, plot $M(R, \pi/\nu)$ versus R . The values of R , for which there is an integer jump in this plot, correspond to the modulus of the roots.

After determining the distance of the roots from the origin, their phase angle can be determined. Let $R_1, \dots, R_n$ be the distance of the roots of $\tilde{\Delta}(w) = 0$ from the origin where by assumption $R_1 < \dots < R_n$. Assume that w_i is a root of $\tilde{\Delta}(w) = 0$ such that $|w_i| = R_i$. In order to determine $\arg(w_i)$, plot $M(R_0, \alpha)$ versus α where $R_0 \in (R_i, R_{i+1})$ is an arbitrarily chosen constant and $0 < \alpha < \pi/\nu$. The values of α for which there is an integer jump in this plot correspond to the phase angle of w_i . Finally, the roots of $\Delta(s) = 0$ can be obtained from the roots of $\tilde{\Delta}(w) = 0$ using the map $s = w^\nu$.

Obviously, $M(\infty, \frac{\pi}{2\nu})$ (as defined in (13)) is equal to the total number of unstable poles (if the integrals converge). See [7] for more details on this subject.

3 Numerical Example

Equation 13 is the key formula of this paper. A convenient approach to evaluate the proper integral $\int_a^b g(x)dx$ is to define $I(b) \triangleq \int_a^b g(x)dx$ and then solve the initial-value problem

$$\frac{dI(x)}{dx} = f(x), \quad I(a) = 0. \quad (15)$$

In the following examples, this approach is used to evaluate the integrals in (13). The MATLAB function `ode23` which is an implementation of an explicit Runge-Kutta (2,3) pair of Bogacki and Shampine [2] is found to be accurate enough for this purpose.

Example. This example shows the application of the proposed method for determining the location of the roots of a fractional-delay equation. Consider the fractional-delay characteristic function

$$\Delta(s) = (\sqrt{s})^2 + K(\sqrt{s} + 1)e^{-\sqrt{s}}, \quad (16)$$

where K is a positive real constant. It has been shown by Ozturk and Uraz [8] that $\Delta(s)$ corresponds to a stable system if $K < 21.51$ and has some roots in the sector of instability $|\arg(w)| \leq \pi/4$ if $K > 21.51$. In the following, we let $K = 22$ and find the location of unstable roots.

First of all, (16) must be transformed to the single-valued characteristic function

$$\tilde{\Delta}(w) = w^2 + 22(w + 1)e^{-w}, \quad (17)$$

with the change of variable $\sqrt{s} = w$. Then, in order to find the modulus of the roots of $\tilde{\Delta}(w) = 0$, $M(R, \pi/2)$ must be calculated for various values of R . Figure 2 shows the plot of $M(R, \pi/2)$ versus R . As it is viewed, $M(3, \pi/2) \approx 0$ and $M(3.1, \pi/2) \approx 2$ which implies that there are two complex conjugate roots in the region defined by

$$3 < |w| < 3.1, \quad |\arg w| < \frac{\pi}{2}. \quad (18)$$

In the same manner, by inspecting the values of R between 3 and 3.1 the exact modulus of the roots is found to be $R_1 \approx 3.085$.

The phase angle of the roots of $\tilde{\Delta}(w) = 0$ can be obtained by plotting $M(R_0, \alpha)$ versus α , where $R_0 > R_1$ is an arbitrarily chosen constant. Figure 3 shows the plot of $M(3.5, \alpha)$ versus α . According to this figure, there are two roots in the region defined by

$$0 < |w| < 3.5, \quad 0.764 < |\arg w| < 0.7954. \quad (19)$$

By inspecting the values of α in the interval $[0.764, 0.7954]$ the absolute value of the phase angle of the roots of $\tilde{\Delta}(w) = 0$ is found to be 0.782 rad. Therefore, the roots in the complex w plane are $w_{1,2} \approx 3.085e^{\pm i0.782}$. The location of the roots on $\mathcal{P}$ can be calculated using the map $s = w^2$, which results in $s_{1,2} \approx 9.5172e^{\pm i1.564}$. As expected, both roots lie in the right-half plane of the first Riemann sheet.

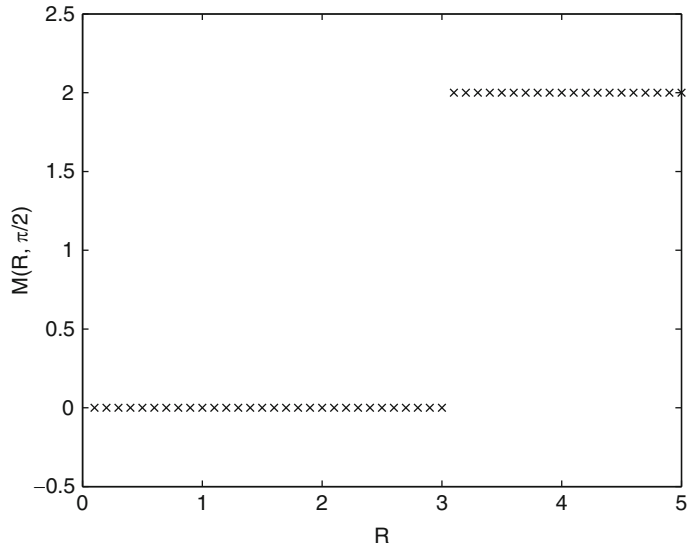


Fig. 2 The plot of $M(R, \pi/2)$ versus R corresponding to (17)

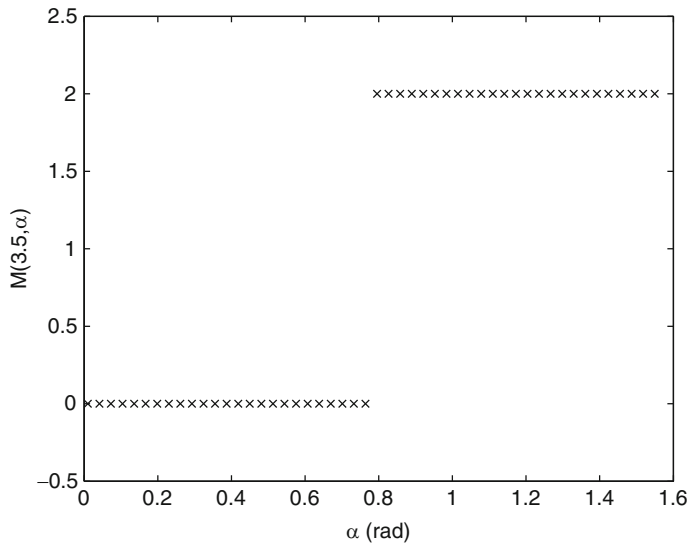


Fig. 3 The plot of $M(3.5, \alpha)$ versus α corresponding to (17)

References

1. Beyer H, Kempfle S (1995) Definition of physically consistent damping laws with fractional derivatives. Zeitschrift fuer Angewandte Mathematik and Mechanik 75(8):623–635
2. Bogacki P, Shampine LF (1989) A 3(2) pair of runge-kutta formulas. Appl Math Lett 2:1–9

3. Curtain RF, Zwart HJ (1995) An introduction to infinite-dimensional linear systems theory. Text in applied mathematics. Springer, Berlin
4. Doyle J, Francis B, Tannenbaum A (1990) Feedback control theory. Macmillan, New York
5. Hwang C, Cheng Y-C (2006) A numerical algorithm for stability testing of fractional delay systems. *Automatica* 42:825–831
6. Merrikh-Bayat F, Karimi-Ghartemani M (2008) On the essential instabilities caused by fractional-order transfer functions. *Math Probl Eng*, Vol 2008, Article ID 419046. doi:10.1155/2008/419046
7. Merrikh-Bayat F, Karimi-Ghartemani M (2009) An efficient numerical algorithm for stability testing of fractional-delay systems. *ISA Trans* 48:32–37
8. Ozturk N, Uraz A (1984) An analytic stability test for a certain class of distributed parameter systems with a distributed lag. *IEEE Trans Automat Contr* 29(4):368–370
9. Podlubny I (1999) Fractional differential equations. Academic, Sandiego
10. Podlubny I, Dorcak L, Kostial I (1999) On fractional derivatives, fractional-order dynamic system and $PI^\lambda D^\mu$ -controllers. In: Proceedings of the 36th IEEE CDC, San Diego

Comparing Numerical Methods for Solving Nonlinear Fractional Order Differential Equations

Farhad Farokhi, Mohammad Haeri, and Mohammad Saleh Tavazoei

Abstract This paper is a result of comparison of some available numerical methods for solving nonlinear fractional order ordinary differential equations. These methods are compared according to their computational complexity, convergence rate, and approximation error. The present study shows that when these methods are applied to nonlinear differential equations of fractional order, they have different convergence rate and approximation error.

1 Introduction

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in physics, fluid mechanics, biology, and engineering. Consequently, considerable attention has been given to the solutions of fractional order ordinary differential equations, integral equations and fractional order partial differential equations of physical interest. Number of literatures concerning the application of fractional order differential equations in nonlinear dynamics has been grown rapidly in the recent years [2, 3, 5, 12–14, 20]. Most fractional differential equations do not have exact analytic solutions and therefore, approximating or numerical techniques are generally applied. There are many different numerical methods such as Predictor Corrector Method (PCM) [8], Quadrature Methods (QM) [22], Kumar-Agrawal Method (KAM) [15], and Lubich Method [17] which have been developed to solve the fractional differential equations. Many new ideas which try to solve these kinds of problems faster and in more convenient way are Nested Memory Principle (NMP) and Fixed Length Integral Principle (FLIP) [7, 10]. These methods are relatively new and provide an approximated solution both for linear and nonlinear equations. There are several papers in

F. Farokhi, M. Haeri (✉), and M.S. Tavazoei
Advanced Control System Lab., Electrical Engineering Department,
Sharif University of Technology, Azadi Ave., P.O. Box 11155-9363, Tehran, Iran
e-mail: farokhi@ee.sharif.edu; haeri@sina.sharif.edu; m.tavazoei@ee.sharif.edu

which some of these methods have been comparatively studied [1, 18, 19, 21, 23]. In most of these works, the existing methods were compared by their implementation difficulties or convergence mean error. Also, these papers discuss issues mostly related to the linear equations and they have paid less attention to the nonlinear cases more specifically chaotic ones. In this paper, we implement the above mentioned methods to solve the nonlinear differential equation of fractional order and then compare them through some numerical examples. The paper is organized as follows. In Sect. 2, there is a review on mathematical concepts of fractional order differential equations and then some different methods to solve the nonlinear versions of these equations are described. The mentioned approaches are applied to some common nonlinear systems and results are discussed. Finally some concluding remarks are given in Sect. 3.

2 Numerical Methods

In this section we want to discuss on the numerical solution of differential equations of fractional order with arbitrary initial conditions;

$$D_*^\alpha y(x) = f(x, y(x)), \quad (1)$$

where $\alpha > 0$ (but not necessarily $\alpha \in \mathbb{N}$), D_*^α is Caputo differential operator of order α [4]. We combine our fractional differential equation (1) with initial conditions;

$$y^{(k)}(0) = y_0^{(k)} \quad k = 0, 1, \dots, m-1. \quad (2)$$

Existence and uniqueness of solution of system of fractional differential equations (1) for a given initial condition (2) have been proven in [6, 9]. In the following sections, methods are introduced and compared in accordance to their computational complexity, convergence rate, and the approximating error. For each method, one example is implemented and the mentioned properties are investigated. These methods can easily be extended to multi-term equations or a system of equations [11].

2.1 Quadrature Method (QM)

This approach is based on the analytical property that the initial value problem (1) and (2) is equivalent to the Volterra integral equation;

$$y(x) = \sum_{k=0}^{[\alpha]-1} y_0^{(k)} \frac{x^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, y(t)) dt, \quad (3)$$

in the sense that a continuous function is a solution of the initial value problem if and only if it is a solution of (3). This method is based on using Quadratures [22] like Simpson in integrating (3) and finding the answer. According to mathematical concepts and numerical methods [22], the following relation exists between the (step size) and the number of flops.

$$\forall \alpha : \text{Flops} = O(h^{-2}), \quad (4)$$

and the following relation exists between the approximation error and the step size;

$$\text{error} = O(h^\alpha), \quad (5)$$

According to (4) and (5), this method is too slow and it is not so accurate and we cannot improve this method because of the theorems in [22].

2.2 *Predictor Corrector Method (PCM)*

In this section the algorithm is a generalization of the classical Adams–Bashforth–Moulton integrator that is well known for the numerical solution of the first-order problems [8]. According to [8], the following relation can be seen between the h (step size) and the number of flops.

$$\forall \alpha : \text{Flops} = O(h^{-2}), \quad (6)$$

The approximation error is related to step size as follows;

$$\text{error} = O(h^\alpha), \quad (7)$$

According to (6) and (7), this algorithm is too slow and its approximation error is high but we can improve its performance (approximation error) [8].

2.3 *Fixed Length Integral Principle (FLIP)*

For performing numerical computation, the simplest approach is to integrate only over a fixed period of recent history [7, 10]. If we can do this, then the computational cost at each step is reduced to $O(1)$, and the total amount of computational cost is reduced to $O(h^{-1})$ [7, 10]. But this method causes a loss of order and changes the nature of the fractional derivative from a non-local operator into a local operator.

2.4 Nested Memory Principle (NMP)

The idea of nested memory concept was introduced by Ford and Simpson. It can be well applied to numerical approximation of (3) in case of $\alpha \in (1, \infty)$, thus the computational cost at each step is reduce to $O(h^{-1} \log(h^{-1}))$ [7, 10].

2.5 Higher Order Methods (Lubich)

This method is based on a new method for approximating the fractional order derivative which has more accurate answers in comparison to the last ones [16, 17]. It can be seen that the numerical method (p -HOFLMSM) is in excellent agreement with the exact solution, and the error between p -HOFLMSM and exact solution is $O(h^p)$ [16, 17]. This method is the most accurate one with computational cost of $O(h^{-2})$ [16]. It can be applied on most of fractional order equations but the implementation of the algorithm is complex.

2.6 Kumar Agrawal Method (KAM)

This method is based on integrating the Voltra equation with translating it in to a nonlinear set of equations and then solving it with nonlinear solver. According to [15], the approximation error is $O(h^\alpha)$ and it has a computational cost of $O(h^{-2})$.

3 Numerical Simulations

The numerical results of these methods are discussed in this section based on two nonlinear fractional differential equations.

Example 1. The following nonlinear fractional-order ordinary differential equation with $\alpha = 1.5$ is considered:

$$D_*^\alpha y(t) + y^2(t) = f(t), \quad y^{(i)}(0) = 0, i = 1, 2, \quad (8)$$

where,

$$f(t) = \frac{\Gamma(6)}{\Gamma(6-\alpha)} t^{5-\alpha} - \frac{3\Gamma(5)}{\Gamma(5-\alpha)} t^{4-\alpha} + \frac{2\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} + (t^5 - 3t^4 + 2t^3)^2. \quad (9)$$

The exact solution for y is

$$y(t) = t^5 - 3t^4 + 2t^3. \quad (10)$$

Numerical results for exact and approximated solutions are illustrated in Figs. 1 and 2 for $\alpha = 1.5$.

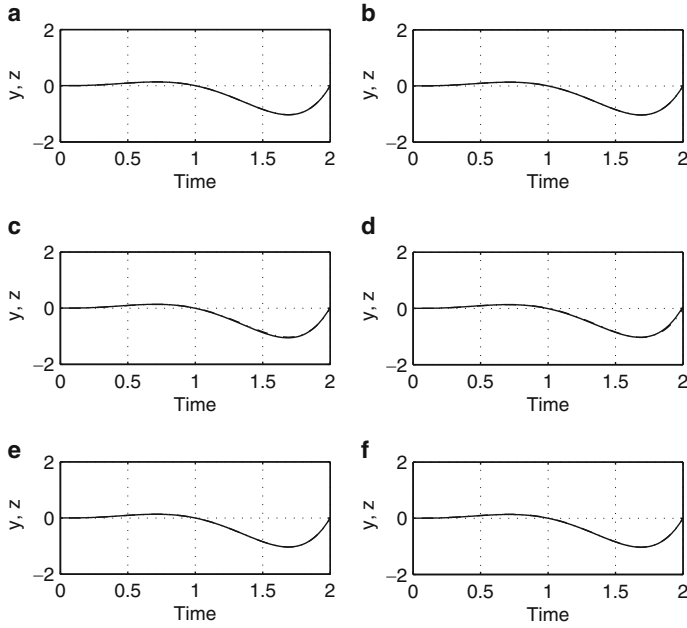


Fig. 1 The exact and the numerical solutions for system in (8) for $\alpha = 1.5$ with (a) QM ($h = 0.002$) (b) PCM ($h = 0.002$) (c) FLIP ($h = 0.002$) (d) NMP ($h = 0.002$) (e) Lubich ($h = 0.02$ and $p = 3$) and (f) KAM ($h = 0.002$)

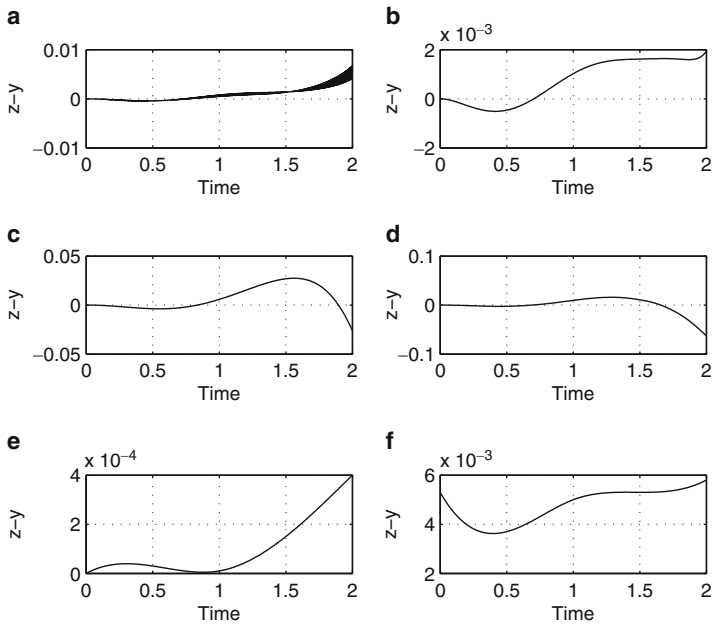


Fig. 2 Error between the exact and the numerical solutions for system in (8) for $\alpha = 1.5$ with (a) QM ($h = 0.002$) (b) PCM ($h = 0.002$) (c) FLIP ($h = 0.002$) (d) NMP ($h = 0.002$) (e) Lubich ($h = 0.02$ and $p = 3$) and (f) KAM ($h = 0.002$)

Example 2. Consider the following fractional differential equation;

$${}_c D_*^\alpha y(t) = \frac{40320}{\Gamma(9-\alpha)} t^{8-\alpha} - 3 \frac{\Gamma(5-\alpha/2)}{\Gamma(5+\alpha/2)} t^{4-\alpha/2} + 2.25\Gamma(\alpha+1) + (1.5t^{\alpha/2} - t^4)^3 - y(t)^{3/2} \quad (11)$$

with the following initial conditions;

$$y(0) = 0, y'(0) = 0. \quad (12)$$

The exact solution of this initial value problem is;

$$y(t) = t^8 - t^{4+\alpha/2} + 2.25t^\alpha. \quad (13)$$

The given equation is solved for $\alpha = 0.9$ and results are shown in Figs. 3 and 4.

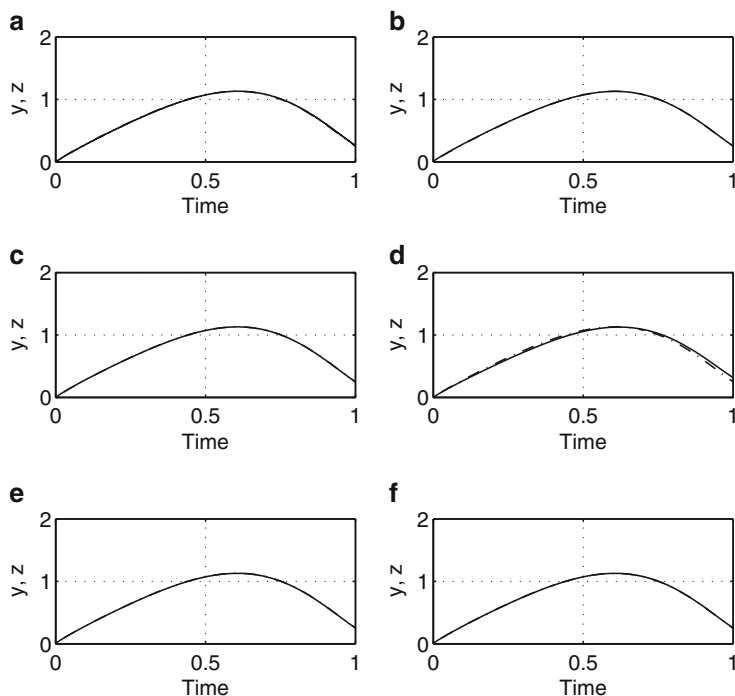


Fig. 3 The exact and the numerical solutions for system in (11) for $\alpha = 0.9$ with (a) QM ($h = 0.002$) (b) PCM ($h = 0.002$) (c) FLIP ($h = 0.002$) (d) NMP ($h = 0.002$) (e) Lubich ($h = 0.02$ and $p = 3$) and (f) KAM ($h = 0.002$)

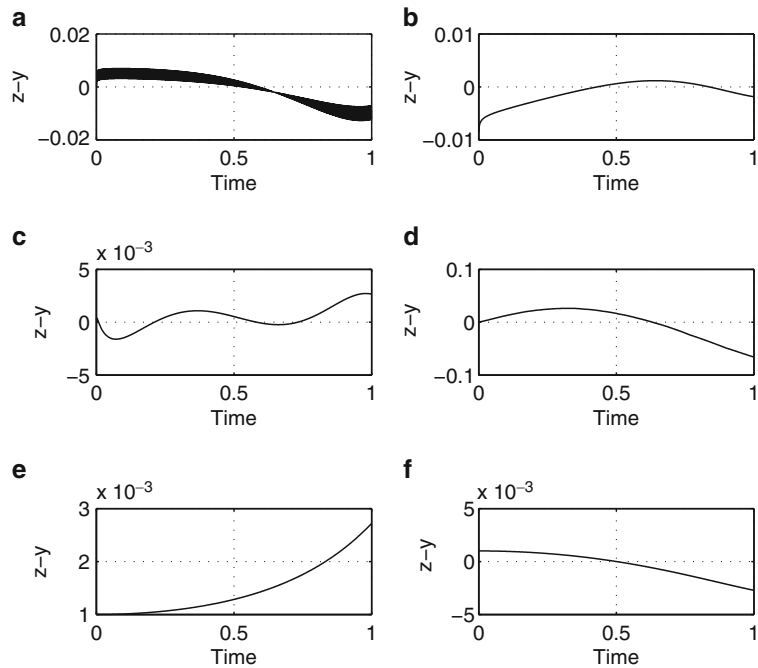


Fig. 4 Error between the exact and the numerical solutions for system in (11) for $\alpha = 0.9$ with (a) QM ($h = 0.002$) (b) PCM ($h = 0.002$) (c) FLIP ($h = 0.002$) (d) NMP ($h = 0.002$) (e) Lubich ($h = 0.02$ and $p = 3$) and (f) KAM ($h = 0.002$)

Table 1 Comparison of some methods in solving fractional order systems		
Method	Computational Cost	Approximation Error
Quadrature method	$O(h^{-2})$	$O(h^\alpha)$
Predictor corrector method	$O(h^{-2})$	$O(h^\alpha)$
Lubich (p^{th} order) method	$O(h^{-2})$	$O(h^p)$
Kumar Agarwal method	$O(h^{-2})$	$O(h^p)$
Fixed length integral principle	$O(h^{-1})$	$E = \begin{cases} \frac{M}{\Gamma(\alpha)} T^{\alpha-1} h & \alpha \in (0, 1) \\ \frac{M}{\Gamma(\alpha)} (t_n^{\alpha+1} - t_{n+1}^{\alpha+1}) h & \alpha \in (1, \infty) \end{cases}$
Nested memory principle	$O(h^{-1} \log(h^{-1}))$	

4 Conclusions

The present analysis exhibits the applicability of the Quadrature Method (QM), Predictor Corrector Method (PCM), Lubich Method, and Kumar-Agrawal Method (KAM) to solve ordinary differential equations of fractional order. In addition to them, we have introduced some new methods which reduce the computational cost of these solving methods, such as, Fixed Length Integral Principle (FLIP) and Nested Memory Principle (NMP). Table 1 compares the studied methods based on

their computational load and the approximating error level. The Lubich method is the most reliable method because of its simplicity and low approximating error but its speed is too low. At the second step one can choose the PCM to solve the fractional order differential equations and as the last chance, one can choose the QM. As one sees the computational cost of all these methods are high and these methods are very slow, but one can use FLIP and NMP in order to reduce the computational cost. NMP is more accurate than FLIP because it is better mapped with fractional order nature and preserves their non-locality property.

References

1. Adomian G (1988) A review of the decomposition method in applied mathematics. *J Math Anal Appl* 135:501–544
2. Bagley RL, Calico RA (1999) Fractional order state equations for the control of viscoelastic structures. *J Guid Control Dyn* 14(2):304–311
3. Benson DA, Wheatcraft SW, Meerschaert MM (2000) Application of a fractional advection-dispersion equation. *Water Resour Res* 36(6):1403–1412
4. Caputo M (1967) Linear models of dissipation whose Q is almost frequency independent. *Geophys J Roy Astr Soc* 13:529–539
5. Chen L, Zhao D (2005) Optical image encryption based on fractional wavelet transform. *Opt Commun* 254:361–367
6. Daftardar-Gejji V, Jafari H (2007) Analysis of a system of non-autonomous fractional differential equations involving Caputo derivatives. *J Math Anal Appl* 328:1026–1033
7. Deng W (2007) Short memory principle and a predictor-corrector approach for fractional differential equations. *J Comput Appl Math* 206:174–188
8. Diethelm K, Ford NJ, Freed (2002) A predictor-corrector approach for the numerical solution of fractional differential equations. *Nonlinear Dynam* 29:3–22
9. Diethelm D, Ford NJ (2002) Analysis of fractional differential equations. *J Math Anal Appl* 265:229–248
10. Ford NJ, Simpson (2001) The numerical solution of fractional differential equations: speed versus accuracy. *Numer Algorithms* 26:336–346
11. Ford NJ, Connolly JA (2008) Systems-based decomposition schemes for the approximate solution of multi-term fractional differential equations. *J Comput Appl Math*. doi:10.1016/j.cam.2008.04.003
12. Hartley TT, Lorenzo CF, Qammer HK (1995) Chaos in a fractional order Chua's system. *IEEE Trans CAS-I* 42:485–490
13. Ichise M, Nagayanagi Y, Kojima T (1971) An analog simulation of non-integer order transfer functions for analysis of electrode process. *J. Electroanal Chem* 33:253–265
14. Koeller RC (1984) Application of fractional calculus to the theory of viscoelasticity. *J Appl Mech* 51:299–307
15. Kumar P, Agrawal OM (2006) An approximate method for numerical solution of fractional differential equations. *Signal Process* 86:2602–2610
16. Lin R, Liu F (2007) Fractional high order methods for the nonlinear fractional ordinary differential equation. *Nonlinear Anal* 66(4):856–869
17. Lubich Ch (1986) Discretized fractional calculus. *SIAM J Math Anal* 17(3):704–719
18. Momani S, Odibat Z (2007) Numerical comparison of methods for solving linear differential equations of fractional order. *Chaos Soliton Fract* 31:1248–1255
19. Momani S, Odibat Z (2007) Numerical approach to differential equations of fractional order. *J Comp Appl Math* 207(1):96–110

20. Oustaloup A, Sabatier J, Lanusse P (1999) From fractal robustness to CRONE control. *Fract Calculus Appl Anal* 2:1–30
21. Shawagfeh N, Kaya D (2004) Comparing numerical methods for the solutions of systems of ordinary differential equations. *Appl Math Lett* 17:323–318
22. Vanloan CF (2000) Introduction to scientific computing: a Matrix-Vector Approach using MATLAB. Prentice-Hall, New Jersey
23. Wazwaz AM (2007) A comparison between the variational iteration method and Adomian decomposition method. *J Comput Appl Math* 207(1):129–136

Fractional-Order Backward-Difference Definition Formula Analysis

Piotr Ostalczyk

Abstract In this paper some properties of coefficients creating a fractional-order backward-difference formula are analyzed. Some relations between the coefficients mentioned are derived due to its evaluation accuracy. Every coefficient is treated as a two variable function. The first continuous-variable is a difference order while the second discrete-variable is a coefficient index. The analysis of the shape of such a two variable function indicates possibilities to minimize calculation errors, especially when a finite number of samples is taken into consideration.

1 Introduction

Practical applications of the Fractional Calculus [2, 6] come up against fractional order differences/sums (FOBD/S) evaluation difficulties. They are caused not only by a finite accuracy of a microprocessor system which is generated by intrinsic errors: processed signal quantization error, time-delay imposing a limit on the maximum sampling frequency, the system finite memory [1]. The latter error related to so called “finite memory principle” [5] seems to be essential in the FOBD calculation.

The paper is organised as follows. First, the basic definitions of FOBD/FOBS [3, 4] are given. In Sect. 3, fundamental relations between the coefficients used in the FOBD/S evaluation are presented. The analysis of the coefficients decreasing and increasing trends is a subject of Sects. 4 and 5. Information about this trends may be helpful in the FOBD/S calculation. This is the main result of the paper.

P. Ostalczyk (✉)
Instytut Automatyki, Politechnika Łódzka, Łódź Poland
e-mail: postalcz@p.lodz.pl

2 Mathematical Preliminaries

2.1 Notation

In the paper the following notation is used. The elements of a set of non-negative integers $\mathbb{Z}_+$ are denoted by latin letters i, j, k, l, m, n whereas the elements of a set of non-negative non-integers $\mathbb{R}_+ \setminus \mathbb{Z}_+$ are denoted by Greek letters ν, μ, ξ . The set $\mathbb{Z}_{+,k_0,k}$ denotes consecutive integers from an interval $[k_0, k]$. For real functions of a discrete variable k letters f, g, h are reserved. Any function f may be treated as a finite sequence $f = f(k) = \{f_{k_0}, f_{k_0+1}, \dots, f_{k-1}, f_k\}$ with function values evaluated for $k_0, k_0 + 1, \dots, k - 1, k$ respectively. Let $\mathbf{A}_k, \mathbf{B}_k$ denote $(k + 1) \times (k + 1)$ real band matrices. An identity matrix of dimensions $(k + 1) \times (k + 1)$ is denoted by $\mathbf{I}_k$.

2.2 Fractional-Order Backward-Difference/Sum

The FOBD of an order ν of a discrete-variable function $f(k)$ is defined as follows

$${}_{k_0}\Delta_k^{(\nu)} f(k) = \sum_{i=k_0}^k a_i^{(\nu)} f_{k-i+k_0} \quad (1)$$

with coefficients $a_i^{(\nu)}$

$$a_i^{(\nu)} = \begin{cases} 0 & \text{for } i < 0 \\ 1 & \text{for } i = 0 \\ (-1)^i \frac{\nu(\nu-1)\cdots(\nu-i+1)}{i!} & \text{for } i = 1, 2, \dots \end{cases} \quad (2)$$

Formula (1) may be also expressed in the vector formula

$${}_{k_0}\Delta_k^{(\nu)} f(k) = \begin{bmatrix} a_0^{(\nu)} & a_1^{(\nu)} & \dots & a_{k-k_0}^{(\nu)} \end{bmatrix} \begin{bmatrix} f_k \\ f_{k-1} \\ \vdots \\ f_{k_0} \end{bmatrix} \quad (3)$$

The FOBS is defined as the FOBD evaluated for negative order ($\nu > 0$).

$${}_{k_0}\Sigma_k^{(\nu)} f(k) = {}_{k_0}\Delta_k^{(\nu)} f(k) \quad (4)$$

Subscripts k_0, k denote the lower and upper summation range. A superscript ν is an order of the FOBD/S. One should realise that (1) and (4) are also valid for integer orders. Then

$${}_{k_0}\Delta_k^{(n)} f(k) = \sum_{i=k_0}^k a_i^{(n)} f_{k-i+k_0} = \sum_{i=k-n}^k a_i^{(n)} f_{2k-i-n}, \quad (5)$$

$${}_{k_0}\Sigma_k^{(n)} f(k) = {}_{k_0}\Delta_k^{(-n)} f(k) = \sum_{i_1=k_0}^k \sum_{i_2=k_0}^{i_1} \cdots \sum_{i_n=k_0}^{i_{n-1}} f_{i_n}, \quad (6)$$

because

$$a_i^{(n)} = 0 \quad \text{for } i > n, \quad (7)$$

$$a_i^{(-n)} \neq 0 \quad \text{for } i \in \mathbb{Z}_{+,0,k}. \quad (8)$$

3 Relation Between Coefficients $a_i^{(v)}$

By direct calculation one may check that coefficients $a_i^{(v)}$ defined above may be evaluated using recurrent formula

$$a_i^{(v)} = a_{i-1}^{(v)} \left(1 - \frac{1+v}{i} \right). \quad (9)$$

First, the FOBD of order μ of a special discrete-variable function

$$g(k) = \{g_0, g_1, \dots, g_k\} = \{a_0^{(v)}, a_1^{(v)}, \dots, a_k^{(v)}\} \quad (10)$$

is evaluated, i.e.

$${}_0\Delta_k^{(\mu)} g(k) = \begin{bmatrix} a_0^{(\mu)} & a_1^{(\mu)} & \cdots & a_k^{(\mu)} \end{bmatrix} \begin{bmatrix} g_k \\ g_{k-1} \\ \vdots \\ g_{k_0} \end{bmatrix} = \begin{bmatrix} a_0^{(\mu)} & a_1^{(\mu)} & \cdots & a_k^{(\mu)} \end{bmatrix} \begin{bmatrix} a_k^{(v)} \\ a_{k-1}^{(v)} \\ \vdots \\ a_0^{(v)} \end{bmatrix}. \quad (11)$$

For $k = 0, 1, 2$ one gets

$${}_0\Delta_k^{(\mu)} g(k) \Big|_{k=0} = a_0^{(\mu)} a_0^{(v)} = 1 = a_0^{(\mu+v)}, \quad (12)$$

$${}_0\Delta_k^{(\mu)} g(k) \Big|_{k=1} = a_0^{(\mu)} a_1^{(v)} + a_1^{(\mu)} a_0^{(v)} = -v - \mu = a_1^{(\mu+v)}, \quad (13)$$

$$\begin{aligned} {}_0\Delta_k^{(\mu)} g(k) \Big|_{k=2} &= a_0^{(\mu)} a_2^{(v)} + a_1^{(\mu)} a_1^{(v)} + a_2^{(\mu)} a_0^{(v)} = \\ &= \frac{v(v-1)}{2!} + (-\mu)(-v) + \frac{\mu(\mu-1)}{2!} = \frac{(\mu+v)(\mu+v-1)}{2!} = \\ &= a_2^{(\mu+v)}. \end{aligned} \quad (14)$$

Continuing the above calculations one finally gets a relation, which is valid for $\nu, \mu > 0, k = 0, 1, 2, \dots$

$${}_0\Delta_k^{(\mu)} g(k) = a_k^{(\mu+\nu)}. \quad (15)$$

Equality (15) can be also evaluated for $k-1, k-2, \dots$. Collecting all such expressions in a matrix equality one gets

$$\mathbf{A}_k^{(\mu)} \mathbf{A}_k^{(\nu)} = \mathbf{A}_k^{(\mu+\nu)}, \quad (16)$$

where

$$\mathbf{A}_k^{(\xi_i)} = \begin{bmatrix} a_0^{(\xi_i)} & a_1^{(\xi_i)} & a_2^{(\xi_i)} & \cdots & a_{k-1}^{(\xi_i)} & a_k^{(\xi_i)} \\ 0 & a_0^{(\xi_i)} & a_1^{(\xi_i)} & \cdots & a_{k-2}^{(\xi_i)} & a_{k-1}^{(\xi_i)} \\ 0 & 0 & a_0^{(\xi_i)} & \cdots & a_{k-3}^{(\xi_i)} & a_{k-2}^{(\xi_i)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_0^{(\xi_i)} & a_1^{(\xi_i)} \\ 0 & 0 & 0 & \cdots & 0 & a_0^{(\xi_i)} \end{bmatrix}, \quad (17)$$

$$\xi_i = \begin{cases} \mu & \text{for } i = 1 \\ \nu & \text{for } i = 2 \\ \mu + \nu & \text{for } i = 3 \end{cases} \quad (18)$$

Now some special cases of (16) are considered. For $\nu = -\mu$ one immediately gets

$$\mathbf{A}_k^{(\mu)} \mathbf{A}_k^{(-\mu)} = \mathbf{A}_k^{(-\mu)} \mathbf{A}_k^{(\mu)} = \mathbf{I}_k. \quad (19)$$

This means, that

$$\left[\mathbf{A}_k^{(\mu)} \right]^{-1} = \mathbf{A}_k^{(-\mu)} \quad \text{and} \quad \left[\mathbf{A}_k^{(-\mu)} \right]^{-1} = \mathbf{A}_k^{(\mu)}. \quad (20)$$

3.1 Numerical Example

One takes into account an integer orders $n_i = i$ for $i = 1, 2, 3$. According to (16) and (17) one obtains

$$\mathbf{A}_k^{(n_1)} = \mathbf{A}_k^{(1)} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad (21)$$

$$\mathbf{A}_k^{(n_2)} = \mathbf{A}_k^{(2)} = \begin{bmatrix} 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad (22)$$

$$\mathbf{A}_k^{(n_3)} = \mathbf{A}_k^{(3)} = \begin{bmatrix} 1 & -3 & 3 & \cdots & 0 & 0 \\ 0 & 1 & -3 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -3 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad (23)$$

It is easy to check that

$$\mathbf{A}_k^{(1)} \mathbf{A}_k^{(2)} = \mathbf{A}_k^{(1+2)} = \mathbf{A}_k^{(3)}, \quad (24)$$

Moreover,

$$\mathbf{A}_k^{(-n_1)} = \mathbf{A}_k^{(-1)} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad (25)$$

$$\mathbf{A}_k^{(-n_2)} = \mathbf{A}_k^{(-2)} = \begin{bmatrix} 1 & 2 & 3 & \cdots & k-1 & k \\ 0 & 1 & 2 & \cdots & k-2 & k-1 \\ 0 & 0 & 1 & \cdots & k-3 & k-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad (26)$$

$$\mathbf{A}_k^{(-n_3)} = \mathbf{A}_k^{(-3)} = \begin{bmatrix} 1 & 3 & 6 & \cdots & \frac{k(k+1)}{2} & \frac{(k+1)(k+2)}{2} \\ 0 & 1 & 3 & \cdots & \frac{k(k-1)}{2} & \frac{k(k+1)}{2} \\ 0 & 0 & 1 & \cdots & \frac{(k-1)(k-2)}{2} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 3 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad (27)$$

One can verify that

$$\left[\mathbf{A}_k^{(1)} \mathbf{A}_k^{(2)} \right]^{-1} = \left[\mathbf{A}_k^{(2)} \right]^{-1} \left[\mathbf{A}_k^{(1)} \right]^{-1} = \mathbf{A}_k^{(-2)} \mathbf{A}_k^{(-1)} = \mathbf{A}_k^{(-3)}. \quad (28)$$

4 Coefficients $a_i^{(\nu)}$ Values Analysis

Now a two variable real function is considered

$$u = u(k, \nu) = a_k^{(\nu)}, \quad \text{where } k \in \mathbb{Z}_+, \nu \in \mathbb{R}_+ \quad (29)$$

The coefficients $a_k^{(\nu)}$ are defined by (2). Plots of the function $u = u(k, \nu)$ for different ranges of a continuous variable (i.e. an order ν) and a constant range $k \in \mathbb{Z}_{+,1,20}$ are presented in Fig. 1. The form of the function $u = u(k, \nu) = a_k^{(\nu)}$ for positive orders from the interval $\nu \in [0, 2]$ shows that $|a_k^{(\nu)}| \rightarrow 0$ for increasing indices k . This means that the FOBD is evaluated by strongly decreasing coefficients. For negative orders $-\nu$ where $\nu \in [0, 2]$ there is a strong increase of positive coefficients

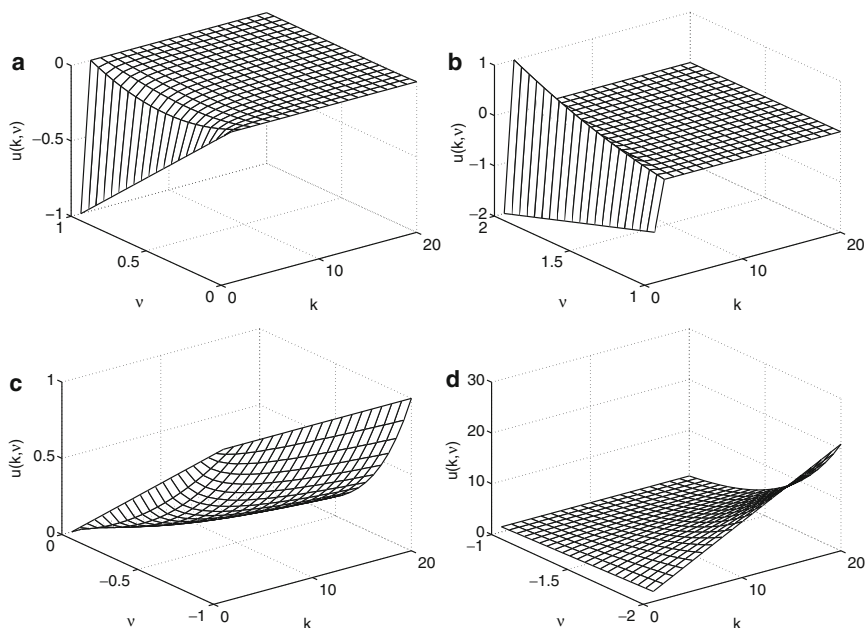


Fig. 1 (a) Plot of (29) for $\nu \in [0, 1]$, $k \in \mathbb{Z}_{+,1,20}$ (b) Plot of (29) for $\nu \in [1, 2]$, $k \in \mathbb{Z}_{+,1,20}$ (c) Plot of (29) for $\nu \in [-1, 0]$, $k \in \mathbb{Z}_{+,1,20}$ (d) Plot of (29) for $\nu \in [-2, -1]$, $k \in \mathbb{Z}_{+,1,20}$

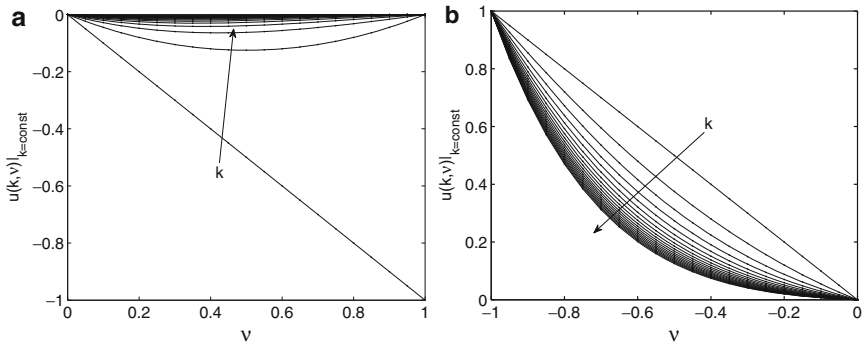


Fig. 2 (a) Plot of (29) for $v \in [0, 1]$, $k \in \mathbb{Z}_{+,1,20}$ (b) Plot of (29) for $v \in [-1, 0]$, $k \in \mathbb{Z}_{+,1,20}$

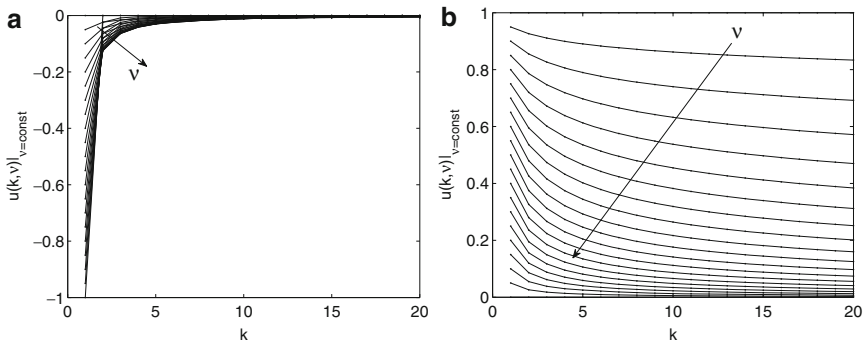


Fig. 3 (a) Plot of (29) for $v \in [0, 1]$, $k \in \mathbb{Z}_{+,1,20}$ (b) Plot of (29) for $v \in [-1, 0]$, $k \in \mathbb{Z}_{+,1,20}$

$a_k^{(-v)} > 0$. Below, in Fig. 2 plots of the function $u = u(k, v)|_{k=const} = a_k^{(v)}|_{k=const}$ are presented. Here the function is treated as one continuous variable function. In Fig. 3 plots of function $u = u(k, v)|_{v=const} = a_k^v|_{v=const}$ are presented.

5 Coefficients $a_i^{(v)}$ Increasing Rate

Now the main interest is focused on the interval of a fractional order $v \in [-1, 1]$. The plots presented in Sect. 4 show that the two variable function $u = u(k, v) = a_k^{(v)}$ is monotonic for $k \in \mathbb{Z}_+ \setminus \{0\}$. To measure an increasing rate, one may analyse its first order derivative. Hence

$$\frac{du}{dv} = \frac{du(k, v)}{dv} \Big|_{k=const} = \frac{d}{dv} \left[a_k^{(v)} \right], \quad \text{where } v \in \mathbb{R}_+ \quad (30)$$

Substitution (9) into (30) yields

$$\frac{d}{dv} [a_k^{(v)}] = \frac{d}{dv} \left[a_{k-1}^{(v)} \left(1 - \frac{1+v}{k} \right) \right] = \frac{d}{dv} [a_{k-1}^{(v)}] \left(1 - \frac{1+v}{k} \right) - \frac{a_{k-1}^{(v)}}{k}. \quad (31)$$

Expression (31) can be transformed into the vector form

$$\left[1 \left(\frac{1+v}{k} - 1 \right) \right] \begin{bmatrix} \frac{da_k^{(v)}}{dv} \\ \frac{da_{k-1}^{(v)}}{dv} \end{bmatrix} = \left[0 - \frac{1}{k} \right] \begin{bmatrix} a_k^{(v)} \\ a_{k-1}^{(v)} \end{bmatrix}. \quad (32)$$

Formula (32) is valid for consecutive values of $k, k-1, \dots, 1, 0$. Hence one can collect all such expressions in the matrix vector form

$$\mathbf{C}_k^{(v)} \begin{bmatrix} \frac{da_k^{(v)}}{dv} \\ \frac{da_{k-1}^{(v)}}{dv} \\ \vdots \\ \frac{da_0^{(v)}}{dv} \end{bmatrix} = \mathbf{D}_k^{(v)} \begin{bmatrix} a_k^{(v)} \\ a_{k-1}^{(v)} \\ \vdots \\ a_0^{(v)} \end{bmatrix}. \quad (33)$$

where

$$\mathbf{C}_k^{(v)} = \begin{bmatrix} 1 \left(\frac{1+v}{k} - 1 \right) & 0 & \cdots & 0 & 0 \\ 0 & 1 & \left(\frac{1+v}{k-1} - 1 \right) & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & v \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad (34)$$

$$\mathbf{D}_k^{(v)} = \begin{bmatrix} 0 - \frac{1}{k} & 0 & \cdots & 0 & 0 \\ 0 & 0 - \frac{1}{k-1} & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad (35)$$

Matrix (34) is always non-singular. The transformation of (33) to a form convenient in a numerical evaluation of (30) yields

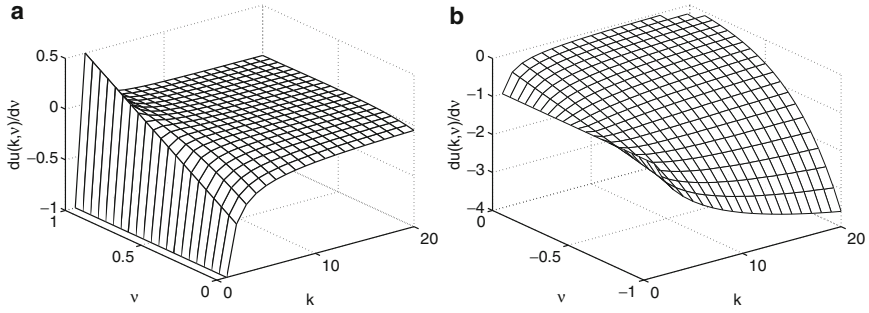


Fig. 4 (a) Plot of (30) for $v \in [0, 1]$, $k \in \mathbb{Z}_{+,1,20}$ (b) Plot of (30) for $v \in [-1, 0]$, $k \in \mathbb{Z}_{+,1,20}$

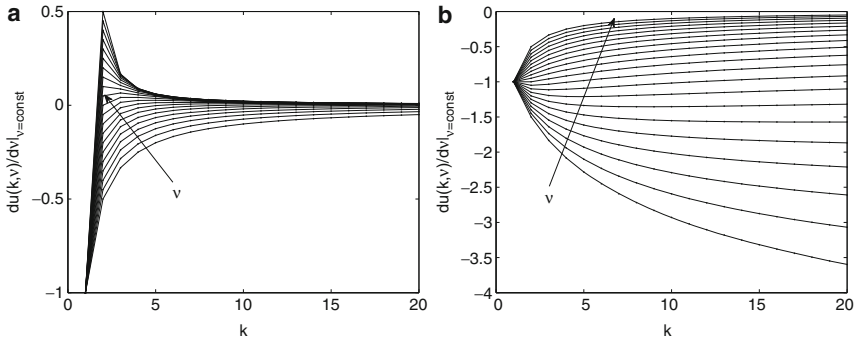


Fig. 5 (a) Plot of (30) for $v \in [0, 1]$, $k \in \mathbb{Z}_{+,1,20}$ (b) Plot of (30) for $v \in [-1, 0]$, $k \in \mathbb{Z}_{+,1,20}$

$$\begin{bmatrix} \frac{da_k^{(v)}}{dv} \\ \frac{da_{k-1}^{(v)}}{dv} \\ \vdots \\ \frac{da_0^{(v)}}{dv} \end{bmatrix} = [\mathbf{C}_k^{(v)}]^{-1} \mathbf{D}_k^{(v)} \begin{bmatrix} a_k^{(v)} \\ a_{k-1}^{(v)} \\ \vdots \\ a_0^{(v)} \end{bmatrix}. \quad (36)$$

In Fig. 4 the surface described by (36) is presented.

Below, in Fig. 5 plots of the function $\left. \frac{da_k^{(v)}}{dv} \right|_{v=const}$ are presented.

To derive the information concerning an increasing rate of coefficients $a_i^{(v)}$ one also evaluates a first order BD.

$${}_{k-1}\Delta_k^{(v)} u(k, v) \Big|_{v=const} = {}_{k-1}\Delta_k^{(v)} a_k^{(v)} \Big|_{v=const} = a_k^{(v)} - a_{k-1}^{(v)} = a_k^{(v-1)}. \quad (37)$$

In Fig. 6 plots of the function ${}_{k-1}\Delta_k^{(v)} u(k, v) \Big|_{v=const}$ are presented.

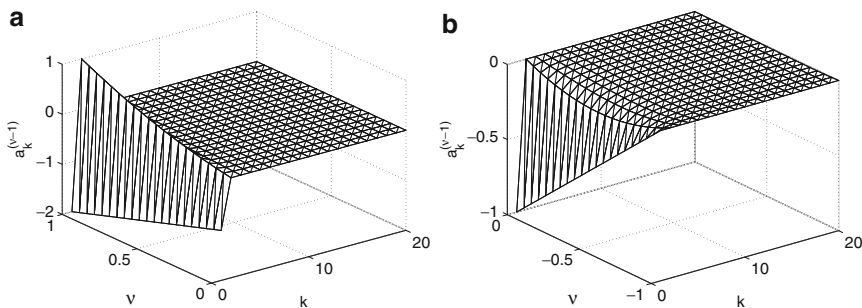


Fig. 6 (a) Plot of (37) for $v \in [0, 1]$, $k \in \mathbb{Z}_{+,1,20}$ (b) Plot of (37) for $v \in [-1, 0]$, $k \in \mathbb{Z}_{+,1,20}$

Finally one evaluates a first order derivative of the function ${}_{k-1}\Delta_k^{(v)}u(k, v) = a_k^{(v-1)}$. Applying the results expressed by (36) one immediately gets

$$\begin{bmatrix} \frac{d[{}_{k-1}\Delta_k^{(v)}u(k, v)]}{dv} \\ \frac{d[{}_{k-2}\Delta_{k-1}^{(v)}u(k, v)]}{dv} \\ \vdots \\ \frac{d[{}_{-1}\Delta_0^{(v)}u(k, v)]}{dv} \end{bmatrix} = \begin{bmatrix} \frac{da_k^{(v-1)}}{dv} \\ \frac{da_{k-1}^{(v-1)}}{dv} \\ \vdots \\ \frac{da_0^{(v-1)}}{dv} \end{bmatrix} = [\mathbf{C}_k^{(v-1)}]^{-1} \mathbf{D}_k^{(v-1)} \begin{bmatrix} a_k^{(v-1)} \\ a_{k-1}^{(v-1)} \\ \vdots \\ a_0^{(v-1)} \end{bmatrix}. \quad (38)$$

where

$$\mathbf{C}_k^{(v-1)} = \begin{bmatrix} 1 \left(\frac{v}{k} - 1 \right) & 0 & \cdots & 0 & 0 \\ 0 & 1 & \left(\frac{v}{k-1} - 1 \right) & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & v-1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad (39)$$

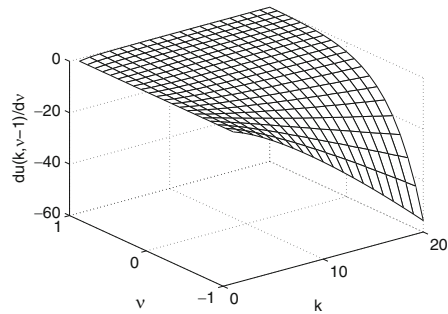
$$\mathbf{D}_k^{(v-1)} = \mathbf{D}_k^{(v)}. \quad (40)$$

In Fig. 7 a plot of the function

$$\frac{d[{}_{k-1}\Delta_k^{(v)}u(k, v)]}{dv} = \frac{d[u(k, v-1)]}{dv} = \frac{d[a_k^{(v-1)}]}{dv} \quad (41)$$

is presented.

Fig. 7 Plot of $\frac{d[u(k, v-1)]}{dv}$



6 Concluding Remarks

The analysis of the shapes of four two-variable functions $u(k, v)$, $\frac{du(k, v)}{dv}$, ${}_{k-1}\Delta_k^{(v)}u(k, v)$, $\frac{d[{}_{k-1}\Delta_k^{(v)}u(k, v)]}{dv}$ defined by formulae (29), (36), (37), (38), respectively, reveals the function $u(k, v)$ decreasing trends. Its various values imply different formula (1) evaluation errors. One should note that according to the results presented, it may be useful to apply an equivalent form of the FOBD form

$${}_{k_0}\Delta_k^{(v)}f(k) = {}_{k_0}\Delta_k^{(v-n)}\left[{}_{k_0}\Delta_k^{(n)}f(k)\right] \quad \text{for } n < v < n + 1$$

where $n \in \mathbb{Z}_+$.

References

1. Ifeakor E, Jervis BW (1993) Digital signal processing. Addison-Wesley, Edinburgh Gate
2. Miller K, Ross B (1993) An introduction to fractional calculus and fractional differential equations. Wiley, New York
3. Ostalczyk P (2000) The non-integer difference of the discrete-time function and its application to the control system synthesis. Int J Syst Sci 31:1551–1561
4. Oustaloup A (1995) La dérivation non entière – théorie, synthèses at applications. Hermes, Paris
5. Podlubny I (1999) Fractional differential equations. Academic, New York
6. Samko S, Kilbas A, Marichev O (1993) Fractional integrals and derivatives: theory and applications. Gordon and Breach, London

Fractional Differential Equations on Algebroids and Fractional Algebroids

Oana Chiş, Ioan Despi, and Dumitru Opreş

Abstract Using Caputo fractional derivative of order α we build the fractional α tangent fiber bundle and its main geometrical structures. We consider Poisson realizations for Maxwell-Bloch and Rabinovich differential equations that allow us to construct a Leibniz algebroid structure on $\mathbb{R}^3$. Fractional differential equations on algebroids are defined, some conclusions and numeric simulations are provided.

1 Introduction

Maxwell-Bloch and Rabinovich equations are dynamical systems that have two Poisson realizations [2, 7, 8]. These realizations allow us to introduce a Leibniz algebroid structure on $\mathbb{R}^3$ and also to define Maxwell-Bloch and Rabinovich fractional differential equations on this structure.

Using Caputo fractional derivative [1, 4, 6] the α -tangent fibre bundle is built on a differential manifold and fractional Leibniz algebroid structure in [5].

Maxwell-Bloch and Rabinovich fractional differential equations are obtained as dynamical systems on this space. In Sect. 2 some definitions are given regarding Leibniz structure on algebroids and the Rabinovich equations on a Leibniz algebroid. In Sect. 3 Caputo fractional derivative is recalled, the α tangent bundle is described, fractional almost Leibniz vector field and the fractional Rabinovich equations are presented. In Sect. 4 we present the notion of fractional Leibniz algebroid over a manifold, the fractional dynamical systems on a fractional Leibniz algebroid and all these notions are exemplified on Rabinovich equations. Section 5 presents conclusions and numeric simulations.

O. Chiş and D. Opreş (✉)

Faculty of Mathematics, West University of Timișoara, Romania

e-mail: chisoana@yahoo.com; opris@math.uvt.ro

I. Despi

School of Science and Technology, University of New England, Armidale, Australia

e-mail: despi@turing.une.edu.au

2 Rabinovich Equations on a Leibniz Algebroid

In this section we refer to the dynamical systems on Leibniz algebroid [5]. Let M be an n -dimensional smooth manifold. We give the following definitions.

A *Leibniz algebroid structure* on a vector bundle $\pi: E \rightarrow M$ is given by a bracket (bilinear operation) $[\cdot, \cdot]$ on the space of sections $\text{Sec}(\pi)$ and two vector bundle morphisms $\rho_1, \rho_2 : E \rightarrow TM$ (called the *left* resp. *right anchor*) such that for all $\sigma_1, \sigma_2 \in \text{Sec}(\pi)$ and $f, g \in C^\infty(M)$, we have

$$[f\sigma_1, g\sigma_2] = f\rho_1(\sigma_1)(g)\sigma_2 - g\rho_2(\sigma_2)(f)\sigma_1 + fg[\sigma_1, \sigma_2].$$

A vector bundle $\pi : E \rightarrow M$ endowed with a Leibniz algebroid structure on E , is called *Leibniz algebroid* over M and is denoted by $(E, [\cdot, \cdot], \rho_1, \rho_2)$.

In the paper [1], it is proved that a Leibniz algebroid structure on a vector bundle $\pi : E \rightarrow M$ is determined by a linear contravariant 2-tensor field on manifold E^* of the dual vector bundle $\pi^* : E^* \rightarrow M$. More precisely, if Λ is a linear 2-tensor field on E^* then the bracket $[\cdot, \cdot]_\Lambda$ of functions is given by $[f, g]_\Lambda = \Lambda(df, dg)$.

Let (x^i) , $i = \overline{1, n}$ be a local coordinate system on M and let $\{e_1, \dots, e_m\}$ be a basis of local sections of E . We denote by $\{e^1, \dots, e^m\}$ the dual basis of local sections of E^* and (x^i, y^a) , (resp. (x^i, ξ_a)) the corresponding coordinates on E (resp. E^*). In a local chart, the linear 2-tensor field Λ has the form:

$$\Lambda = C_{ab}^d \xi_d \frac{\partial}{\partial \xi_a} \otimes \frac{\partial}{\partial \xi_b} + \rho_{1a}^i \frac{\partial}{\partial \xi_a} \otimes \frac{\partial}{\partial x^i} - \rho_{2a}^i \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial \xi_a} \quad (1)$$

with $C_{ab}^d, \rho_{1a}^i, \rho_{2a}^i \in C^\infty(M)$, $i = \overline{1, n}$, $a, b, d = \overline{1, m}$.

We call a *dynamical system on Leibniz algebroid* $\pi : E \rightarrow M$ the dynamical system associated to vector field X_H with $H \in C^\infty(M)$ given by $X_H(f) = \Lambda(df, dH)$, $\forall f \in C^\infty(M)$. In a local chart, the above dynamical system is given by:

$$\dot{\xi}_a = [\xi_a, H]_\Lambda = C_{ab}^d \xi_d \frac{\partial H}{\partial \xi_b} + \rho_{1a}^i \frac{\partial H}{\partial x^i}, \quad \dot{x}^i = [x^i, H]_\Lambda = -\rho_{2a}^i \frac{\partial H}{\partial \xi_a}. \quad (2)$$

Let $\pi : E = \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the vector bundle and let $\pi^* : E = \mathbb{R}^3 \times (\mathbb{R}^3)^* \rightarrow \mathbb{R}^3$ be its dual, where E^* is the dual of E . We consider on E^* the linear 2-tensor field P , with the components given by the matrix

$$P = (C_{ab}^d \xi_d) = \begin{pmatrix} 0 & -\xi_3 x^3 & \xi_2 x^2 \\ \xi_3 x^3 & 0 & -\xi_1 x^1 \\ -\xi_2 x^2 & \xi_1 x^1 & 0 \end{pmatrix}. \quad (3)$$

In this paper we will consider the anchors $\rho_1, \rho_2 : \text{Sec}(\pi) \rightarrow T(\mathbb{R}^3)$ given by the following matrices:

$$\rho_{12} = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & 0 \\ -x^2 & 0 & 0 \end{pmatrix}, \quad \rho_{22} = \rho_{12} = -\rho_{11}, \quad \rho_{22} = \begin{pmatrix} 0 & 0 & \frac{1}{2}x^2 \\ 0 & 0 & -\frac{1}{2}x^1 \\ -\frac{1}{2}x^2 & \frac{1}{2}x^1 & 0 \end{pmatrix}, \quad (4)$$

$$\rho_{13} = \rho_{22}, \quad \rho_{23} = -\rho_{22} \quad (5)$$

or

$$\rho_{14} = \rho_{11}, \quad \rho_{24} = -\rho_{22}. \quad (6)$$

The following results are true:

Proposition 1. (a) On the Leibniz algebroid $(\mathbb{R}^3 \times \mathbb{R}^3, P, \rho_{12}, \rho_{22})$, with P, ρ_{12}, ρ_{22} given by (3) and (4) and associated to function $H = 2x^2\xi_2 + 2x^3\xi_3$, the dynamical system (2) is given by:

$$\begin{cases} \dot{\xi}_1 = 2x^3(1 + x^2)\xi_2 - 2x^2(1 + x^3)\xi_3, \\ \dot{\xi}_2 = -2x^1x^3\xi_1, \\ \dot{\xi}_3 = -2x^1x^2\xi_1 + x^1\xi_2, \\ \dot{x}^1 = x^2x^3, \\ \dot{x}^2 = -x^1x^3, \\ \dot{x}^3 = x^1x^2. \end{cases} \quad (7)$$

(b) For P, ρ_{13}, ρ_{23} given by (3) and (5), the dynamical system (2) on the Leibniz algebroid $(\mathbb{R}^3 \times \mathbb{R}^3, P, \rho_{13}, \rho_{23})$ and associated to function $H = 2x^2\xi_2 - 2x^3\xi_3$, is given by:

$$\begin{cases} \dot{\xi}_1 = 2x^2(x^3\xi_2 - \xi_3), \\ \dot{\xi}_2 = -x^1(2x^3 + 1)\xi_2 - 2x^1x^3\xi_1, \\ \dot{\xi}_3 = x^2(2x^1\xi_2 + \xi_1), \\ \dot{x}^1 = x^2x^3, \\ \dot{x}^2 = -x^1x^3, \\ \dot{x}^3 = x^1x^2. \end{cases} \quad (8)$$

(c) The dynamical system (2) on the Leibniz algebroid $(\mathbb{R}^3 \times \mathbb{R}^3, P, \rho_{14}, \rho_{24})$ with P, ρ_{14}, ρ_{24} defined by (3) and (6), associated to function $H = 2x^2\xi_2 + 2x^3\xi_3$, has the following form:

$$\begin{cases} \dot{\xi}_1 = 2x^2(x^3\xi_2 + \xi_3), \\ \dot{\xi}_2 = -2x^3x^1\xi_3 - 2x^3\xi_1(x^1 + 1), \\ \dot{\xi}_3 = 2x^2(x^1\xi_2 + \xi_1), \\ \dot{x}^1 = x^2x^3, \\ \dot{x}^2 = -x^1x^3, \\ \dot{x}^3 = x^1x^2. \end{cases} \quad (9)$$

□

The equations (7), (8) and (9) are called *Rabinovich equations on Leibniz algebroids*.

3 Fractional Rabinovich Equations

For $\alpha \in (0, 1)$ and a manifold M , let $(T^\alpha, \pi^\alpha, M)$ be the fractional tangent bundle to M [2]. If $x_0 \in U \subset M$ and $c : I \rightarrow M$ is a curve given by $x^i = x^i(t)$, $\forall t \in I$ on $(\pi^\alpha)^{-1}(U) \in T^\alpha(M)$, then the coordinates of the class $([c]_{x_0}^\alpha) \in T^\alpha(M)$ are $(x^i, y^{i(\alpha)})$, where $x^i = x^i(0)$, $y^{i(\alpha)} = \frac{1}{\Gamma(1+\alpha)} D_t^\alpha x^i(t)$, $i = \overline{1, n}$, where $D_t^\alpha x^i(t)$ is Caputo fractional derivative of the function $x^i(t)$, as in [1].

Let $\mathcal{D}^\alpha(U)$ be the module of 1-forms on U . The fractional exterior derivative $d^\alpha : C^\infty(U) \rightarrow \mathcal{D}^\alpha(U)$, $f \rightarrow d^\alpha(f)$, is given by $d^\alpha(f) = d(x^i)^\alpha D_{x^i}^\alpha(f)$ where D^α is the fractional order derivative and $D_{x^i}^\alpha$ is the partial fractional order derivative.

We denote by $\mathcal{X}^\alpha(U)$ the module of fractional vector fields generated by $\{D_{x^i}^\alpha, i = \overline{1, n}\}$. The fractional differentiable equations associated to X^α , where $X^\alpha = X^{\alpha i} D_{x^i}^\alpha$ with $X^{\alpha i} \in C^\infty(U)$, is defined by: $D_t^\alpha x^i(t) = X^{\alpha i}(x(t))$, $i = \overline{1, n}$. Let P^α , resp. g^α be a skew-symmetric, resp. symmetric fractional 2-tensor field on M . We define the bracket $[\cdot, (\cdot, \cdot)]^\alpha : C^\infty(M) \times (C^\infty(M) \times C^\infty(M)) \rightarrow C^\infty(M)$ by $[f, (H_1, H_2)]^\alpha = P^\alpha(d^\alpha f, d^\alpha H_1) + g^\alpha(d^\alpha f, d^\alpha H_2)$, $\forall f, H_1, H_2 \in C^\infty(M)$. The fractional vector field $X_{H_1 H_2}^\alpha$ defined by $X_{H_1 H_2}^\alpha = [f, (H_1, H_2)]^\alpha$, $\forall f \in C^\infty(M)$ is called *the fractional almost Leibniz vector field*. In a local chart, *the fractional almost Leibniz system associated to $(P^\alpha, g^\alpha, H_1, H_2)$ on M* is the differential system associated to $X^\alpha_{H_1 H_2}$, that is:

$$D_t^\alpha x^i(t) = P^{\alpha ij} D_{x^j}^\alpha H_1 + g^{\alpha ij} D_{x^j}^\alpha H_2, \quad i, j = 1, \dots, n. \quad (10)$$

In the following we will consider matrices $P^\alpha = P$ given by (3) and g^α given by:

$$g^\alpha = g_1^\alpha = \begin{pmatrix} -(x^2)^2 & x^1 x^2 & 0 \\ x^1 x^2 & -(x^1)^2 & 0 \\ 0 & 0 & -((x^1)^2 + (x^2)^2) \end{pmatrix}, \quad (11)$$

$$g^\alpha = g_2^\alpha = \begin{pmatrix} -4((x^2)^2 + (x^3)^2) & 0 & 0 \\ 0 & -4(x^3)^2 & 4x^2 x^3 \\ 0 & 4x^2 x^3 & -4(x^2)^2 \end{pmatrix}, \quad (12)$$

$$g^\alpha = g_3^\alpha = \begin{pmatrix} (x^3)^2 & 0 & -4x^1 x^3 \\ 0 & -((x^1)^2 + (x^3)^2) & 0 \\ -4x^1 x^3 & 0 & -(x^1)^2 \end{pmatrix}. \quad (13)$$

Using fractional derivative properties and (10) we have the following results.

Proposition 2. (a) For $H_{11}^\alpha = \frac{1}{\Gamma(1+\alpha)}((x^1)^{1+\alpha} + (x^2)^{1+\alpha})$ and $H_{21}^\alpha = \frac{1}{\Gamma(1+\alpha)}((x^1)^{1+\alpha} + (x^3)^{1+\alpha})$, the fractional Leibniz system on $(\mathbb{R}^3, P, g_1^\alpha)$ is defined in the following way:

$$\begin{cases} D_t^\alpha x^1 = \frac{1}{2}(x^3 + x^1 x^2)x^2, \\ D_t^\alpha x^2 = -\frac{1}{2}(x^1 x^2 + x^3)x^1, \\ D_t^\alpha x^3 = \frac{1}{2}[x^1 x^2 - ((x^1)^2 + (x^2)^2)x^3]. \end{cases} \quad (14)$$

(b) The fractional Leibniz system on $(\mathbb{R}^3, P, g_2^\alpha)$ and associated to $H_{12}^\alpha = \frac{1}{\Gamma(1+\alpha)}((x^2)^{1+\alpha} + (x^3)^{1+\alpha})$ and $H_{22}^\alpha = \frac{1}{\Gamma(1+\alpha)}((x^1)^{1+\alpha} + (x^2)^{1+\alpha})$ can be written as:

$$\begin{cases} D_t^\alpha x^1 = \frac{1}{2}x^2 x^3 - 4x^1[x^2]^2 + (x^3)^2, \\ D_t^\alpha x^2 = -\frac{1}{2}x^1 x^3 - 4x^2(x^3)^2, \\ D_t^\alpha x^3 = \frac{1}{2}x^1 x^2 + 4(x^2)^2 x^3. \end{cases} \quad (15)$$

(c) For P and g_3^α , given by (3) and (12), the fractional Leibniz system on $(\mathbb{R}^3, P, g_3^\alpha)$ associated to the functions $H_{13}^\alpha = \frac{1}{\Gamma(1+\alpha)}((x^1)^{1+\alpha} - (x^3)^{1+\alpha})$ and $H_{23}^\alpha = \frac{1}{\Gamma(1+\alpha)}((x^1)^{1+\alpha} + (x^2)^{1+\alpha})$ has the following form:

$$\begin{cases} D_t^\alpha x^1 = \frac{1}{2}x^2 x^3 + x^1(x^3)^2, \\ D_t^\alpha x^2 = -\frac{1}{2}x^1 x^3 + x^2[-(x^1)^2 + (x^3)^2], \\ D_t^\alpha x^3 = \frac{1}{2}x^1 x^2 - 4x^1 x^2 x^3. \end{cases} \quad (16)$$

□

The differential systems (14), (15) and (16) are called the revised fractional Rabinovich equations associated to the considered Hamilton-Poisson realizations.

4 The Fractional Equations on a Fractional Leibniz Algebroid

A Leibniz algebroid structure (pseudo-Lie algebroid structure) on a vector bundle $\pi : E \rightarrow M$ is given by the bracket $[\cdot, \cdot]$ on the space of sections Σ and two vector bundle morphisms $\rho_1, \rho_2 : E \rightarrow TM$, (called the left and the right anchors), such that: $[f\sigma_1, g\sigma_2] = f\rho_1(\sigma_1)(g)\sigma_2 - g\rho_2(\sigma_2)(f)\sigma_1 + fg[\sigma_1, \sigma_2]$, $\forall \sigma_1, \sigma_2 \in \Sigma$ and $f, g \in C^\infty(M)$. A vector bundle $\pi : E \rightarrow M$ endowed with a Leibniz algebroid structure $([\cdot, \cdot], \rho_1, \rho_2)$ on E is called Leibniz algebroid over M and it is denoted by $(E, [\cdot, \cdot], \rho_1, \rho_2)$.

If E is a Leibniz algebroid over M , then in the description of fractional Leibniz algebroid, the role of the tangent bundle is played by the fractional tangent bundle $T^\alpha M$ to M . For more details about this subject see [6].

A fractional Leibniz algebroid structure on a vector bundle $\pi : E \rightarrow M$ is given by a bracket $[\cdot, \cdot]^\alpha$ on the space of sections $Sec(\pi)$ and two vector bundle morphisms

$\rho_1^\alpha, \rho_2^\alpha : E \rightarrow T^\alpha M$ (called the *left*, resp. *right fractional anchor*) such that for all $\sigma_1, \sigma_2 \in \text{Sec}(\pi)$ and $f, g \in C^\infty(M)$ we have $[e_a, e_b]^\alpha = C_{ab}^c e_c$,

$$[f\sigma_1, g\sigma_2]^\alpha = f^\alpha \rho_1(\sigma_1)(g)\sigma_2 - g^\alpha \rho_2(\sigma_2)(f)\sigma_1 + fg[\sigma_1, \sigma_2]^\alpha.$$

A vector bundle $\pi : E \rightarrow M$, endowed with a fractional Leibniz algebroid structure on E , is called *fractional Leibniz algebroid over M* and it is denoted by $(E, [\cdot, \cdot]^\alpha, \rho_1^\alpha, \rho_2^\alpha)$.

A fractional Leibniz algebroid structure on a vector bundle $\pi : E \rightarrow M$ is determined by a linear fractional 2-tensor field $\Lambda^{\alpha\beta}$ on the dual vector bundle $\pi^* : E^* \rightarrow M$ [2], and hence the bracket $[\cdot, \cdot]_{\Lambda^{\alpha\beta}}$ is defined by:

$$[f, g]_{\Lambda^{\alpha\beta}} = \Lambda^{\alpha\beta}(d^{\alpha\beta} f, d^{\alpha\beta} g) \forall f, g \in C^\infty(E^*), \quad (17)$$

where $d^{\alpha\beta} f = d(x^i)^\alpha D_{x^i}^\alpha f + d(\xi_a)^\beta D_{\xi_a}^\beta f = d^\alpha(f) + d^\beta(f)$.

If (x^i) , $i = \overline{1, n}$, (x^i, y^a) , resp. (x^i, ξ_a) , for $i = \overline{1, n}$, $a = \overline{1, m}$ are coordinates on M , E , resp. E^* , then the linear fractional tensor $\Lambda^{\alpha\beta}$ on E^* has the form

$$\Lambda^{\alpha\beta} = C_{ab}^d \xi_d D_{\xi_a}^\beta \otimes D_{\xi_b}^\beta + \rho_{1a}^{\alpha i} D_{\xi_a}^\beta \otimes D_{x_i}^\alpha - \rho_{2a}^{\alpha i} D_{x_i}^\alpha \otimes D_{\xi_a}^\beta.$$

A *fractional dynamical system* on $(E, [\cdot, \cdot]^\alpha, \rho_1^\alpha, \rho_2^\alpha)$ is the fractional system associated to the vector field $X_H^{\alpha\beta}$ with $H \in C^\infty(E^*)$ and given by:

$$X_H^{\alpha\beta}(f) = \Lambda^{\alpha\beta}(d^{\alpha\beta} f, d^{\alpha\beta} H), \quad \forall f \in C^\infty(E^*). \quad (18)$$

In a local chart, the dynamical system (18) reads:

$$\begin{aligned} D_t^{\alpha\beta} \xi_a &= [\xi_a, H]_{\Lambda^{\alpha\beta}} = C_{ab}^d \xi_d D_{\xi_b}^\beta H + \rho_{1a}^{\alpha i} D_{x_i}^\alpha H, \\ D_t^\alpha x^i &= [x_a^i, H]_{\Lambda^{\alpha\beta}} = -\rho_{2a}^{\alpha i} D_{\xi_a}^\beta H. \end{aligned} \quad (19)$$

Let $\pi^* : E = \mathbb{R}^3 \times (\mathbb{R}^3)^* \rightarrow \mathbb{R}^3$ be the dual of the vector bundle $\pi : E = \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $\alpha > 0$, $\beta > 0$ and let $\Lambda^{\alpha\beta}$ be defined by (3). Let H be given by:

$$\begin{aligned} H &= \frac{1}{\Gamma(1+\alpha)}((x^1)^{\alpha+1} + (x^2)^{\alpha+1} + (x^3)^{\alpha+1} \\ &\quad + \frac{1}{\Gamma(1+\beta)}((\xi_1)^{\beta+1} + (\xi_2)^{\beta+1} + (\xi_3)^{\beta+1})). \end{aligned} \quad (20)$$

Then the following result is true.

Proposition 3. (a) *The fractional dynamical system (18) on the fractional Leibniz algebroid $(E, P, \rho_{11}, \rho_{21})$ associated to the function H given by*

$$H = \frac{1}{\Gamma(1+\beta)}((x^2)^{\alpha+1}(\xi_2)^{\beta+1} + (x^3)^{\alpha+1}(\xi_3)^{\beta+1}) \quad (21)$$

is:

$$\begin{cases} D_t^\beta \xi_1 = x^2 x^3 ((x^3)^\alpha - (x^2)^\alpha) \xi_2 \xi_3 \\ \quad + B x^2 x^3 ((\xi_3)^{\beta+1} - (\xi_2)^{\beta+1}), \\ D_t^\beta \xi_2 = -\xi_1 \xi_3 x^1 (x^3)^{\alpha+1}, \\ D_t^\beta \xi_3 = \xi_1 \xi_2 x^1 (x^2)^{\alpha+1}, \\ D_t^\alpha x^1 = (x^2)^{\alpha+1} \xi_2, \\ D_t^\alpha x^2 = x^1 (x^3)^{\alpha+1} \xi_3, \\ D_t^\alpha x^3 = -x^1 (x^2)^{\alpha+1} \xi_2. \end{cases} \quad (22)$$

where $B =: \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)}$. $\square$

(b) For the function H given by (21), we can define the fractional dynamical system (18) on the fractional Leibniz algebroid $(E, P, \rho_{11}, \rho_{22})$ in the following way:

$$\begin{cases} D_t^\beta \xi_1 = x^2 x^3 ((x^3)^\alpha - (x^2)^\alpha) \xi_2 \xi_3 \\ \quad + B x^2 x^3 ((\xi_3)^{\beta+1} - (\xi_2)^{\beta+1}), \\ D_t^\beta \xi_2 = -\xi_1 \xi_3 x^1 (x^3)^{\alpha+1}, \\ D_t^\beta \xi_3 = \xi_1 \xi_2 x^1 (x^2)^{\alpha+1}, \\ D_t^\alpha x^1 = -x^2 (x^3)^{\alpha+1} \xi_3, \\ D_t^\alpha x^2 = x^1 (x^3)^{\alpha+1} \xi_3, \\ D_t^\alpha x^3 = -x^1 (x^2)^{\alpha+1} \xi_3, \end{cases} \quad (23)$$

where $B =: \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)}$. $\square$

(c) Let us consider the fractional Leibniz algebroid $(E, P, \rho_{22}, \rho_{24})$ associated to the function H given by (21). The fractional dynamical system (18) on the considered fractional Leibniz algebroid is given by:

$$\begin{cases} D_t^\beta \xi_1 = -x^2 x^3 \xi_3 ((x^3)^\alpha - \frac{1}{2} B \xi_3^\beta), \\ D_t^\beta \xi_2 = \xi_1 \xi_3 x^1 x^3 ((x^1)^\alpha + (x^3)^\alpha) + \frac{1}{2} B x^1 x^3 \xi_3^{\beta+1}, \\ D_t^\beta \xi_3 = -x^1 x^2 \xi_1 (\xi_2 (x^1)^\alpha + \frac{1}{2} B \xi_1^\beta), \\ D_t^\alpha x^1 = -x^2 (x^3)^{\alpha+1} \xi_3, \\ D_t^\alpha x^2 = x^1 (x^3)^{\alpha+1} \xi_3, \\ D_t^\alpha x^3 = -x^2 (x^1)^{\alpha+1} \xi_1, \end{cases} \quad (24)$$

where $B =: \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)}$. $\square$

5 Numerical Simulations and Conclusions

Numerical simulations for fractional differential equations are done using Adams-Moulton method given in [3] and [4]. For the initial conditions $(0.2, 0.1, 0.6, 0.2, 0.1, 0.2)$ and $\alpha = 0.8$, $\beta = 0.2$, equations (24) are represented in the coordinate systems of coordinates $Ox^1 x^2 x^3$ and $O\xi_1 \xi_2 \xi_3$ in Figs. 1 and 2.

Fig. 1 Equations (Eqs. 24) represented in system of coordinates $Ox^1x^2x^3$

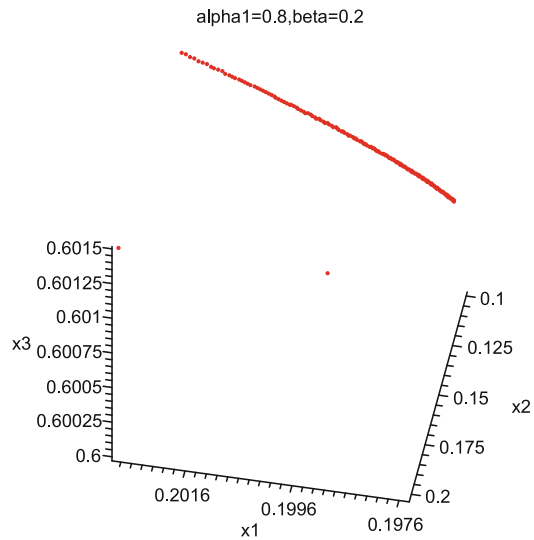
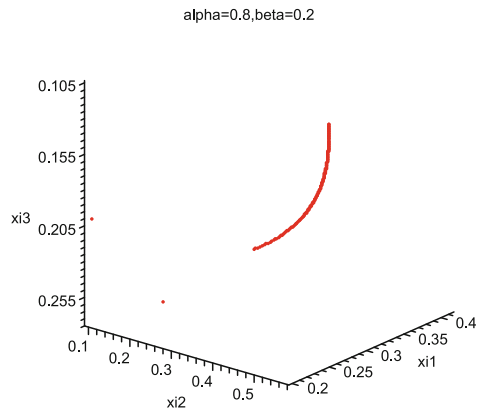


Fig. 2 Equations 24 represented in system of coordinates $O\xi_1\xi_2\xi_3$



In this paper we used Poisson realizations for Rabinovich dynamical systems. We considered a Leibniz algebroid structure on $\mathbb{R}^3$. Rabinovich fractional dynamical systems are obtained as dynamical systems on these spaces. The qualitative study for these systems will be presented in our future work.

References

1. Baleanu D, Agrawal P (2006) Fractional Hamilton formalism within Caputo's derivative. Czech J Phys 56(10–11):1087–1092
2. Chiş O, Puta M (2008) The dynamics of Rabinovich system. Differential geometry-dynamical systems, Vol. 10. Geometry Balkan Press, pp 91–98
3. Deng WH, Li CP (2005) Chaos synchronize of fractional Lü system. Physica A 353:61–72

4. Diethelm K (2003) Fractional differential equations, theory and numerical treatment, preprint. Technical University of Braunschweig
5. Ivan Gh, Ivan M, Oprea D (2007) Fractional dynamical systems on fractional Leibniz algebroids, *Analele Științifice ale Universității "Al. I. Cuza" din Iași (S.N.)*, Matematica, Tomul LIII, Suppliment, 223–234
6. Podlubny I (1999) Fractional differential equations. Academic, San Diego
7. Puta M (1993) Hamiltonian mechanical systems and geometric quantization, *Mathematics and its applications*, Vol. 260, Kluwer, Amsterdam
8. Tarasov VE (2006) Fractional variations for dynamical systems: Hamilton and Lagrange Approaches. *J Phys A* 39(26):8409–8425

Generalized Hankel Transform and Fractional Integrals on the Spaces of Generalized Functions

Kuldeep Singh Gehlot and Dinesh N. Vyas

Abstract The aim of present paper is to extend the definition of generalized Hankel transform (GHT) to certain spaces of generalized functions. A transformable generalized function is defined for the present purpose. An analyticity theorem is proved as a mark of justification. The products of GHT and certain Fractional Integral operators of generalized functions have also been studied. A few particular/special cases are also pointed out.

1 Introduction

Numerous research papers and monographs have been published on the theory and application of generalized functions to the transform analysis. One may refer to [5, 10], extended the theory of fractional integration to the spaces of generalized functions [8], Studied the fractional integrals earlier defined by [7], that involve Gauss hypergeometric functions, on the spaces $F_{p,\mu}$ and $F'_{p,\mu}$ of McBride. In two subsequent papers [2, 3] studied the product of Laplace transform and fractional integration and Stieltjes transform respectively on the same spaces. In the present paper our aim at the discussion of Hankel transform, an important integral transform to deal with, together with its association with fractional integration on the spaces of certain generalized functions defined by McBride.

Section 2 of the paper deals with certain definition and notations, that are used in the sequel, for the space $F_{p,\mu}$. In Sect. 3 we recall the definitions of fractional integration and their mapping properties on $F_{p,\mu}$ and $F'_{p,\mu}$. Section 4 comprises of

K.S. Gehlot

Department of Mathematics, Bangur Government, P.G. College Pali, Rajasthan, India
e-mail: drksgehlot@yahoo.com

D.N. Vyas (✉)

M.L.V. Textile Institute Bhilwara – 311001, Rajasthan, India
e-mail: drdnvyas@yahoo.co.in

the development of the theory of GHT of generalized functions on these spaces justified under an analyticity theorem. In two subsequent sections we give the GHT of fractional integrals and fractional integrals of GHT on $F'_{p,\mu}$.

In support of the present results we report certain special cases of them (i.e. relationships of GHT with the operators of fractional integration due to Riemann–Liouville, Erdelye–Kober and Weyl).

2 Notations and Definitions

2.1 Notations

$\mathbb{N}$ (resp. $\mathbb{N}_0$) denote the set of positive (resp. non-negative) integers, $\mathbb{R}^+$ (resp. $\mathbb{R}_0^+$) the set of positive (resp. non-negative) real numbers, and $\mathbb{R}$ (resp. $\mathbb{C}$) the field of real (resp. complex) numbers. Following [5], the sets C^+ , H_0 and $\overline{H_0}$ are defined as

$$\begin{aligned} C^+ &= \{s : s \in \mathbb{C} \setminus (-\infty, 0]\}, \\ H_0 &= \{s : s \in \mathbb{C}, \operatorname{Re}(s) > 0\}, \\ \overline{H_0} &= \{s : s \in \mathbb{C}, \operatorname{Re}(s) \geq 0\} \end{aligned}$$

$L_p(\mathbb{R}^+)$ is the space of measurable functions on $\mathbb{R}^+$, such that

$$\|f\|_p = \left[\int_0^\infty |f(t)|^p dt \right]^{1/p}, \quad (1 \leq p < \infty)$$

or

$$\|f\|_\infty = \operatorname{ess\,sup}_{0 < t < \infty} |f(t)|, \quad (p = \infty)$$

is finite.

As usual let p, q be positive real numbers or ∞ with $p^{-1} + q^{-1} = 1$.

By D_z^k we denote the differentiation operator

$$D_z^k \equiv \frac{d^k}{dz^k}, \quad (k \in \mathbb{N}_0)$$

Further more, we shall use the terminology of [10], in case of the terms that are not defined explicitly for the present purpose.

2.2 The Spaces $F_{p,\mu}$ and Their Duals

The spaces $F_{p,\mu}$ of test functions given by [5, 6] are:

Definition 1. For $\mu \in C$ and $\varphi \in C^\infty(R^+)$, set

$$F_{p,o} = \left\{ \phi : x^K \frac{d^K \phi}{dx^K} \in L_P(R^+), k \in N_0, (1 \leq P < \infty) \right\}$$

and

$$F_{\infty,o} = \left\{ \phi : x^k \frac{d^k \phi}{dx^k} \rightarrow 0 \text{ as } x \rightarrow +0 \text{ and as } x \rightarrow +\infty, k \in N_0 \right\},$$

If $1 \leq p \leq \infty$, let $F_{p,\mu} = \{ \varphi : x^{-\mu} \varphi(x) \in F_{p,0} \}$

The topology in $F_{p,\mu}$ is induced by the system of seminorms $\gamma_k^{p,\mu}$, $k \in N_0$ defined as

$$\gamma_k^{p,\mu}(\varphi) = \left\| x^k \frac{d^k}{dx^k} (x^{-\mu} \varphi(x)) \right\|_p \quad (1)$$

Remark 1. $F_{p,\mu}$ can be proved to be a countable multinormed space.

$$F_{p,\mu} = F_{p,\text{Re}(\mu)} \quad (2)$$

Defining operators x^λ ($\lambda \in C$) and D on $F_{p,\mu}$ as usual

$$(x^\lambda \varphi)(x) = x^\lambda \varphi(x), (D\varphi)(x) = \frac{d\varphi}{dx} \quad (3)$$

Lemma 1. (i) The operator x^λ is a homeomorphism of $F_{p,\mu}$ on to $F_{p,\mu+\lambda}$. (ii) The operator D is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu-1}$ and it is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu-1}$ iff $\text{Re}(\mu) \neq P^{-1}$.

Definition 2. $F'_{p,\mu}$ denote the space of continuous linear functionals on $F_{p,\mu}$ equipped with the weak topology. The value of $f \in F'_{p,\mu}$ at a function $\varphi \in F_{p,\mu}$ is denoted by (f, φ) .

By Lemma 1, we write

$$\begin{aligned} (x^\lambda f, \varphi) &= (f, x^\lambda \varphi), f \in F'_{p,\mu}, \varphi \in F_{p,\mu-\lambda}, \\ (Df, \varphi) &= (f, -D\varphi), f \in F'_{p,\mu}, \varphi \in F_{p,\mu+1} \end{aligned}$$

Lemma 2. (i) The operator x^λ is a homeomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-\lambda}$. (ii) The operator D is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu+1}$ and it is a homeomorphism if $\text{Re}(\mu) \neq -q^{-1}$.

3 Fractional Integrals

[7], Introduced and studied the generalized fractional integrals involving Gauss' Hypergeometric function in the kernel. For the sake of our convenience we use them as:

Definition 3. Let $\alpha, \beta, \eta \in C, \operatorname{Re}(\alpha) > 0$. The integral operators $I^{\alpha, \beta, \eta}$ and $J^{\alpha, \beta, \eta}$ are defined by

$$I^{\alpha, \beta, \eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x}\right) f(t) dt \quad (4)$$

and

$$J^{\alpha, \beta, \eta} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t}\right) f(t) dt \quad (5)$$

Where ${}_2F_1(\cdot)$ is Gauss' Hypergeometric function.

Definition 4. Let $A_{p, \mu}$ (resp. $B_{p, \mu}$) denote the set of complex number ξ satisfying

$$\operatorname{Re}(\xi) \neq \operatorname{Re}(\mu) - p^{-1} + n \text{ (resp. } \operatorname{Re}(\xi) \neq \operatorname{Re}(\mu) - p^{-1} + 1 - n) \\ \text{for all } n \in N$$

Theorem 1. The operators $I^{\alpha, \beta, \eta}$ (resp. $J^{\alpha, \beta, \eta}$) is a continuous linear mapping of $F_{p, \mu}$ into $F_{p, \mu-\beta}$ provided $\{0, \beta - \eta\} \subset A_{p, \mu}$ (resp. $\{\beta, \eta\} \subset B_{p, \mu}$). (cf. [8]).

Definition 5. Let $f \in F'_{p, \mu}$ and let $\{0, \beta - \eta\} \subset A_{q, -\mu}$ (resp. $\{\beta, \eta\} \subset B_{q, -\mu}$). Then the operator $I^{\alpha, \beta, \eta}$ (resp. $J^{\alpha, \beta, \eta}$) is defined on $F'_{p, \mu}$ by

$$\left(I^{\alpha, \beta, \eta} f, \varphi\right) = \left(f, J^{\alpha, \beta, \eta} \varphi\right), \varphi \in F_{p, \mu+\beta} \quad (6)$$

$$\text{resp. } \left(J^{\alpha, \beta, \eta} f, \varphi\right) = \left(f, I^{\alpha, \beta, \eta} \varphi\right), \varphi \in F_{p, \mu+\beta}. \quad (7)$$

Theorem 2. The operator $I^{\alpha, \beta, \eta}$ (resp. $J^{\alpha, \beta, \eta}$) is a continuous linear mapping of $F'_{p, \mu}$ into $F'_{p, \mu+\beta}$ provided $\{0, \beta - \eta\} \subset A_{q, -\mu}$ (resp. $\{\beta, \eta\} \subset B_{q, -\mu}$)

4 Generalized Hankel Transform on $F'_{p, \mu}$

[4], gave a generalization of the Hankel transform as follows

$$H[f(t)] = \bar{f}(s) = \int_0^\infty (st)^v J_\lambda^\tau(st) f(t) dt \quad (8)$$

Where $J_\lambda^\tau(st)$ is the Maitland's generalized Bessel function given by

$$J_\lambda^\tau(st) = \sum_{r=0}^{\infty} \frac{(-st)^r}{r! \Gamma(1 + \lambda + \tau r)} \quad (9)$$

which processes the following H - function representation.

$$J_\lambda^\tau(st) = H_{0,2}^{1,0} \left[st \left| \begin{matrix} \cdots, \cdots \\ (0, 1), (-\lambda, \tau) \end{matrix} \right. \right] \quad (10)$$

and $H_{0,2}^{1,0}(\cdot)$ is the well known Fox's H-function (cf. [9]).

Proposition 1. Let $t \in R^+$ and $s \in C_+$ then $g(st) \in F_{p,\mu}$ if $(\nu) < \text{Re}(\mu) < p^{-1}$

where

$$g(st) = (st)^\nu J_\lambda^\tau(st) \quad (11)$$

Proof. Since $F_{p,\mu} = F_{p,\text{Re}(\mu)}$, we assume $\mu \in R$. Let, $1 \leq p < \infty$, then we have

$$t^k \frac{d^k}{dt^k} (t^{-\mu} g(st)) = t^k \frac{d^k}{dt^k} (t^{-\mu} (st)^\nu J_\lambda^\tau(st))$$

An apply (9) leads to

$$t^k \frac{d^k}{dt^k} (t^{-\mu} g(st)) = t^k s^\nu \sum_{r=0}^{\infty} \frac{(-s)^r (r + \nu - \mu)! t^{r+\nu-\mu-k}}{r! \Gamma(1 + \lambda + \tau r) (r + \nu - \mu - k)!} \quad (12)$$

The system of seminorms in the countable multinormed space $F_{p,\mu}$ of testing functions is defined by

$$y_k^{p,\mu} [g(st)] = \left\| t^k D_t^k (t^{-\mu} g(st)) \right\|_p$$

therefore from (12) we conclude that $y_k^{p,\mu} [g(st)]_p < \infty$. if $p\mu > 1$ (behaviour at zero) and $(\nu - \mu) p < 0$ (behaviour at $+\infty$).

For the case when $p = \infty$, we demand $|t^k D_t^k (t^{-\mu} g(st))| \rightarrow 0$ which leads to the restrictions $\nu < \text{Re}(\mu) < p^{-1}$

Proposition 2. Let $f \in F'_{p,\mu}$, $s \in C_+$ and $\nu < \text{Re}(\mu) < p^{-1}$, The GHT, H of f is defined by

$$H[f](s) = \bar{f}(s) = (f(t), (st)^\nu J_\lambda^\tau(st)) \quad (13)$$

Analiticity theorem:

The GHT, H of $f \in F'_{p,\mu}$ is a holomorphic function of s on C_+ and it holds that

$$D_s^k f(s) = \left(f(t), D_s^k g(st) \right), \quad k \in N_0 \quad (14)$$

Proof. Let $s, s+h \in C_+, x \in R^+$. We consider the function

$$\psi_h(t) = \frac{1}{h} [g(s+h)t - g(st)] - \frac{\partial}{\partial s} \{g(st)\}$$

we prove, for the justification of above theorem, the following

$$\lim_{h \rightarrow 0} \gamma_k^{p,\mu}(\psi_h) = 0, \quad k \in N_0$$

we proceed to write

$$\begin{aligned} D_t^k [t^{-\mu} \psi_h(t)] &= \frac{\partial^k}{\partial t^k} \left[t^{-\mu} \left\{ \frac{g(s+h)t - g(st)}{h} - \frac{\partial}{\partial s} g(st) \right\} \right] \\ &= \sum_{j=0}^k D_t^{k-j} t^{-\mu} D_t^j \left(\frac{g(s+h)t - g(st)}{h} - \frac{\partial}{\partial s} g(st) \right) \\ &= \sum_{j=0}^k C_j t^{-\mu-k+j} \left\{ \frac{g_{j_t}((s+h)t) - g_{j_t}(st)}{h} - \frac{\partial}{\partial s} g_{j_t}(st) \right\} \end{aligned}$$

where C_j are certain constants and suffix j_t with g indicates j times partial differentiation with respect to 't'.

Further, let L denotes a closed circle with centre at s and radius r_1 . The Cauchy's integral formula suggests the following.

$$\begin{aligned} &= \sum_{j=0}^k C_j t^{-\mu-k+j} \frac{1}{2\pi i} \int_L g_{j_t}(zt) \left[\frac{1}{h} \left\{ \frac{1}{z-s-h} - \frac{1}{z-s} \right\} - \frac{1}{(z-s)^2} \right] dz \\ &= \sum_{j=0}^k C_j t^{j-\mu-k} \frac{h}{2\pi i} \int_L \frac{g_{j_t}(zt) dz}{(z-s-h)(z-s)^2} \end{aligned}$$

Now, we consider the estimates, such that

$$\begin{aligned} y_k^{p,\mu}(\psi_h) &= \left| t^k D_t^k \{t^{-\mu} \psi_h(t)\} \right| \leq \sum_{j=0}^k \frac{|h| |M| |C_j| 2\pi r_1}{2\pi(r-r_1)r_1^2} \\ &\leq \sum_{j=0}^k \frac{|h| |M| |C_j|}{r_1(r-r_1)} \end{aligned}$$

where $|t^{j-\mu} g_{j_t}(zt)| \leq M$. Hence the left hand side of above expression tends to zero as $h \rightarrow 0$.

5 GHT of Fractional Integrals on $F'_{p,\mu}$

Theorem 3. Let $f \in F'_{p,\mu}$, $\operatorname{Re}(\alpha) > 0$

$$\text{and } \operatorname{Re}(v - \beta) < \operatorname{Re}(\mu) < \frac{1}{p} - \operatorname{Re}(\beta)$$

Then for $\{0, \beta - \eta\} \subset A_{q,-\mu}$

$$H \left[I^{\alpha,\beta,\eta} f \right] (s) = (f(x), \Phi_1(s, x)) \quad (15)$$

And for $\{\beta, \eta\} \subset B_{q,-\mu}$

$$H \left[J^{\alpha,\beta,\eta} f \right] (s) = (f(x), \Phi_2(s, x)) \quad (16)$$

where

$$\Phi_1(s, x) = J^{\alpha,\beta,\eta} \left[(st)^v J_\lambda^\tau(st) \right] \quad (17)$$

$$= x^{-\beta} H_{2,4}^{1,2} \left[sx \left| \begin{matrix} (1-\beta, -1), (1-\eta, -1), & \text{---}, & \text{---} \\ (v, 1), (v\tau - \lambda, \tau), (1, -1), (1-\alpha - \beta - \eta, -1) \end{matrix} \right. \right]$$

$$\Phi_2(s, x) = I^{\alpha,\beta,\eta} \left[(st)^v J_\lambda^\tau(st) \right] \quad (18)$$

$$= x^{-\beta} H_{2,4}^{1,2} \left[sx \left| \begin{matrix} (0, 1), (\beta - \eta, 1), & \text{---}, & \text{---} \\ (v, 1), (v\tau - \lambda, \tau), (\beta, 1), (-\alpha - \eta, 1) \end{matrix} \right. \right]$$

Proof. The expressions (15) and (16) are readily expressible with the aid of definitions 3.1, 3.3, Proposition 4.1 and Theorems 3.1, 3.2 combined.

For the proof of (17) and (18), we proceed as follows

$$\Phi_1(s, x) = J^{\alpha,\beta,\eta} \left[(st)^v J_\lambda^\tau(st) \right]$$

use of (5), (9) and ([1], Eq. 1.1.33, p.156) permit us to write

$$= x^{-\beta} \sum_{r=0}^{\infty} \frac{(sx)^{v+r} (-1)^r \Gamma(\beta - v - r) \Gamma(\eta - v - r)}{r! \Gamma(1 + \lambda + \tau r) \Gamma(-v - r) \Gamma(\alpha + \beta + \eta - v - r)}$$

using ([9], Eq. 2.6.11 p.19) we obtain

$$= x^{-\beta} (sx)^v H_{2,4}^{1,2} \left[sx \left| \begin{matrix} (v - \beta + 1, -1), (v + 1 - \eta, -1), & \text{---}, & \text{---} \\ (0, 1), (-\lambda, \tau), (v + 1, -1), (v + 1 - \alpha - \beta - \eta + v, -1) \end{matrix} \right. \right]$$

which finally shapes into

$$\Phi_1(s, x) = x^{-\beta} H_{2,4}^{1,2} \left[sx \left| \begin{matrix} (1 - \beta, -1), (1 - \eta, -1), & \text{---}, & \text{---} \\ (v, 1), (-\lambda + v\tau, \tau), (1, -1), (1 - \alpha - \beta - \eta, -1) \end{matrix} \right. \right]$$

where we have used ([9], Eq. 2.3.6, p.15).

Now

$$\Phi_2(s, x) = I^{\alpha, \beta, \eta} \left[(st)^\nu J_\lambda^\tau(st) \right]$$

Using (4), (9) and ([1], Eq. 1.1.33, p.156) we get

$$= x^{-\beta} \sum_{r=0}^{\infty} \frac{(sx)^{r+\nu} (-1)^r \Gamma(\nu+r+1) \Gamma(\nu+r+\eta+1-\beta)}{r! \Gamma(1+\lambda+\tau r) \Gamma(\nu+r+1-\beta) \Gamma(\alpha+\nu+\eta+r+1)}$$

By using ([9], Eq. 2.6.11, p.19), permit us to write

$$x^{-\beta} (sx)^\nu H_{2,4}^{1,2} \left[sx \left| \begin{matrix} (-\nu, 1), (\beta-\nu-\eta, 1), & \text{---}, & \text{---} \\ (0, 1), (-\lambda, \tau), (\beta-\nu, 1), & (-\alpha-\nu-\eta, 1) \end{matrix} \right. \right]$$

Referring to ([9], Eq. 2.3.6, p.15), we write

$$\Phi_2(s, x) = x^{-\beta} H_{2,4}^{1,2} \left[sx \left| \begin{matrix} (0, 1), (\beta-\eta, 1), & \text{---}, & \text{---} \\ (\nu, 1), (\nu\tau-\lambda, \tau), (\beta, 1), & (-\alpha-\eta, 1) \end{matrix} \right. \right]$$

In what follows as a mark of usefulness of our results, we report a few special cases of (15) and (16).

1. If we set $\beta = 0$ in (15) and (17), we get

$$\begin{aligned} H[E^{\alpha, \eta} f](s) &= H[I^{\alpha, 0, \eta} f](s) \\ &= \left(f(x), H_{1,3}^{1,1} \left[sx \left| \begin{matrix} (1-\eta, -1), & \text{---}, & \text{---} \\ (\nu, 1), (\nu\tau-\lambda, \tau), (1-\alpha-\eta, -1) \end{matrix} \right. \right] \right) \end{aligned}$$

(Erdely–Kober)

2. If we put $\beta = 0$ in (16) and (18), we get

$$\begin{aligned} H[K^{\alpha, \eta} f](s) &= H[J^{\alpha, 0, \eta} f](s) \\ &= \left(f(x), H_{1,3}^{1,1} \left[sx \left| \begin{matrix} (-\eta, 1), & \text{---}, & \text{---} \\ (\nu, 1), (\nu\tau-\lambda, \tau), (-\alpha-\eta, 1) \end{matrix} \right. \right] \right) \end{aligned} \quad (\text{Erdely–Kober})$$

3. Put $\beta = -\alpha$ in (15) and (17), we obtain

$$\begin{aligned} H[R^\alpha f](s) &= H[I^{\alpha, -\alpha, \eta} f](s) \\ &= \left(f(x), x^\alpha H_{1,3}^{1,1} \left[sx \left| \begin{matrix} (1+\alpha, -1), & \text{---}, & \text{---} \\ (\nu, 1), (\nu\tau-\lambda, \tau), (-1, 1) \end{matrix} \right. \right] \right) \end{aligned}$$

(Riemann–Liouville)

4. Put $\beta = -\alpha$ in (16) and (18), we get

$$= \left(f(x), x^\alpha H_{1,3}^{1,1} \left[sx \left| \begin{matrix} (0, 1), & \text{-----}, & \text{-----} \\ (v, 1), & (v\tau - \lambda, \tau), & (-\alpha, -1) \end{matrix} \right. \right] \right)$$

(Weyl)

6 Fractional Integral of GHT on $F'_{p,\mu}$

Theorem 4. Let $f \in F'_{p,\mu}$, $\operatorname{Re}(\alpha) > 0$ and $\operatorname{Re}(v - \beta) < \operatorname{Re}(\mu) < \frac{1}{p} - \operatorname{Re}(\beta)$, then

$$\begin{aligned} & \text{for } \{\beta, \eta\} \subset B_{q,-\mu} \text{ and } \operatorname{Re}(\beta - \eta) < 0 \\ & I^{\alpha,\beta,\eta} [H f](x) = (f(t), \Phi_2(x, t)) \end{aligned} \quad (19)$$

And for $\{0, \beta - \eta\} \subset A_{q,-\mu}$.

$$J^{\alpha,\beta,\eta} [H f](x) = (f(t), \Phi_1(x, t)) \quad (20)$$

Where Φ_1 and Φ_2 are given in (17) and (18)

Proof. Above two results (19) and (20) can be proved on the guidelines of the proofs of propositions (19) and (20) of [3].

Special Cases.

1. Put $\beta = 0$ in (19) we get

$$\begin{aligned} E^{\alpha,\eta} [H f](x) &= I^{\alpha,0,\eta} [H f](x) \\ &= \left(f(t), H_{1,3}^{1,1} \left[xt \left| \begin{matrix} (-\eta, 1), & \text{-----}, & \text{-----} \\ (v, 1), & (v\tau - \lambda, \tau), & (-\alpha - \eta, 1) \end{matrix} \right. \right] \right) \end{aligned}$$

(Erdely–Kober)

2. Put $\beta = 0$ in (20) we get

$$\begin{aligned} K^{\alpha,\eta} [H f](x) &= J^{\alpha,0,\eta} [H f](x) \\ &= \left(f(t), H_{1,3}^{1,1} \left[xt \left| \begin{matrix} (1 - \eta, -1), & \text{-----}, & \text{-----} \\ (v, 1), & (v\tau - \lambda, \tau), & (1 - \alpha - \eta, -1) \end{matrix} \right. \right] \right) \end{aligned}$$

(Erdely–Kober)

3. Put $\beta = -\alpha$ in (19) we get

$$\begin{aligned} R^\alpha [H f](x) &= I^{\alpha,-\alpha,\eta} [H f](x) \\ &= \left(f(t), t^\alpha H_{1,3}^{1,1} \left[xt \left| \begin{matrix} (0, 1), & \text{-----}, & \text{-----} \\ (v, 1), & (v\tau - \lambda, \tau), & (-\alpha, 1) \end{matrix} \right. \right] \right) \end{aligned}$$

(Riemann–Liouville)

4. Put $\beta = -\alpha$ in (20) we get

$$W^\alpha [Hf](x) = J^{\alpha, -\alpha, \eta} [Hf](x) \\ = \left(f(t), t^\alpha H_{1,3}^{1,1} \left[xt \left| \begin{matrix} (1+\alpha, -1), -\dots, -\dots \\ (v, 1), (v\tau - \lambda, \tau), (1, -1) \end{matrix} \right. \right] \right)$$

(Weyl)

7 Concluding Remarks

The results of the present paper are supposed to be new and more general in character. By suitably adjusting the parameters and constants involved in the results involving GHT, a number of known and new results can be obtained in terms of certain well known integral transforms like Laplace, Stieltjes and generalized Stieltjes transforms. (cf. [2, 3]).

References

1. Exton H (1978) Handbook of hypergeometric integrals: theory applications. Tables, computer programmes. Ellis Horwood, Chichester
2. Glaeske HJ, Saigo M (1992) Product of Laplace transform and fractional integrals on spaces of generalized functions. Math Japonica 37(2):373–382
3. Glaeske HJ, Saigo M (1994). Stieltjes transform and fractional integrals on spaces of generalized functions. Math Japonica 39(1):127–135
4. Kumar R (1959) Certain convergence theorems connected with a generalized Hankel transform. J Indian Math Soc XXIII, 3 & 4 (126)
5. McBride AC (1975) A theory of fractional integration for generalized functions. SIAM J Math Anal 6(3):583–599
6. McBride AC (1979) Fractional Calculus and Integral Transforms of Generalized Functions. Pitman (Res. Notes in Math. 31), London
7. Saigo M (1978) A remark on integral operators involving the Gauss' hypergeometric function. Math Rep Coll Gen Edu 11:135–143, Kyushu University
8. Saigo M, Glaeske HJ (1990) Fractional calculus operators involving the Gauss function on spaces $F_{p,\mu}$ and $F'_{p,\mu}$. Math Nachr 147:285–306
9. Srivastava HM, Gupta KC, Goyal SP (1982) The H-functions of one and two variables with applications. South Asian Publishers, 36, Netaja Subhash Marg, Daryaganj, New Delhi
10. Zemanian AH (1968) Generalized integral transformations. Interscience, New York

Some Bounds on Maximum Number of Frequencies Existing in Oscillations Produced by Linear Fractional Order Systems

Sadegh Bolouki, Mohammad Haeri, Mohammad Saleh Tavazoei, and Milad Siami

Abstract In this paper, it has been studied that how the inner dimension of a fractional order system influences the maximum number of frequencies which may exist in oscillations produced by this system. Both commensurate and incommensurate systems have been considered to clarify the relationship between the inner dimension and maximum number of frequencies. It has been shown that although in commensurate fractional order systems, like integer order systems, the maximum number of frequencies is half of the inner dimension, in incommensurate systems the problem is significantly more complicated and the relationship between the inner dimension and maximum frequencies can not precisely determined. However, in this article, some upper and lower bounds, depending on the inner dimension, have been provided for the maximum frequencies of incommensurate systems.

1 Introduction

Study on dynamical behaviours of the fractional order systems has attracted increasing attention in the recent years. Analysis of oscillatory behaviour is one of the main issues in these studies. It has been found that fractional order systems can generate undamped oscillations like as the classical integer order systems. The generated oscillations may be regular or irregular (chaotic). Existence of regular oscillatory behaviours in the nonlinear fractional order systems and its basic analysis is subject of some papers such as [1–3]. Moreover, oscillations generated by marginally stable linear fractional order systems have been analyzed in [4,5]. Also, existence of chaos in fractional order models is another topic in this category which has been studied in some papers such as [5–7].

S. Bolouki, M. Haeri (✉), M.S. Tavazoei, and M. Siami
Advanced Control System Lab., Electrical Engineering Department, Sharif University of Technology, Azadi Ave., P.O. Box 11155-9363, Tehran, Iran
e-mail: saadegh@ee.sharif.edu; haeri@sina.sharif.edu; m_tavazoei@ee.sharif.edu; msiami@ee.sharif.edu

In this paper, the oscillations generated by linear fractional order systems are analyzed according to their frequency content. The outcome of the paper could be disclosing the relation between the inner dimension of a system and maximum number of frequencies which can exist in oscillations generated by this system. This paper is organized as follows. In Sect. 2, the question which reveals the relation between the inner dimension and maximum number of frequencies is mathematically formulated. The answer of this question for fractional order systems with commensurate order is obvious and is presented in Sect. 2. Although for fractional order systems with incommensurate order the exact relation between the inner dimension and maximum number of frequencies is not specified, two bounds are given for the maximum number of frequencies according to the inner dimension. The upper bound and lower bound are respectively found in Sects. 3 and 4. Finally, the paper is concluded in Sect. 5.

2 Problem Statement

There are some different definitions for fractional derivative as extension of ordinary derivative. The most commonly used definitions are Riemann–Liouville, Grunwald–Letnikov, and Caputo definitions. The Riemann–Liouville definition of derivative with terminal value 0 is given as

$$D^\alpha f(t) = \frac{d^m}{dt^m} \{J^{m-\alpha} f(t)\}, \quad (1)$$

where m is the first integer which is not less than α , and J^α denotes the α^{th} order fractional integral operator defined as follows [8].

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \quad (2)$$

A fractional order system may be defined in the following state space form

$$\frac{d^{\alpha_i} x_i}{dt^{\alpha_i}} = f_i(x_1, x_2, \dots, x_n), \quad x_i(0) = x_{i0}, \quad i = 1, 2, \dots, n. \quad (3)$$

where $0 < \alpha_i \leq 1$ for $i = 1, 2, \dots, n$, and $\frac{d^{\alpha_i}}{dt^{\alpha_i}}$ refers to smooth fractional derivation [9], i.e., $\frac{d^{\alpha_i} f(t)}{dt^{\alpha_i}} = D^{\alpha_i} f(t) - \frac{t^{-\alpha_i}}{\Gamma(1-\alpha_i)} f(0)$. n is called the inner dimension of system (3) [9]. If $\alpha_1 = \alpha_2 = \dots = \alpha_n$, system (3) is called a commensurate order system, otherwise system (3) indicates an incommensurate order system. Existence and uniqueness of solution of system of fractional differential Eq. 3 for a given initial condition have been proved in [10]. Consider the following linear fractional order system

$$\left\{ \begin{array}{l} \frac{d^{\alpha_1} x_1}{dt^{\alpha_1}} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \frac{d^{\alpha_2} x_2}{dt^{\alpha_2}} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ \frac{d^{\alpha_n} x_n}{dt^{\alpha_n}} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n, \end{array} \right. \quad (4)$$

where all α_i 's are rational numbers between 0 and 1. In this case, system (4) generally is an incommensurate fractional order system which is reducible to commensurate fractional order system. Let M be the lowest common multiple of the denominators u_i 's of α_i 's, where $\alpha_i = v_i/u_i$, $(u_i, v_i) = 1$, $u_i, v_i \in N$ for $i = 1, 2, \dots, n$. The zero solution of system (4) is globally asymptotically stable in the Lyapunov sense if all roots λ 's of the equation

$$P(\lambda) = \begin{bmatrix} \lambda^{M\alpha_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda^{M\alpha_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda^{M\alpha_n} - a_{nn} \end{bmatrix} = 0, \quad (5)$$

satisfy $|\arg(\lambda)| > \pi/2M$ [11]. When $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, this condition converts to Matignon criterion [9] which has been proved for linear fractional order systems with commensurate order. $P(s)$ is called the characteristic polynomial of system (4) [11]. It has been shown that for each pair of conjugate roots $\sigma \pm j\kappa$ of Eq. 5, which are settled on the stability boundary $|\arg(\lambda)| = \pi/2M$, an asymptotic periodic term with frequency $(\sigma^2 + \kappa^2)^{1/2M}$ is appeared in the steady state solutions of (4) [5]. Let us rewrite Eq. 5 in the following form

$$P(s) = \det \left(\begin{bmatrix} s^{N_1} & & 0 \\ & \ddots & \\ 0 & & s^{N_n} \end{bmatrix} \right) = 0, \quad (6)$$

where $N_i = M\alpha_i$'s ($i = 1, 2, \dots, n$) are constant positive integers and A is a matrix in $R^{n \times n}$. It can be easily shown that if $N_1 = N_2 = \cdots = N_n$, Eq. 6 can maximally have $\lfloor n/2 \rfloor$ pairs of conjugate roots settling on the stability boundary $|\arg(s)| = \pi/2M$. Hence, the Fourier spectrum of asymptotic oscillations generated by a linear commensurate order system with the inner dimension n indicates maximum $\lfloor n/2 \rfloor$ different frequencies. But if all N_i 's ($i = 1, 2, \dots, n$) are not equal, the number of pairs of conjugate roots of Eq. 6 settling on the stability boundary $|\arg(s)| = \pi/2M$ may be more than $\lfloor n/2 \rfloor$. For example, in [5] a class of incommensurate fractional order systems with the inner dimension 3 has been introduced which can generate oscillations containing three different frequency harmonics. Now, this question may be struck: "What is the maximum number of different frequencies which can be available in Fourier spectrum of asymptotic

oscillations generated by an incommensurate fractional order system with the inner dimension n ?" Or in other words, "For a given n , what is the maximum number of pairs of conjugate roots of Eq. 6 settling on the boundary $|\arg(s)| = \pi/2M$ where $A \in R^{n \times n}$ is an arbitrary real matrix and N_i 's ($i = 1, 2, \dots, n$) and M are arbitrary positive integers which satisfy $N_i \leq M$ for $i = 1, 2, \dots, n$?" Suppose that $m(n)$ denotes the maximum number mentioned above. In the next sections, an upper bound and a lower bound will be found for $m(n)$.

3 An Upper Bound

In this section, we show that for a given n , there is an upper bound for $m(n)$ which is not infinite. Let the curve γ is defined as follows.

$$\gamma = \{re^{j\pi/2M} | r \in R, r > 0\}. \quad (7)$$

In fact, $m(n)$ is the maximum number of the roots of the Eq. 6 on γ for all possible $A \in R^{n \times n}$, $N_1, N_2, \dots, N_n$ and M where $N_i \leq M$ for $i = 1, 2, \dots, n$. The Eq. 6 can be written as

$$P(s) = \sum_{\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}} c_{i_1, \dots, i_k} s^{N_{i_1} + \dots + N_{i_k}} = 0, \quad (8)$$

where the coefficients are real and depend on the matrix A . To find an upper bound for $m(n)$, we begin with the following theorem.

Theorem 1. Assume that p is a positive integer. If $Q(s)$ is a polynomial with real coefficient such that $Q(s)$ has p nonzero coefficients, then the equation has at most $p - 1$ roots in $R^+ = \{r > 0 | r \in R\}$.

Proof. We use the mathematics induction on p . For $p = 1$, we have $Q(s) = c_1 s^{t_1}$. Therefore, the equation $Q(s) = 0$ has the unique solution $s = 0$, so it has no solution in R^+ . Now, suppose that the theorem is true for $p - 1$ and we prove it for p . Let $Q(s)$ be a polynomial with p nonzero coefficients, that is $Q(s) = c_1 s^{t_1} + c_2 s^{t_2} + \dots + c_p s^{t_p}$, $t_1 > t_2 > \dots > t_p$. Obviously, the number of solutions of the equation $Q(s) = 0$ in R^+ is equal to the number of solutions of $Q(s)/s^{t_p} = 0$ in R^+ . Thus, without loss of generality we can assume that $t_p = 0$. On the other hand, $Q'(s) = dQ(s)/ds$ has $p - 1$ nonzero coefficients, and by hypothesis of the induction, $Q'(s) = 0$ has at most $p - 2$ roots in R^+ . Since $Q'(s) = 0$ has at most $p - 2$, from the mean value Theorem [12] one can conclude that $Q(s)/s^{t_p} = 0$ and therefore $Q(s) = 0$ have at most $p - 1$ roots in R^+ . This concludes proof of the Theorem 1.

The following result can be obtained from the conclusion of Theorem 1.

Corollary 1. Let $Q(s)$ be a polynomial with p nonzero real coefficients, where $p > 0$. If M is an arbitrary positive integer and $r_1 e^{j\pi/2M}, \dots, r_k e^{j\pi/2M}$ are the roots of the equation $Q(s) = 0$, then $k \leq p - 1$.

Proof. Suppose that $Q(s) = c_1 s^{t_1} + \dots + c_p s^{t_p}$, $t_1 > t_2 > \dots > t_p$. We have

$$Q(r_i e^{j\pi/2M}) = c_1 r_i^{t_1} e^{j\pi t_1/2M} + \dots + c_p r_i^{t_p} e^{j\pi t_p/2M}, \quad i = 1, \dots, k.$$

Therefore,

$$\operatorname{Re}\{Q(r_i e^{j\pi/2M})\} = c_1 \cos\left(\frac{\pi t_1}{2M}\right) r_i^{t_1} + \dots + c_p \cos\left(\frac{\pi t_p}{2M}\right) r_i^{t_p} = 0, \quad i = 1, 2, \dots, k.$$

Let us define

$$H(s) = c_1 \cos\left(\frac{\pi t_1}{2M}\right) s^{t_1} + \dots + c_p \cos\left(\frac{\pi t_p}{2M}\right) s^{t_p}.$$

Hence, $H(r_i) = 0$, $i = 1, 2, \dots, k$. On the other hand $H(s)$ is a polynomial with real coefficients such that it has at most p nonzero coefficients (If all coefficients of $H(s)$ are zero, then we consider $c_1 \sin(\frac{\pi t_1}{2M}) s^{t_1} + \dots + c_p \sin(\frac{\pi t_p}{2M}) s^{t_p}$ as $H(s)$, which is the imaginary part instead of the real part, and the proof is continued). Thus, by Theorem 1, the equation $H(s) = 0$ has at most $p - 1$ roots on γ defined by (7). So $k \leq p - 1$, as desired.

Note that $P(s)$ is a polynomial with at most 2^n nonzero coefficients, because the set $\{1, 2, \dots, n\}$ has 2^n subsets, and $\{i_1, i_2, \dots, i_k\}$ which has appeared in the Eq. 8 is a subset of the set $\{1, 2, \dots, n\}$. By Corollary 1, the equation $P(s)$ given by (6) has at most $2^n - 1$ roots on the curve γ . Therefore,

$$m(n) \leq 2^n - 1. \quad (9)$$

4 A Lower Bound

We claim the following lower bound for $m(n)$.

$$\lfloor \frac{1}{2} \left(\frac{n^2}{4} + n \right) \rfloor \leq m(n). \quad (10)$$

The proof of inequality (10) is presented in the remainder. First, let n be an even integer. We put $N - 1 = \dots = N_{n/2} = 1$ and $N_{n/2+1} = \dots = N - n = n/2 + 1$. For simplicity of writing we define $k = n/2 + 1$. Let us define the matrix A as follows

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where A_i 's ($i = 1, 2, 3, 4$) are $n/2 \times n/2$ matrices defined as follows.

$$A_1 = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -a_{k-1} & -a_{k-2} & \cdots & -a_1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \ddots & \\ 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -a_{2k-1} & -a_{2k-2} & \cdots & -a_{k+1} \\ -a_{3k-1} & -a_{3k-2} & \cdots & -a_{2k+1} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{k.k-1} & -a_{k.k-2} & \cdots & -a_{(k-1)k+1} \end{bmatrix} + \begin{bmatrix} a_k \\ a_{2k} \\ \vdots \\ a_{(k-1)k} \end{bmatrix} [a_{k-1} \ a_{k-2} \ \cdots \ a_1],$$

$$A_4 = \begin{bmatrix} -a_k & 1 & 0 \\ -a_{2k} & & \ddots \\ \vdots & 0 & 1 \\ -a_{(k-1)k} & 0 & \cdots & 0 \end{bmatrix}.$$

By a little calculation, it is found that Eq. 8 in this case is equivalent to the equation

$$\sum_{i=0}^{\frac{n^2}{4}+n} a_i s^i = 0, \quad (11)$$

where $a_0 = 1$. The left hand side of Eq. 11 is a polynomial of degree $n^2/4 + n$. Since each coefficient of this polynomial can be arbitrarily chosen, we can choose them such that $\lfloor \frac{1}{2}(\frac{n^2}{4} + n) \rfloor$ of the polynomial roots settle on the curve γ . Hence the inequality (10) is proved for even n . Now, we prove the lower bound for odd integers. If n is an odd integer, put $N_1 = \cdots = N_{(n-1)/2} = 1$ and $N_{(n+1)/2} = \cdots = N_n = (n+1)/2$. For simplicity of writing we define $k = (n+1)/2$. Suppose the matrix A is defined as follows

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where A_i 's ($i = 1, 2, 3, 4$) are $(n-1)/2 \times (n-1)/2$, $(n-1)/2 \times (n+1)/2$, $(n+1)/2 \times (n-1)/2$, and $(n+1)/2 \times (n+1)/2$ matrices respectively and are defined as follows

$$A_1 = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -a_{k-1} & -a_{k-2} & \cdots & -a_1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \ddots & \\ 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -a_{2k-1} & -a_{2k-2} & \cdots & -a_{k+1} \\ -a_{3k-1} & -a_{3k-2} & \cdots & -a_{2k+1} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{(k+1)k-1} & -a_{(k+1)k-2} & \cdots & -a_{k^2+1} \end{bmatrix} + \begin{bmatrix} a_k \\ a_{2k} \\ \vdots \\ a_{k^2} \end{bmatrix} [a_{k-1} \ a_{k-2} \ \cdots \ a_1],$$

$$A_4 = \begin{bmatrix} -a_k & 1 & 0 \\ -a_{2k} & & \ddots \\ \vdots & 0 & 1 \\ -a_{k^2} & 0 & \cdots & 0 \end{bmatrix}.$$

By a little calculation, it is found that Eq. 8 in this case is equivalent to the following equation

$$\sum_{i=0}^{\frac{n^2-1}{4}+n} a_i s^i = 0, \quad (12)$$

with $a_0 = 1$. Since the coefficients of Eq. 12 are arbitrary, we can choose them such that $\lfloor \frac{1}{2}(\frac{n^2-1}{4} + n) \rfloor$ roots of this equation settle on the curve γ . It can be easily verified that $\lfloor \frac{1}{2}(\frac{n^2-1}{4} + n) \rfloor = \lfloor \frac{1}{2}(\frac{n^2}{4} + n) \rfloor$ for odd n , thus inequality (10) is proved for odd n .

5 Conclusion

It is clear that in a marginally stable linear integer order system the maximum number of frequencies which can be observed in undamped oscillations generated by the system (after transient time), is half of its inner dimension. This point was also confirmed for fractional order system with commensurate order. It was shown that a commensurate order system with the inner dimension n can generate oscillations containing maximally $\lfloor n/2 \rfloor$ different frequencies in the steady state. Moreover, two upper and lower bounds are found for the maximum number of frequencies which can exist in oscillations generated by a fractional order system with incommensurate order. It was proved that for an incommensurate fractional order system with the inner dimension n this mentioned maximum denoted by $m(n)$ is between the following lower and upper bounds

$$\lfloor \frac{1}{2}(\frac{n^2}{4} + n) \rfloor \leq m(n) \leq 2^n - 1. \quad (13)$$

To find an exact relation for $m(n)$ or reducing the conservativeness of the given bounds for $m(n)$ could be interesting topics for future work in this field.

References

1. Ahmad W, El-Khazali R, El-Wakil A (2001) Fractional-order Wien-bridge oscillator. *Electr Lett* 37:1110–1112
2. Barbosa RS, Machado JAT, Vingare BM, Calderon AJ (2007) Analysis of the Van der Pol oscillator containing derivatives of fractional order. *J Vib Control* 13(9–10):1291–1301
3. Tavazoei MS, Haeri M (2008) Regular oscillations or chaos in a fractional order system with any effective dimension. *Nonlinear Dynam* 54(3):213–222
4. Radwan AG, El-Wakil AS, Soliman AM (2008) Fractional-order sinusoidal oscillators: design procedure and practical examples. *IEEE Trans Circ Syst I* 55(7):2051–2063
5. Tavazoei MS, Haeri M (2008) Chaotic attractors in incommensurate fractional order systems. *Physica D* 237(20):2628–2637
6. Hartley TT, Lorenzo CF, Qammer HK (1995) Chaos in a fractional order Chua's system. *IEEE Trans Circ Syst I* 42:485–490
7. Tavazoei MS, Haeri M (2007) A necessary condition for double scroll attractor existence in fractional order systems. *Phys Lett A* 367(1–2):102–113
8. Podlubny I (1999) Fractional differential equations. Academic, San Diego
9. Matignon D (1996) Stability result on fractional differential equations with applications to control processing. In: IMACS-SMC Proceedings. Lille, France, pp 963–968
10. Daftardar-Gejji V, Jafari H (2007) Analysis of a system of nonautonomous fractional differential equations involving Caputo derivatives. *J Math Anal Appl* 328:1026–1033
11. Deng W, Li C, L J (2007) Stability analysis of linear fractional differential system with multiple time delays. *Nonlinear Dynam* 48:409–416
12. Adams RA (2006) *Calculus: a complete course*, 6th edn. Addison Wesley, Toronto

Fractional Derivatives with Fuzzy Exponent

Witold Kosiński

Abstract Fractional derivatives with fuzzy exponent defined with the help of new type of fuzzy numbers are introduced. New model of fuzzy numbers, called ordered fuzzy numbers (OFN), invented in 2003 by the author and his coworkers, is also shortly presented. The model contains all convex fuzzy numbers and allows to withdraw main drawbacks of classical operations based on the extension principle of Zadeh. It forms a tool in which calculation on fuzzy concept is rigorous and the same as with real numbers. Formulas for fractional and integral operators presented in the paper are based on Caputo definition. They generalize the concept of interval fractional derivatives. Some examples are also given.

1 Introduction

Fractional Calculus (FC) stems from the beginning of theory of differential and integral calculus [21, 23]. In recent years FC has been a fruitful field of research in science and engineering. It seems that in the last two decades, fractional differentiation has played an increasing role in various fields such as mechanics (viscoelasticity/damping), electricity, electronics, chemistry, biology, economics and notably control theory, robotics, image and signal processing, diffusion and wave propagation [7, 20, 25, 28, 29].

Recently, the application of the theory of FC to robotics revealed promising aspects for future developments. Another and very broad field of recent applications of FC is biomechanics. Some diseases such as osteoporosis are caused by biochemical and hormonal changes in human body. They lead to modification of the structure and composition of bones such as porosity and thickness of trabeculae, as well

W. Kosiński (✉)

Department of Computer Science, Polish-Japanese Institute of Information Technology
ul. Koszykowa 86, 02-008 Warsaw, Poland

Kazimierz Wielki University in Bydgoszcz, ul. Chodkiewicza 30, 85-064 Bydgoszcz, Poland
e-mail: wkos@pjwstk.edu.pl; wkos@ukw.edu.pl

as to mineral density changes. Since trabecular bone is an inhomogeneous porous medium with viscoelastic properties, to measure those changes ultrasonic methods are used. The interaction between ultrasound and bone is highly complex. Modelling ultrasonic propagation through trabecular tissue has been considered using porous media theories, such as Biot's theory. Many authors have used fractional calculus as an empirical method to describe the properties of porous and viscoelastic materials [7, 27].

It was shown that fractional-order models capture phenomena and properties that classical integer-order simply neglect. On the other hand there are problems, especially in non-stationary cases, when fixed and precise orders are not sufficient in modelling. A number of papers have recently appeared in which stability of linear time invariant systems with interval fractional orders and interval coefficients was discussed (cf. [3] and the literature cited there). Moreover, they have proposed an extension of FC from the constant-order to variable-order differential and integrals (see [4, 24] and the literature cited there) where a variable order differential operator $\mathcal{D}^q(t)$ acting on a time-dependent function $x(t)$ has appeared. We will go further and propose the next generalization by admitting that the order is a fuzzy number.

In the paper we joint the concept to the fractional derivatives with the new type of fuzzy numbers. If the order of a fractional differential operator becomes an ordered fuzzy number then we face with a *fuzzy fractional derivative*. In the paper such differential operators are defined, their working formulas are presented together with some examples; they generalize the concept of interval fractional derivatives appearing somewhere else.

The organization of the paper is as follows. In Sect. 2 we review main concept of fuzzy sets and fuzzy numbers and give main arguments presented in the series of papers [12, 15, 17, 19] which lead to a generalization of the classical concept of convex fuzzy numbers (CFN). Then the definition of the *ordered fuzzy numbers* (OFN) and their algebra are given in Sect. 4. In Sect. 5 we show that OFN form a Banach space together with the recent concepts related to the defuzzification. Then some interpretations, examples and applications in modelling are presented in Sect. 6. In Sect. 7 fractional differential and integral operators in the framework of ordered fuzzy numbers are defined and shortly discussed for some elementary function.

2 Fuzzy Sets and Numbers

Classical fuzzy numbers are very special fuzzy sets defined on the universe of all real numbers. Fuzzy numbers are of great importance in fuzzy systems. There are two commonly accepted methods of dealing with fuzzy numbers, both basing on the classical concept of fuzzy sets, namely on the membership functions. The first, more general one deals with the so-called convex fuzzy numbers of Nguyen [22], while the second one deals with shape functions and $L - R$ numbers, set up by Dubois and Prade [5].

When operating on convex fuzzy numbers we have the interval arithmetic for our disposal. However, the approximations of shape functions and operations are needed, if one wants to remain within the $L - R$ numbers while following the Zadeh's extension principle [32]. In this representation in most cases the calculation results are not exact and are questionable if some rigorous and exact data are needed, as for example in the control or modelling problems. This can be regarded as a drawback of properties of fuzzy algebraic operations. Moreover, it is well known in the literature that unexpected and uncontrollable results of repeatedly applied operations, caused by the need of making intermediate approximations (remarked in [30]) can appear. This rises a heavy argument for those who still criticize the fuzzy number calculus. Fortunately, it was already noticed by both Dubois and Prade in their recent publications [6] that something is missing in the definition of the fuzzy numbers and the operations on them. The author of the present paper together with his two co-workers: Dominik Ślęzak and Piotr Prokopowicz proposed in 2002 a generalization of the existing concept of fuzzy numbers in order to withdraw their main drawbacks and to form a tool in which new calculation on fuzzy concept will be rigorous and at the same time it will make possible to deal with fuzzy inputs quantitatively, exactly in the same way as with real numbers [16].

Now we would like to refer to one of the very first representations of a fuzzy set defined on a universe X of discourse (the real axis $\mathbf{R}$, say), i.e., on the set of all feasible numerical values (observations, say) of a fuzzy concept (say: variable or physical measurement). In that representation (cf. [31]) a fuzzy set (read here: a fuzzy number) A is defined as a set of ordered pairs $\{(x, \mu_x)\}$, where $x \in X$, and $\mu_x \in [0, 1]$ has been called the grade (or level) of membership of x in A . At this stage, no other assumptions concerning μ_x have been made. Later on, it was assumed that μ_x is (or must be) a function of x .

Assume for the moment that a membership function exists of a (convex) fuzzy number A . Its partial invertibility allows to define two functions a_1, a_2 on $[0, 1]$

$$a_1(\omega) = \mu_A|_{incr}^{-1}(\omega) \text{ and } a_2(\omega) = \mu_A|_{decr}^{-1}(\omega) . \quad (1)$$

that give lower and upper bounds of each ω -cut $A[\omega] := \{x \in \mathbf{R} : \mu_A(x) \geq \omega\} = [a_1(\omega), a_2(\omega)]$. Here the symbol $\mu_A|_{incr}^{-1}$ (or $\mu_A|_{decr}^{-1}(\omega)$) denotes the inverse function of the increasing part (or decreasing part) of the membership function $\mu_A|_{incr}$ (or $\mu_A|_{decr}$). Then the membership function μ_A of A is completely defined by two functions $a_1 : [0, 1] \rightarrow \mathbf{R}$ and $a_2 : [0, 1] \rightarrow \mathbf{R}$. In most cases one assumes that a typical membership function μ_A of a fuzzy number A satisfies convexity assumptions required by Nguyen [22]. As long as one works with CFN that possess continuous membership functions, the two procedures: the Zadeh's extension principle [32] and the ω -cut with the interval arithmetic method [10], give the same results (cf. [1, 2]).

The results of multiple operations on the convex fuzzy numbers are leading to a large growth of the fuzziness, and depend on the order of the operations since the distributive law, which involves the interaction of addition and multiplication, does hold there. Moreover, the use of the extension principle to define the arithmetic

operations on fuzzy numbers is generally numerically inefficient. In our opinion the main drawback is the lack of a solution X to the most simple fuzzy arithmetic equation $A + X = C$ with known fuzzy numbers A and C . If the support of C is greater than that of A a unique solution in the form of a fuzzy number X exists. However, this is the only case. Another drawback is related to the fact that in general $A + B - A$ is not equal to B (cf. [11]).

3 Ordered Fuzzy Numbers

In our approach [12,13,15,17] the concept of membership functions has been weakened by requiring a mere *membership relation*.

Definition 1. By an ordered fuzzy number A we mean an ordered pair (f, g) of functions such that $f, g : [0, 1] \rightarrow \mathbf{R}$ are continuous.

Notice that f and g need not be inverse functions of some membership function. On the other hand if an ordered fuzzy number (f, g) is given in which f is an increasing function and g – decreasing, and such that $f \leq g$, then to this pair a continuous (with the convex graph) function μ can be attached, which can be regarded as a membership function of a convex fuzzy number with an extra arrow which denotes the orientation of the number (cf. Fig. 1). It can be done by the formula, inverse to (1), namely $f^{-1} = \mu|_{incr}$ and $g^{-1} = \mu|_{decr}$. To be in agreement with classical denotations of fuzzy sets (numbers) the independent variable of f and g is denoted by y in Fig. 1 (or s), and the values of them by x . Notice that pairs

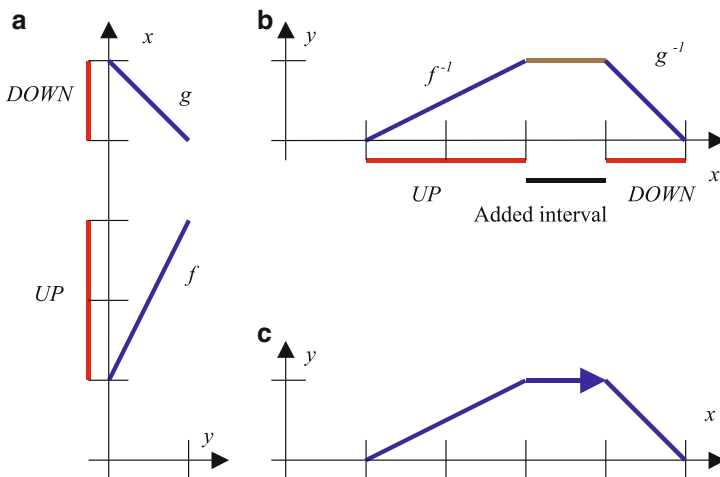


Fig. 1 (a) An example of the ordered fuzzy number, (b) its transformation to a classical view (membership function), (c) short notation of direction in the OFN

(f, g) and (g, f) represents two different ordered fuzzy numbers, unless $f = g$. They differ by their orientations.

Definition 2. Let $A = (f_A, g_A)$, $B = (f_B, g_B)$ and $C = (f_C, g_C)$ are mathematical objects called ordered fuzzy numbers. The sum $C = A + B$, subtraction $C = A - B$, product $C = A \cdot B$, and division $C = A \div B$ are defined by formula

$$f_C(y) = f_A(y) \star f_B(y) \quad g_A(y) \star g_B(y) \quad (2)$$

where “ $\star$ ” works for “+”, “-”, “ $\cdot$ ”, and “ $\div$ ”, respectively, and where $A \div B$ is defined, if the functions $|f_B|$ and $|g_B|$ are bigger than zero. Scalar multiplication by real $r \in \mathbf{R}$ is defined in natural way: $r \cdot A = (rf_A, rg_A)$.

Notice that the subtraction of B is the same as the addition of the opposite of B . It means that subtraction is not compatible with the extension principle, if we confine OFNs to convex fuzzy numbers. However, the addition operation is compatible, if its components have the same orientations. Moreover, any fuzzy algebraic equation $A + X = C$ has a unique solution X within the space $\mathcal{F}$ of OFNs.

If for any pair of affine functions $(f(s), g(s))$ of $s \in [0, 1]$ we form a quaternion (tetrad) of real numbers according to the rule $[f(0), f(1), g(1), g(0)]$, then this tread uniquely determines the ordered fuzzy number $A = (f, g)$. If $(e, h) =: B$ is another pairs of affine functions then the sum $A + B = (f + e, g + h) =: C$ will be uniquely represented by the tread $[f(0) + e(0), f(1) + e(1), g(1) + h(1), g(0) + h(0)]$. This mnemotechnic method makes the calculation easy. However, that addition, as well as subtraction, of two OFNs that are represented by affine functions and possess classical membership functions, i.e. they are convex fuzzy numbers, may lead to result which may not possess its membership functions. We call them in [17] improper fuzzy numbers. Notice that in general $f(1)$ needs not be less than $g(1)$. In this way we can reach improper intervals, as supports of ordered fuzzy numbers, which have been already discussed in the framework of the extended interval arithmetic by Kaucher in [9].

Operations introduced in the space $\mathcal{F}$ make it an algebra with a sup norm $\|A\| = \max(\sup_{s \in I} |f_A(s)|, \sup_{s \in I} |g_A(s)|)$ if $A = (f_A, g_A)$. Moreover, $\mathcal{F}$ becomes a Banach space,¹ isomorphic to a cartesian product of $C(0, 1)$ – the space of continuous functions on $[0, 1]$. Continuous, linear, functionals on $\mathcal{F}$ give a class of defuzzification functionals. Each of them, say ϕ , has the representation by the sum of two Stieltjes integrals with respect to functions h_1, h_2 of bounded variation,

$$\phi(f, g) = \int_0^1 f(s)dh_1(s) + \int_0^1 g(s)dh_2(s). \quad (3)$$

Notice that with if for $h_1(s)$ and $h_2(s)$ we put $1/2H(s)$ with $H(s)$ as the Heaviside function with the unit jump at $s = 1$ then the defuzzification functional in (3) will

¹ In [8] the authors for the first time introduced a linear structure to convex fuzzy numbers.

represent the classical MOM – middle of maximum.

$$\phi(f, g) = 1/2(f(1) + g(1)). \quad (4)$$

New model gives a continuum number of defuzzification operators both linear and nonlinear, which map ordered fuzzy numbers into reals. Nonlinear functional can be defined, cf. [14]. It is worthwhile to point out that OFNs represent the whole class of convex fuzzy numbers with continuous membership functions. Some generalization of the present Definition 1 to include discontinuous functions is proposed in [13].

Neutral element of addition in $\mathcal{F}$ is a pair of the constant functions $(0^\dagger, 0^\dagger)$ equal to zero, i.e. $0^\dagger(s) = 0, \forall s \in [0, 1]$, which is identified with the crisp zero, while the multiplication has a neutral element – the pair of the two constant functions $(1^\dagger, 1^\dagger)$ equal to one, identified with the crisp one.

Adopting the last denotation for the constant function we will write for any real number $r \in \mathbf{R}$ the corresponding pair of constant functions with the value r as $\hat{r} = (r^\dagger, r^\dagger)$. It means that any crisp $r \in \mathbf{R}$ is written as $\hat{r}$ as an order fuzzy number and an element of $\mathcal{F}$. The relation of partial order in $\mathcal{F}$ can be introduced by

Definition 3. We say that the fuzzy number $A = (f, g)$ is not less than zero, and write

$$A \geq 0 \text{ iff } f \geq 0 \text{ and } g \geq 0. \quad (5)$$

Hence for two ordered fuzzy numbers B, C the relation $B \geq C$ holds if $B - C \geq 0$, which makes $\mathcal{F}$ a partial ordered ring. In the situation when for two numbers B, C the above inequality holds and, moreover, $B \neq C$ we will write $B > C$. According to the above denotations we can write shortly $A < 1$ for the case when $\hat{1} - A > 0$.

Let us notice that each real valued function $\psi(z)$ of a real variable $z \in \mathbf{R}$ may be transformed to the function on $\mathcal{F}$. Since each OFN is a pair of continuous functions, say (f, g) , the fuzzy counterpart of the function ψ at the fuzzy variable (f, g) will be the pair $(\psi(f), \psi(g))$, where each element of the pair as a composition of two functions is defined on the interval $[0, 1]$, i.e. $\psi(f) : [0, 1] \rightarrow \mathbf{R}$ and $\psi(g) : [0, 1] \rightarrow \mathbf{R}$.

4 Fuzzy Fractional Derivative

Now we combine the concept to fractional derivatives with the new type of fuzzy numbers. If the order of a fractional differential operator becomes an ordered fuzzy number then we face with a *fuzzy fractional derivative*. Now fractional differential operators are defined, their properties are presented together with some examples.

There are two different view-points to the fractional calculus: the continuous view-point based on the Riemann–Liouville fractional integral [23] and the discrete view-point based on the Grünwald–Letnikov fractional derivative [25]. Both approaches turn out to be useful in treating situations of practical application in different fields, including numerical analysis, physics, engineering, biology, economics and finance.

Let us begin with the superposition of the gamma function with an order fuzzy number, its results should be an OFN. In fact, if $\alpha \in \mathbf{R}$ then $\Gamma(\alpha) \in \mathbf{R}$; in view of the remark made at the end of Sect. 3 now $\alpha \in \mathcal{F}$ is represented by a pair of continuous functions, say,

$$\alpha = (\alpha_{up}, \alpha_{down}) \quad (6)$$

then its composition with the Euler function Γ leads to ordered fuzzy number

$$\Gamma(\alpha)(s) = (\Gamma(\alpha_{up}(s)), \Gamma(\alpha_{down}(s))), s \in [0, 1] \quad (7)$$

which is just a pair of continuous functions of s variable, in view of Definition 1, i.e. the composition of real-valued function Γ with an ordered fuzzy number $\alpha = (\alpha_{up}, \alpha_{down})$ gives the pair $(\Gamma(\alpha_{up}), \Gamma(\alpha_{down})) \in \mathcal{F}$ which is an ordered fuzzy number. As previously in Sect. 3, each crisp number α , say, can be identified with an order fuzzy number by its representation in the form of a pair of constant functions, denoted by the hat $\hat{\alpha} = (\alpha^{\dagger}, \alpha^{\dagger})$ and defined on the interval $[0, 1]$. To omit some misunderstanding we will often use this identification.

To pass to our main definition based on the Riemann–Liouville formula we should remember that the relation of partial order has been introduced in the space $\mathcal{F}$ of OFN's thanks to Definition 3 and (5). Now we can introduce a new derivative of a sufficiently smooth function $h : [0, \infty) \rightarrow \mathbf{R}$ for any natural number n .

Definition 4. By a fuzzy fractional derivative of order $\alpha \in \mathcal{F}$ of h we understand a function defined on $[0, \infty) =: \mathbf{R}^+$ with its value in $\mathcal{F}$, i.e. for each $t \in [0, \infty)$ it is an ordered fuzzy number given by the classical Caputo definition [25] for fractional derivative of order α of the function $h(t)$

$$\frac{d^{\alpha}h(t)}{dt^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad (8)$$

for $n-1 < \alpha \leq n$. If $\alpha = n$ then

$$\frac{d^{\alpha}h(t)}{dt^{\alpha}} = h^{(n)}(t), \quad (9)$$

here $h^{(n)}(\tau)$ denotes the n -order derivative of the function $h(\tau)$, $\tau \in \mathbf{R}^+$.

In Definition 4 the inequality $n-1 < \alpha \leq n$ is understood as a relation in $\mathcal{F}$ between ordered fuzzy numbers: $\widehat{n-1}$, α and $\hat{n}$, i.e. in view of Eqs. 5 and 6 for each $s \in [0, 1]$

$$\alpha_{up}(s) - (n-1) > 0 \text{ and } n - \alpha_{up}(s) \geq 0 \quad (10)$$

for the up-part of α , and

$$\alpha_{down}(s) - (n - 1) > 0 \text{ and } n - \alpha_{down}(s) \geq 0 \quad (11)$$

for the down-part of OFN α , where $s \in [0, 1]$. Since for $\Gamma(\alpha)$ we will have the representation (7) and for $h^\alpha := \frac{d^\alpha h(t)}{dt^\alpha}$, we obtain a new representation $(h_{up}^\alpha, h_{down}^\alpha)$ as a pair of functions of two variables t and s , given by

$$h_{up}^\alpha(t, s) = \frac{1}{\Gamma(n - \alpha_{up}(s))} \int_0^t \frac{h^{(n)}(\tau)}{(t - \tau)^{\alpha_{up}(s) + 1 - n}} d\tau \quad (12)$$

and

$$\begin{aligned} h_{down}^\alpha(t, s) &= \\ &= \frac{1}{\Gamma(n - \alpha_{down}(s))} \int_0^t \frac{h^{(n)}(\tau)}{(t - \tau)^{\alpha_{down}(s) + 1 - n}} d\tau. \end{aligned} \quad (13)$$

The above formulas could be discussed in particular cases of the functions $(h_{up}^\alpha, h_{down}^\alpha)$. It will be the subject of the further paper. Here, the case of the elementary function $h(t) = t^m$ is presented in the case of the negative fractional order α . Then we get

$$h_{up}^\alpha(t, s) = \frac{\Gamma(m + 1)}{\Gamma(m + 1 - \alpha_{up}(s))} t^{m - \alpha_{up}(s)}, \quad (14)$$

for the up-part of h^α , and

$$h_{down}^\alpha(t, s) = \frac{\Gamma(m + 1)}{\Gamma(m + 1 - \alpha_{down}(s))} t^{m - \alpha_{down}(s)} \quad (15)$$

for the down-part of h^α .

For $m = 0$ we obtain the fuzzy fractional derivative of constant function $h(t) = 1$ in the form

$$h_{up}^\alpha(t, s) = \frac{1}{\Gamma(1 - \alpha_{up}(s))} t^{-\alpha_{up}(s)}, \quad (16)$$

for the up-part of h^α , and

$$h_{down}^\alpha(t, s) = \frac{1}{\Gamma(1 - \alpha_{down}(s))} t^{-\alpha_{down}(s)} \quad (17)$$

for the down-part of h^α .

Now a fuzzy fractional integral can be defined.

Definition 5. By a fuzzy fractional integral of order $\alpha \in \mathcal{F}$ of h we understand a function defined on $[0, \infty)$ with its value in $\mathcal{F}$, i.e. for each $t \in [0, \infty)$ it is an ordered fuzzy number given by the classical Riemann–Liouville formula [25] for fractional integral of order α of the function $h(t)$

$$\mathcal{D}^{-\alpha}h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t h(\tau)(t-\tau)^{\alpha-1} d\tau, \text{ for } 0 < \alpha. \quad (18)$$

As in the previous case the fuzzy fractional integral of the fuzzy order α of the real-valued function $h(t)$ is a pair of functions of two variables: t and s . It is possible to derive a definition for the fractional derivative in terms of the fractional integral.

5 Conclusions

In the paper the definition of Caputo's of fractional derivatives has been adopted. In [26] discussion concerning its relations to other definition is given.

The above concepts of fuzzy fractional derivatives and integral may naturally be applied in modelling problems in different fields of science. It is obvious that the interval fractional derivative [3] as well as the integral are naturally included in the above setup, since any interval $[a, b]$ may be identified with the ordered fuzzy number represented by a pair of constant functions $(a^\dagger, b^\dagger)$. The appearance of the non-trivial fuzzy number α of the form (6) as the order of fractional derivatives and integrals in both recent definitions allows for more flexibility in modeling, for example, biological systems as well as viscoelastic behaviour of materials. We should, however, remember, that the fuzzy fractional derivative (or integral) needs the defuzzification procedure at the end, since the fuzzy fractional derivative (or integral) of a function $h(t)$ of t variable is a function of two variables t and $s \in [0, 1]$ (cf. (12), (13) and (18)). As a result of the defuzzification procedure the dependence on s disappears. The choice of the appropriate defuzzification procedure is the next problem and will be included in forthcoming papers.

Acknowledgements The author thanks Professors Tadeusz Kaczorek, Andrzej Turski and Dr. Ewa Turska for inspiring discussions on the subject of fractional derivatives.

References

1. Buckley James J, Eslami E (2005) An introduction to fuzzy logic and fuzzy sets. Physica, Springer, Heidelberg
2. Chen Guanrong, Pham Trung Tat (2001) Fuzzy sets, fuzzy logic, and fuzzy control systems. CRC, Boca Raton, London, New York, Washington DC
3. Chen YQ, Ahn H-S, Xue D (2006) Robust controllability of interval fractional order linear time invariant systems. Signal Process 86:2794–2802

4. Coimbra CFM (2003) Mechanics with variable-order differential operators. *Annalen der Physik* 12:692–703
5. Dubois D, Prade HH (1978) Operations on fuzzy numbers. *Int J System Sci* 9(6):613–626
6. Dubois D, Prade HH (2008) Gradual elements in a fuzzy set. *Soft Comput* 12:165–175. doi: 10.1007/s00500-007-0187-6
7. Fellah M, Fellah ZEA, Depollier C (2006) Transient wave propagation in inhomogeneous porous materials: Application of fractional derivatives. *Signal Process* 86:2658–2667
8. Goetschel R Jr, Voxman W (1986) Elementary fuzzy calculus. *Fuzzy Sets Syst* 18(1):31–43
9. Kaucher E (1980) Interval analysis in the extended interval space IR. *Computing Suppl* 2:33–49
10. Kaufman A, Gupta MM (1991) Introduction to fuzzy arithmetic. Van Nostrand Reinhold, New York
11. Klir GJ (1997) Fuzzy arithmetic with requisite constraints. *Fuzzy Sets Syst* 91(2):165–175
12. Kosiński W (2004) On defuzzification of ordered fuzzy numbers. In: Rutkowski L, Jörg Siekmann, Ryszard Tadeusiewicz, Lofti A. Zadeh (eds) ICAISC 2004, 7th international conference, Zakopane, Poland, June 2004. LNAI, vol 3070, pp 326–331, Springer, Berlin, Heidelberg
13. Kosiński W (2006) On fuzzy number calculus. *Int J Appl Math Comput Sci* 16(1):51–57
14. Kosiński W (2007) Evolutionary algorithm determining defuzzification operators. *Eng Appl Artif Intell* 20(5):619–627. doi:10.1016/j.engappai.2007.03.003
15. Kosiński W, Prokopowicz P (2004) Algebra of fuzzy numbers (In Polish: Algebra liczb rozmytych). *Matematyka Stosowana. Matematyka dla Społeczeństwa* 5(46):37–63
16. Kosiński W, Prokopowicz P, Ślęzak D (2002) Fuzzy numbers with algebraic operations: algorithmic approach. In: Kłopotek M, Wierzchoń ST, Michalewicz M (Eds) Intelligent information systems 2002. Proceedings of IIS'2002, Sopot, 3–6 June, 2002, pp 311–320, Poland. Physica, Heidelberg
17. Kosiński W, Prokopowicz P, Ślęzak D (2003) Ordered fuzzy numbers. *Bull Polish Acad Sci, Ser Sci Math* 51(3):327–338
18. Kosiński W, Prokopowicz P, Ślęzak D (2003) On algebraic operations on fuzzy numbers. In: Kłopotek M, Wierzchoń ST, Trojanowski K (eds) Intelligent information processing and web mining. Proceedings of the international IIS: IIPWM'03 conference held in Zakopane, Poland, 2–5 June 2003. Physica, Heidelberg, pp 353–362
19. Koleśnik R, Prokopowicz P, Kosiński W, Fuzzy calculator – usefull tool for programming with fuzzy algebra. In: Rutkowski L, Jörg Siekmann, Ryszard Tadeusiewicz, Lofti A Zadeh (eds) ICAISC 2004, 7th International Conference, Zakopane, Poland, June 2004. LNAI, vol 3070, pp 320–325. Springer, Berlin, Heidelberg
20. Machado JAT (1997) Analysis and design of fractional-order digital control systems. *SAMS - J Syst Anal-Modelling-Simul* 27:107–122
21. Miller KS, Ross B (1993) An introduction to the fractional calculus and fractional differential equations. Wiley, New York
22. Nguyen HT (1978) A note on the extension principle for fuzzy sets. *J Math Anal Appl* 64: 369–380
23. Oldham KB, Spanier J (1974) The fractional calculus: theory and application of differentiation and integration to arbitrary order. Academic, New York, London
24. Pedro HT, Kobayashi MH, Pereira JMC, Coimbra CFM (2006) Variable order modeling of diffusive-convective effects in the oscillatory flow past a sphere. In: Proceedings 2ed IFAC workshop on fractal differentiation and its applications. Porto, Portugal, 19–21 July 2006
25. Podlubny I (1999) Fractional differential equations. Academic, San Diego
26. Rudolf Gorenflo, Francesco Mainardi (May, 2008) Essentials of fractional calculus. <http://www.maphysto.dk/oldpages/events/LevyCAC2000/MainardiNotes/fm2k0a.ps>
27. Sebaa N, Fellah ZEA, Lauriksa W, Depollier C (2006) Application of fractional calculus to ultrasonic wave propagation in human cancellous bone. *Signal Process* 86:2668–2677
28. Torvik PJ, Bagley RL (1984) On the appearance of the fractional derivative in the behaviour of real materials. *ASME J Appl Mech* 51:294–298
29. Turski AJ, Atamaniuk B, Turska E (2003) Fractional derivative analysis of Helmholtz and paraxial-wave equations. *J Tech Phys* 44(2):193–206

30. Wagenknecht M, Hampel R, Schneider V (2001) Computational aspects of fuzzy arithmetic based on archimedean t -norms. *Fuzzy Sets Syst* 123(1):49–62
31. Zadeh LA (1965) Fuzzy sets. *Inform Control* 8(3):338–353
32. Zadeh LA (1975) The concept of a linguistic variable and its application to approximate reasoning, Part I. *Inform Sciences* 8(3):199–249

Game Problems for Fractional-Order Systems

Arkadii Chikrii and Ivan Matychyn

Abstract The paper concerns game problems for controlled systems with arbitrary Riemann–Liouville fractional derivatives, regularized Dzhrbashyan–Caputo derivatives, and sequential Miller–Ross derivatives. Under fixed controls of players, solution to such systems is presented in the form of a Cauchy formula analog. On the basis of the method of resolving functions, sufficient conditions for the finite-time game termination from given initial states are derived. Theoretical results are illustrated on the model example where the dynamics of the pursuer and the evader are described by the systems of order π and e respectively.

1 Fractional-Order Systems

Let $\mathbb{R}^m$ be the m -dimensional Euclidean space, $\mathbb{R}_+$ the positive semi-axis. The Riemann–Liouville fractional derivative of arbitrary order α ($n-1 < \alpha < n, n \in \mathbb{N}$) is defined as follows [17]:

$$\begin{aligned} D^\alpha f(t) &= {}_0\mathbf{D}_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \\ &= \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0) + \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \end{aligned} \quad (1)$$

where the function $f(t)$, $f: \mathbb{R}_+ \rightarrow \mathbb{R}^m$, has absolutely continuous derivatives up to order $n-1$ and $\Gamma(\cdot)$ is the Gamma function satisfying $\Gamma(z+1) = z\Gamma(z)$.

A. Chikrii (✉) and I. Matychyn

Glushkov Institute of Cybernetics NASU, 40 Glushkov Prsp 03680, MSP Kiev 187 Ukraine
e-mail: chikrii@gmail.com

The regularized Caputo derivative of order α ($n - 1 < \alpha < n$) has the form [16]:

$$D^{(\alpha)} f(t) = {}^C \mathbf{D}_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau. \quad (2)$$

The Riemann–Liouville fractional derivatives have a singularity at zero due to (1). That is why differential equations of fractional order in the sense of Riemann–Liouville require initial conditions of special form lacking clear physical interpretation. These shortcomings do not occur with the regularized Caputo derivative. Both the Riemann–Liouville and Caputo derivatives possess neither semigroup nor commutative property, i.e. in general,

$$\mathbf{D}^{\alpha+\beta} f(t) \neq \mathbf{D}^\alpha \mathbf{D}^\beta f(t), \quad \mathbf{D}^\alpha \mathbf{D}^\beta f(t) \neq \mathbf{D}^\beta \mathbf{D}^\alpha f(t),$$

where $\mathbf{D}^\alpha$ stands for the Riemann–Liouville or Caputo fractional differentiation operator of order α .

This fact motivated introduction by Miller and Ross [15] of sequential derivatives, defined as follows:

$$\mathcal{D}^\alpha f(t) = \mathbf{D}^{\alpha_1} \mathbf{D}^{\alpha_2} \dots \mathbf{D}^{\alpha_k} f(t), \quad (3)$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ is a multi-index, and function $f(t)$ is sufficiently smooth. In general, the operator $\mathbf{D}^\alpha$ underlying sequential Miller–Ross derivative can be either the Riemann–Liouville or Caputo or any other kind of integro-differentiation operator. In particular, in the case of integer α it is conventional differentiation operator $\left(\frac{d}{dt}\right)^\alpha$.

Let us choose some ν , $n - 1 < \nu < n$, $n \in \mathbb{N}$, and let us study the case when

$$\alpha = (j, \nu - n + 1, n - 1 - j), \quad j = 0, \dots, n - 1.$$

Let us introduce the following notation

$$\begin{aligned} \mathcal{D}_j^\nu f(t) &= \left(\frac{d}{dt}\right)^j D^{\nu-n+1} \left(\frac{d}{dt}\right)^{n-1-j} f(t), \\ \mathcal{D}_j^{(\nu)} f(t) &= \left(\frac{d}{dt}\right)^j D^{(\nu-n+1)} \left(\frac{d}{dt}\right)^{n-1-j} f(t), \end{aligned}$$

where $j = 0, 1, \dots, n - 1$.

The following lemma shows a relationship between the sequential derivatives $\mathcal{D}_j^\nu f(t)$, $\mathcal{D}_j^{(\nu)} f(t)$ and classical derivatives of Riemann–Liouville and Caputo.

Lemma 1 *Let $n - 1 < \nu < n$, $n \in \mathbb{N}$, and the function $f(t)$ have absolutely continuous derivatives up to the order $(n - 1)$. Then the following equalities hold true*

$$\begin{aligned}
\mathcal{D}_0^{(v)} f(t) &= D^{(v)} f(t) = D^v f(t) - \sum_{k=0}^{n-1} \frac{t^{k-v}}{\Gamma(k-v+1)} f^{(k)}(0), \\
\mathcal{D}_1^{(v)} f(t) &= \mathcal{D}_0^v f(t) = D^{(v)} f(t) + \frac{t^{n-1-v}}{\Gamma(n-v)} f^{(n-1)}(0) \\
&= D^v f(t) - \sum_{k=0}^{n-2} \frac{t^{k-v}}{\Gamma(k-v+1)} f^{(k)}(0), \\
&\dots\dots\dots \\
\mathcal{D}_j^{(v)} f(t) &= \mathcal{D}_{j-1}^v f(t) = D^{(v)} f(t) + \sum_{k=n-j}^{n-1} \frac{t^{k-v}}{\Gamma(k-v+1)} f^{(k)}(0) \\
&= D^v f(t) - \sum_{k=0}^{n-1-j} \frac{t^{k-v}}{\Gamma(k-v+1)} f^{(k)}(0), \\
&\dots\dots\dots \\
\mathcal{D}_{n-1}^{(v)} f(t) &= \mathcal{D}_{n-2}^v f(t) = D^{(v)} f(t) + \sum_{k=1}^{n-1} \frac{t^{k-v}}{\Gamma(k-v+1)} f^{(k)}(0) \\
&= D^v f(t) - \frac{t^{-v}}{\Gamma(1-v)} f(0), \\
\mathcal{D}_{n-1}^v f(t) &= D^{(v)} f(t) + \sum_{k=0}^{n-1} \frac{t^{k-v}}{\Gamma(k-v+1)} f^{(k)}(0) = D^v f(t).
\end{aligned}$$

Taking into account the equalities $\mathcal{D}_j^{(v)} f(t) = \mathcal{D}_{j-1}^v f(t)$ and setting $\mathcal{D}_n^{(v)} f(t) = D^v f(t)$ one can introduce common notation

$$\mathfrak{D}_j^v f(t) = \mathcal{D}_j^{(v)} f(t) = \mathcal{D}_{j-1}^v f(t),$$

where $n-1 < v < n$, $j = 0, \dots, n$. Obviously $\mathfrak{D}_0^v f(t) = D^{(v)} f(t)$ and $\mathfrak{D}_n^v f(t) = D^v f(t)$.

In [6] the Mittag-Leffler generalized matrix function was introduced:

$$E_\rho(B; \mu) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma(k\rho^{-1} + \mu)}, \quad (4)$$

where $\rho > 0$, $\mu \in \mathbb{C}$, and B is an arbitrary square matrix of order m .

Suppose $g(t) \in L_{\text{loc}}^1(\mathbb{R}_+)$. Consider a dynamic system whose evolution is described by the equation:

$$D^\alpha z = Az + g, \quad n-1 < \alpha < n, \quad (5)$$

under the initial conditions

$$D^{\alpha-k} z(t) \big|_{t=0} = z_k^0, \quad k = 1, \dots, n. \quad (6)$$

Lemma 2 *The trajectory of the system (5), (6) has the form:*

$$z(t) = \sum_{k=1}^n t^{\alpha-k} E_{\frac{1}{\alpha}}(At^\alpha; \alpha-k+1) z_k^0 + \int_0^t (t-\tau)^{\alpha-1} E_{\frac{1}{\alpha}}(A(t-\tau)^\alpha; \alpha) g(\tau) d\tau. \quad (7)$$

Now consider a dynamic system of fractional order in the sense of Caputo described by the equation:

$$D^{(\alpha)} z = Az + g, \quad n-1 < \alpha < n, \quad (8)$$

under the initial conditions

$$z^{(k)}(0) = z_k^0, \quad k = 0, \dots, n-1. \quad (9)$$

Lemma 3 *The trajectory of the system (8), (9) has the form:*

$$z(t) = \sum_{k=0}^{n-1} t^k E_{\frac{1}{\alpha}}(At^\alpha; k+1) z_k^0 + \int_0^t (t-\tau)^{\alpha-1} E_{\frac{1}{\alpha}}(A(t-\tau)^\alpha; \alpha) g(\tau) d\tau. \quad (10)$$

Finally, let us study systems involving sequential derivatives of special form $\mathfrak{D}_j^\alpha$. Consider a dynamic system whose evolution is described by the equation:

$$\mathfrak{D}_j^\alpha z = Az + g, \quad n-1 < \alpha < n, \quad j \in \{0, 1, \dots, n\} \quad (11)$$

under the initial conditions

$$\begin{aligned} \mathfrak{D}_{j-k-1}^{\alpha-k-1} z(t) \big|_{t=0} &= \tilde{z}_k^0, \quad k = 0, \dots, j-1, \\ z^{(l)}(0) &= z_l^0, \quad l = 0, \dots, n-j-1. \end{aligned} \quad (12)$$

Lemma 4 *The trajectory of the system (11), (12) has the form:*

$$\begin{aligned} z(t) = \sum_{l=0}^{n-j-1} t^l E_{\frac{1}{\alpha}}(At^\alpha; l+1) z_l^0 + \sum_{k=0}^{j-1} t^{\alpha-k-1} E_{\frac{1}{\alpha}}(At^\alpha; \alpha-k) \tilde{z}_k^0 \\ + \int_0^t (t-\tau)^{\alpha-1} E_{\frac{1}{\alpha}}(A(t-\tau)^\alpha; \alpha) g(\tau) d\tau. \end{aligned} \quad (13)$$

The proofs of Lemmata 2–4 employ the Laplace transform and can be found in [8].

2 Game Problems

Consider a conflict-controlled process whose evolution is defined by the fractional-order system

$$\mathbf{D}^\alpha z = Az + \varphi(u, v), \quad n-1 < \alpha < n. \quad (14)$$

Here $\mathbf{D}^\alpha$ as before stands for the operator of fractional differentiation in the sense of Riemann–Liouville, Caputo or Miller–Ross. It will be clear from the context, which type of the fractional differentiation operator is meant. The state vector z belongs to the m -dimensional real Euclidean space $\mathbb{R}^m$, A is a square matrix of order m , the control block is defined by the jointly continuous function $\varphi(u, v)$, $\varphi : U \times V \rightarrow \mathbb{R}^m$, where u and v , $u \in U$, $v \in V$, are control parameters of the first and second players respectively, and the control sets U and V are from the set $K(\mathbb{R}^m)$ of all nonempty compact subsets of $\mathbb{R}^m$.

When $\mathbf{D}^\alpha$ is the operator of fractional differentiation in the sense of Riemann–Liouville, i.e. $\mathbf{D}^\alpha = D^\alpha$, the initial conditions for the process (14) are given in the form (6). In this case denote $z^0 = (z_{11}^0, \dots, z_{n1}^0)$. When the derivative in (14) is understood in Caputo's sense, $\mathbf{D}^\alpha = D^{(\alpha)}$, the initial conditions are of the form (9) and $z^0 = (z_{02}^0, \dots, z_{n-1,2}^0)$. For sequential derivatives of special form $\mathbf{D}^\alpha = \mathfrak{D}_j^\alpha$ the initial conditions are given by (12) and $z^0 = (\tilde{z}_0^0, \dots, \tilde{z}_{j-1}^0, z_0^0, \dots, z_{n-j-1}^0)$.

Along with the process dynamics (14) and the initial conditions a terminal set of cylindrical form is given

$$M^* = M_0 + M, \quad (15)$$

where M_0 is a linear subspace of $\mathbb{R}^m$, $M \in K(L)$, and $L = M_0^\perp$ is the orthogonal complement of the subspace M_0 in $\mathbb{R}^m$.

When the controls of the both players are chosen in the form of Lebesgue measurable functions $u(t)$ and $v(t)$ taking values from U and V , respectively, the Cauchy problem for the process (14) with corresponding initial values has a unique absolutely continuous solution [15, 17].

Consider the following dynamic game. The first player aims to drive a trajectory of the process (14) to the set (15), while the other player strives to delay the moment of hitting the terminal set as much as possible. We assume that the second player's control is an arbitrary measurable function $v(t)$ taking values from V , and the first player at each time instant t , $t \geq 0$, forms his control on the basis of information about z^0 and $v(t)$:

$$u(t) = u(z^0, v(t)), \quad u(t) \in U. \quad (16)$$

Therefore, $u(t)$ is Krasovskii's counter-control [13] prescribed by the O. Hajek stroboscopic strategies [11].

By solving the problem stated above we employ the Method of Resolving Functions [3–7, 9, 14]. Usually this method implements the pursuit process in the class of quasistrategies. However in this paper we use the results from [9] providing sufficient conditions for the termination of the pursuit in the aforementioned method with the help of counter-controls.

Denote by Π the orthoprojector from $\mathbb{R}^m$ onto L and by $C(t)$ a measurable and bounded with respect to t matrix function. Set $\varphi(U, v) = \{\varphi(u, v) : u \in U\}$ and consider set-valued maps

$$W^*(t, v) = \Pi t^{\alpha-1} E_{\frac{1}{\alpha}}(At^{\alpha}; \alpha) \varphi(U, C(t)v), \quad W^*(t) = \bigcap_{v \in V} W^*(t, v),$$

$$M(t) = M^* - \int_0^t \tau^{\alpha-1} \Pi E_{\frac{1}{\alpha}}(A\tau^{\alpha}; \alpha) \varphi^*(\tau, U, V) d\tau,$$

where $\varphi^*(t, u, v) = \varphi(u, v) - \varphi(u, C(t)v)$ and $X \stackrel{*}{-} Y = \{z : z + Y \subset X\} = \bigcap_{y \in Y} (X - y)$ is the Minkowski (geometrical) subtraction [2]. By the integral of a set-valued map we mean the Aumann integral, i.e. a union of integrals of all possible measurable selections of the given map [12]. Hereafter, we will say that the modified Pontryagin condition is fulfilled whenever a measurable bounded matrix function $C(t)$ exists such that

$$W^*(t) \neq \emptyset, \quad M(t) \neq \emptyset \quad \forall t \in \mathbb{R}_+. \quad (17)$$

By virtue of the properties of the process (14) parameters the set-valued map $\varphi(U, C(t)v)$, $v \in V$, is continuous in the Hausdorff metric. Therefore, taking into account the analytical properties of the Mittag-Leffler generalized matrix function, the set-valued map $W^*(t, v)$ is measurable with respect to t , $t \in \mathbb{R}_+$, and closed with respect to v , $v \in V$. Then [1], the set-valued map $W^*(t)$ is measurable with respect to t and closed-valued.

It follows from the condition (17) and from the measurable selection theorem [1] that there exists at least one measurable selection $\gamma(\cdot)$ such that $\gamma(t) \in W^*(t)$, $t \in \mathbb{R}_+$. Denote by Γ the set of all such selections.

Let us also denote by $g(t, z^0)$ the solution of homogeneous system (14) for $\varphi(u, v) \equiv 0$, i.e. $\mathbf{D}^{\alpha} z = Az$. Let us introduce the function

$$\xi(t) = \Pi g(t, z^0) + \int_0^t \gamma(\tau) d\tau, \quad t \in \mathbb{R}_+, \quad (18)$$

where $\gamma(\cdot) \in \Gamma$ is a certain fixed selection. By virtue of the assumptions made, the selection $\gamma(\cdot)$ is summable.

Consider the set-valued map

$$\mathfrak{A}(t, \tau, v) = \{\alpha \geq 0 : [W^*(t - \tau, v) - \gamma(t - \tau)] \cap \alpha[M(t) - \xi(t)] \neq \emptyset\}, \quad (19)$$

defined on $\Delta \times V$, where $\Delta = \{(t, \tau) : 0 \leq \tau \leq t < \infty\}$. Let us study its support function in the direction of $+1$

$$\alpha(t, \tau, v) = \sup\{\alpha : \alpha \in \mathfrak{A}(t, \tau, v)\}, \quad (t, \tau) \in \Delta, \quad v \in V.$$

This function is called the resolving function [3].

Taking into account the modified Pontryagin condition (17), the properties of the process (14) parameters, as well as characterization and inverse image theorems, one can show that the set-valued map $\mathfrak{A}(t, \tau, v)$ is $\mathcal{L} \times \mathcal{B}$ -measurable [1] with respect to $\tau, v, \tau \in [0, t], v \in V$, and the resolving function $\alpha(t, \tau, v)$ is $\mathcal{L} \times \mathcal{B}$ -measurable in τ, v by virtue of the support function theorem [1] when $\xi(t) \notin M(t)$.

It should be noted that for $\xi(t) \in M(t)$ we have $\mathfrak{A}(t, \tau, v) = [0, \infty)$ and hence $\alpha(t, \tau, v) = +\infty$ for any $\tau \in [0, t], v \in V$.

Denote

$$\mathfrak{T} = \left\{ t \in \mathbb{R}_+ : \inf_{v(\cdot)} \int_0^t \alpha(t, \tau, v(\tau)) d\tau \geq 1 \right\}. \quad (20)$$

Since the function $\alpha(t, \tau, v)$ is $\mathcal{L} \times \mathcal{B}$ -measurable with respect to τ, v , it is superpositionally measurable [10]. If $\xi(t) \in M(t)$, then $\alpha(t, \tau, v) = +\infty$ for $\tau \in [0, t]$ and in this case it is natural to set the value of the integral in (20) to be equal $+\infty$. Then the inequality in (20) is fulfilled by default. In the case when the inequality in braces in (20) fails for all $t > 0$, we set $\mathfrak{T} = \emptyset$. Let $T \in \mathfrak{T} \neq \emptyset$.

Condition 1 The map $\mathfrak{A}(T, \tau, v)$ is convex-valued for all $\tau \in [0, T], v \in V$.

Theorem 1 Let for the game problem (14), (15) there exist a bounded measurable matrix function $C(t)$ such that the conditions (17) hold true and the set M be convex. If there exists a finite number $T, T \in \mathfrak{T} \neq \emptyset$, such that Condition 1 is satisfied, then the trajectory of the process (14) can be brought to the set (15) from the initial position z^0 at the time instant T using the control of the form (16).

3 Example

Here we consider an example of fractional order pursuit-evasion dynamic game. Let the dynamics of the Pursuer be described by the equation:

$$\mathbf{D}^\pi x = u, \quad \|u\| \leq 1, \quad (21)$$

where $\pi = 3.14159 \dots$ is the ratio of a circle's circumference to its diameter.

The dynamics of the Evader is governed by the equation:

$$\mathbf{D}^e y = v, \quad \|v\| \leq 1, \quad (22)$$

where $e = 2.71828 \dots$ is the base of the natural logarithm.

Denote by $g_x(x^0, t)$, $g_y(y^0, t)$ solutions to $\mathbf{D}^\pi x = 0$, $\mathbf{D}^e y = 0$, respectively, under appropriate initial conditions x^0, y^0 .

The goal of the pursuer is to achieve the fulfillment of the inequality

$$\|x(T) - y(T)\| \leq \varepsilon, \quad \varepsilon > 0, \quad (23)$$

for some finite time instant T . The goal of the evader is to prevent the fulfillment of the inequality (23) or, provided it is impossible, to maximally postpone it.

The use of the Method of Resolving Functions described above makes it possible to derive sufficient condition for the solvability of the formulated pursuit problem. This condition is of the form

$$\varepsilon \geq \frac{(\Gamma(\pi)/\Gamma(e))^{\frac{e}{\pi-e}}}{\Gamma(e+1)} - \frac{(\Gamma(\pi)/\Gamma(e))^{\frac{\pi}{\pi-e}}}{\Gamma(\pi+1)}. \quad (24)$$

If (24) holds, the time T of the game termination can be found as the least positive root of the equation

$$\frac{t^\pi}{\Gamma(\pi+1)} - \frac{t^e}{\Gamma(e+1)} + \varepsilon = \|g_x(x^0, t) - g_y(y^0, t)\|.$$

References

1. Aubin J-P, Frankowska H (1990) Set-valued analysis. Birkhäuser, Boston
2. Chikrii AA (1997) Conflict-controlled processes. Kluwer, Boston
3. Chikrii AA (2006) Game problems for systems with fractional derivatives of arbitrary order. In: Haurie A, Muto S, Petrosyan, L, Raghavan T (eds) Advances in dynamic games 8(II): 105–113, Birkhäuser, Boston
4. Chikrii AA (2008) Game dynamic problems for systems with fractional derivatives. In: Chinchuluun A, Pardalos PM, Migdalas, A, Pitsoulis L (eds) Pareto optimality, game theory and equilibria, vol 17. Springer, New York
5. Chikrii AA (2008) Optimization of game interaction of fractional-order controlled systems. J Optim Methods Softw 23(1):39–72
6. Chikrii AA, Eidelman SD (2000) Generalized Mittag-Leffler matrix functions in game problems for evolutionary equations of fractional order. Cybern Syst Anal 36(3):315–338
7. Chikrii AA, Eidelman SD (2002) Game problems for fractional quasilinear systems. Int J Comput Math Appl 44(7):835–851
8. Chikrii AA, Matichin II (2008) Presentation of solutions of linear systems with fractional derivatives in the sense of Riemann–Liouville, Caputo and Miller-Ross. J Autom Inform Sci 40(6):1–11
9. Chikrii AA, Rappoport IS, Chikrii (2007) KA Multivalued mappings and their selectors in the theory of conflict-controlled processes. Cybern Syst Anal 43(5):719–730
10. Clarke F (1983) Optimization and nonsmooth analysis. Wiley-Interscience, New York
11. Hajek O (1975) Pursuit games. Mathematics in science and engineering, vol 120. Academic, New York
12. Ioffe AD, Tikhomirov VM (1979) Theory of extremal problems. North-Holland, Amsterdam
13. Krasovskii NN, Subbotin AI (1988) Game-theoretical control problems. Springer, New York

14. Krivonos Yu G, Matychyn II, Chikrii AA (2005) Dynamic games with discontinuous trajectories. Naukova Dumka, Kiev (in Russian)
15. Miller KS, Ross B (1993) An introduction to the fractional calculus and fractional differential equations. Wiley, New York
16. Podlubny I (1999) Fractional differential equations. Academic, San Diego
17. Samko SG, Kilbas AA, Marichev OI (1993) Fractional integrals and derivatives. Gordon and Breach, Amsterdam

Synchronization Analysis of Two Networks

Changpin Li and Weigang Sun

Abstract In this chapter we study synchronization for two coupled networks. Synchronization conditions between two networks which have same topological connectivity are derived. Then numerical examples are given to demonstrate that synchronization between two networks with directed or undirected topological connections can be achieved.

1 Introduction

Presently researches on complex networks have attracted much attention, which mainly focuses on the network model and network dynamics. Especially explaining the relations between topological structures and dynamics of networks is an active topic. The small-world model [1] and scale-free network [2] are most noticeable network models, which was introduced by Watts and Barabási in 1998 and 1999, respectively.

When studying the dynamics of networks, synchronizing all the nodes is an interesting thing [3] and many references cited therein. Though there are much work on the synchronization of all kinds of networks, such as introducing nonlinear coupling function [4], time delays [5], weighted connections [6], time-dependence in the coupling matrices [7], etc., we find synchronization all happens inside a network, strictly speaking, we call it “inner synchronization” [8].

A natural question is, “does synchronization between networks also happen?” We may call “outer synchronization” of networks if such a synchronization exists. In fact, if network nodes are of similar properties, we can regard it as one network;

C. Li (✉)

Department of Mathematics, Shanghai University, Shanghai, 200444 China
e-mail: lcp@shu.edu.cn

W. Sun

The School of Science, Hangzhou Dianzi University, Hangzhou, 310018 China
e-mail: wgsun@hdu.edu.cn

otherwise, as more networks. For example, how the infectious diseases (Mad Cows, SARS, AIDS) spread between the human beings and animals, here regarding the human and animals as two networks. Hence studying the dynamics between networks is very meaningful.

In this chapter, we study the synchronization between two networks, synchronization analysis between them which have the same topological connectivity is derived and numerical examples are given to show the efficiency of our derived results.

2 Synchronization Analysis

Consider a general system:

$$\frac{dy}{dt} = F(y), \quad y \in R^n. \quad (1)$$

The original OPCL [9] offers a driving for system (1) to achieve a desired goal dynamics $x(t) \in R^n$. The driving is in the following form:

$$D(y, x) = D_1(x) + D_2(y, x), \quad (2)$$

where

$$\begin{aligned} D_1(x) &= \frac{dx}{dt} - F(x), \\ D_2(y, x) &= (H - \frac{\partial F(x)}{\partial x})(y - x), \end{aligned}$$

where H is an arbitrary constant Hurwitz matrix (a matrix with negative real part eigenvalues), whose elements can be chosen as simple as possible. When $(\partial F(x)/\partial x)_{ik}$ is a constant, we can then choose $H_{ik} = (\partial F(x)/\partial x)_{ik}$ such that $(H - \partial F(x)/\partial x)_{ik}$ is zero. When we cannot find such a Hurwitz matrix, we introduce one or two or more parameters guaranteeing that H is a Hurwitz matrix. From this viewpoint we can conclude that the coupling form may be simpler if $F(x)$ has fewer nonlinear terms.

The driving system $\frac{dy}{dt} = F(y) + D(y, x)$ has an asymptotic behavior $y(t) \rightarrow x(t)$ if $\|y(0) - x(0)\|$ is small enough. This strategy can be applied to master-slave synchronization. The master-slave systems are as follows:

$$\frac{dx}{dt} = F(x), \quad (3)$$

$$\frac{dy}{dt} = F(y) + \left(H - \frac{\partial F(x)}{\partial x}\right)(y - x). \quad (4)$$

Now we apply the above coupling form to investigate the synchronization between two coupled networks. Here we take the driving network in the following form

$$\dot{x}_i(t) = F(x_i(t)) + \varepsilon \sum_{j=1}^N a_{ij} \Gamma x_j(t), \quad i = 1, 2, \dots, N, \quad (5)$$

and the response network as

$$\begin{aligned} \dot{y}_i(t) &= G(y_i(t)) + \varepsilon \sum_{j=1}^N b_{ij} \Gamma y_j(t) + D(y_i, x_i), \\ i &= 1, 2, \dots, N, \end{aligned} \quad (6)$$

where $D(y_i, x_i) = (H - \frac{\partial F(x_i)}{\partial x_i})(y_i(t) - x_i(t))$. Here $x_i = (x_{i1}, x_{i2}, \dots, x_{in})^T \in R^n$ and $y_i = (y_{i1}, y_{i2}, \dots, y_{in})^T \in R^n$ are the state variables of node i , N is the number of the network nodes. $F, G : R^n \rightarrow R^n$ are continuously differentiable functions which determine the dynamical behavior of the nodes in networks $\mathbb{X}$ and $\mathbb{Y}$, respectively. $\varepsilon > 0$ is the coupling strength, and $\Gamma \in R^{n \times n}$ is a constant 0-1 matrix linking coupled variables. For simplicity, we often assume that Γ is an identity matrix. $A = (a_{ij})_{N \times N}$, $B = (b_{ij})_{N \times N}$ are outer connection matrices of networks $\mathbb{X}$ and $\mathbb{Y}$, which are both irreducible, and symmetric or asymmetric matrices which satisfy zero row-sums. The entries $a_{ij}(b_{ij})$ are defined as follows: if there is a connection between node i and node j ($j \neq i$), then $a_{ij}(b_{ij}) = 1$, otherwise $a_{ij}(b_{ij}) = 0$ ($j \neq i$); the diagonal elements of A, B are defined as $a_{ii} = -\sum_{j=1, j \neq i}^N a_{ij}$, $b_{ii} = -\sum_{j=1, j \neq i}^N b_{ij}$.

Definition 1. Hereafter, network (5) and network (6) are said to achieve synchronization if

$$\lim_{t \rightarrow +\infty} \|y_i(t) - x_i(t)\| = 0, \quad i = 1, 2, \dots, N. \quad (7)$$

In this paper, we assume the networks $\mathbb{X}$ and $\mathbb{Y}$ have the same dynamics ($F = G$) and same topological structures ($A = B$).

Letting $\delta_i = y_i - x_i$, and linearizing the error system around x_i , we get

$$\dot{\delta}_i = H \delta_i + \varepsilon \sum_{j=1}^N a_{ij} \Gamma \delta_j, \quad i = 1, 2, \dots, N. \quad (8)$$

Equation 8 can be written as

$$\dot{\delta} = H \delta + \varepsilon \Gamma \delta A^T, \quad (9)$$

where ^{T} stands for matrix transpose and $\delta = [\delta_1, \delta_2, \dots, \delta_N]$ denotes a $n \times N$ matrix. Decompose the coupling matrix according to $A^T = SJS^{-1}$, where J is a Jordan canonical form with complex eigenvalues $\lambda \in \mathcal{C}$ and S contains the

corresponding eigenvectors. Multiplying (9) from the right with S , and denoting $\eta = \delta S$, we obtain

$$\dot{\eta} = H\eta + \varepsilon \Gamma \eta J, \quad (10)$$

where J is a block diagonal matrix,

$$\begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_l \end{bmatrix} \quad (11)$$

and J_k is a block corresponding to the m_k multiple eigenvalue λ_k of A :

$$\begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix} \quad (12)$$

Let $\eta = [\eta_1, \eta_2, \dots, \eta_l]$ and $\eta_k = [\eta_{k,1}, \eta_{k,2}, \dots, \eta_{k,m_k}]$. Due to the fact that the sum of every line of the matrix A is zero, we can assume $\lambda_1 = 0$, and J_1 is a 1×1 matrix. If $\lambda_1 = 0$, we get $\dot{\eta}_1 = H\eta_1$. Since H is a Hurwitz matrix, the zero solution $\eta_1 = \mathbf{0}$ is asymptotically stable. Next, we discuss the cases $k = 2, 3, \dots, l$. We can rewrite (10) in a component form.

$$\dot{\eta}_{k,1} = (H + \varepsilon \lambda_k \Gamma) \eta_{k,1}, \quad (13)$$

$$\begin{aligned} \dot{\eta}_{k,r+1} &= (H + \varepsilon \lambda_k \Gamma) \eta_{k,r+1} + \varepsilon \Gamma \eta_{k,r}, \\ 1 \leq r &\leq m_k - 1. \end{aligned} \quad (14)$$

Firstly we consider the stability of (13). Let $\eta_{k,1} = u_{k,1} + jv_{k,1}$ and $\lambda_k = \alpha_k + j\beta_k$, where $u_{k,1}, v_{k,1}, \beta_k \in \Re$, $\alpha_k < 0$, $k = 2, \dots, l$, and j is the imaginary unit. Equation 13 can be rewritten as

$$\begin{aligned} \dot{u}_{k,1} &= (H + \varepsilon \alpha_k \Gamma) u_{k,1} - \varepsilon \beta_k \Gamma v_{k,1}, \\ \dot{v}_{k,1} &= (H + \varepsilon \alpha_k \Gamma) v_{k,1} + \varepsilon \beta_k \Gamma u_{k,1}. \end{aligned} \quad (15)$$

We define the Lyapunov function as

$$V(t) = u_{k,1}^T u_{k,1} + v_{k,1}^T v_{k,1}. \quad (16)$$

Then we get

$$\begin{aligned} \dot{V}(t) &= u_{k,1}^T (P^T + P) u_{k,1} + v_{k,1}^T (P^T + P) v_{k,1} \\ &\quad + \varepsilon \beta_k u_{k,1}^T (\Gamma^T - \Gamma) v_{k,1} + \varepsilon \beta_k v_{k,1}^T (\Gamma^T - \Gamma) u_{k,1}, \end{aligned}$$

$$= \begin{bmatrix} u_{k,1} \\ v_{k,1} \end{bmatrix}^T Q_k \begin{bmatrix} u_{k,1} \\ v_{k,1} \end{bmatrix}, \quad (17)$$

where

$$Q_k = \begin{bmatrix} P^T + P & \\ & P^T + P \end{bmatrix}, \quad (18)$$

with $P = H + \varepsilon \alpha_k \Gamma$. If $Q_k < 0$, $k = 2, \dots, l$, i.e., these matrices are negative definite, then the zero solution of (13) is asymptotically stable. Obviously this stable condition ($Q_k < 0$) is equivalent to $P^T + P < 0$.

Secondly we study the stability of (14). Without loss of generality, we take $r = 1$, (14) reads as

$$\dot{\eta}_{k,2} = (H + \varepsilon \lambda_k \Gamma) \eta_{k,2} + \varepsilon \Gamma \eta_{k,1} \quad (19)$$

Since the norm of Γ is bounded, and $\eta_{k,1}$ is asymptotically stable, we know the $\varepsilon \Gamma \eta_{k,1}$ is exponentially small. Then the same condition $P^T + P < 0$ can guarantee (19) is stable ($\eta_{k,2} \rightarrow 0$, as $t \rightarrow +\infty$), which shows the zero solution of (14) is asymptotically stable. From this, it follows that the synchronization between the drive network (5) and the response one (6) is achieved.

Theorem 1. Consider the networks (5) and (6). Assume $F = G$, $A = B$ and H is a Hurwitz matrix. Let $\lambda_k = \alpha_k + j\beta_k$ be the nonzero eigenvalues of the coupling matrix A , where $\beta_k \in \mathbb{R}$, $\alpha_k < 0$. If $P^T + P < 0$, $P = H + \varepsilon \alpha_k \Gamma$, then the synchronization between the drive network (5) and the response network (6) is achieved.

The above criterion is derived through the linear matrix equality, in fact, this condition $\text{Re}\mathbb{L}(H + \varepsilon \lambda_k \Gamma) < 0$ can also make the synchronization between these two networks be achieved, where $\mathbb{L}(\cdot)$ denotes all the eigenvalues of a matrix, and $\text{Re}(\cdot)$ is the real part of a complex number.

If we let $\delta = [\delta_1^T, \delta_2^T, \dots, \delta_N^T]^T \in \mathbb{R}^{nN}$, Eq. 8 can be written as

$$\dot{\delta} = (\overline{H} + \varepsilon A \otimes \Gamma) \delta. \quad (20)$$

where $\overline{H} = \text{diag}(H, H, \dots, H)$, and the signal $\otimes$ denotes the Kronecker product. If the real parts of all the eigenvalues of $\overline{H} + \varepsilon A \otimes \Gamma$ are negative, the zero solution of (20) is asymptotically stable, which shows these two networks achieve synchronization.

In a word, there exist three criteria on synchronization between two networks.

1. $P^T + P < 0$, where $P = H + \varepsilon \alpha_k \Gamma$
2. $\text{Re}\mathbb{L}(H + \varepsilon \lambda_k \Gamma) < 0$
3. $\text{Re}\mathbb{L}(\overline{H} + \varepsilon A \otimes \Gamma) < 0$

Using the knowledge of Kronecker product and Γ is an identity matrix, we can know the criterion 2 and 3 are equivalent, but are not equivalent to criterion 1. Criterion 1 is usually used, since the dimension of $(\overline{H} + \varepsilon A \otimes \Gamma)$ is so large.

3 Numerical Examples

In the considered networks below, the dynamics at every node follows the well-known Lorenz system:

$$\begin{cases} \dot{x}_{i1} = \sigma(x_{i2} - x_{i1}), \\ \dot{x}_{i2} = \gamma x_{i1} - x_{i1}x_{i3} - x_{i2}, \\ \dot{x}_{i3} = x_{i1}x_{i2} - bx_{i3}, \end{cases} \quad (21)$$

where σ, γ, b are parameters, we always use in the following $\sigma = 10, \gamma = 28, b = 8/3$, i.e., the system has a chaotic attractor.

We take H as the following form

$$H = \begin{bmatrix} -\sigma & \sigma & 0 \\ \mu & -1 & 0 \\ 0 & 0 & -B \end{bmatrix}. \quad (22)$$

where μ is a parameter. If $\mu < 1$, it is easy to check that H is a Hurwitz matrix. Here we take $\Gamma = \text{diag}(1, 1, 1)$.

In what follows, we consider two cases, the connection matrix is undirected and directed.

3.1 Directed Connection

For simplicity, we analyze networks with 10 nodes, at this time a combination of systems (5) and (6) is 60-dimensional. For the first case, we suppose that $A = B = A_{\text{direc}}$, where

$$A_{\text{direc}} = \begin{bmatrix} -4 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -4 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -7 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -5 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & -6 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & -4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & -6 \end{bmatrix}. \quad (23)$$

In the following, the initial values are chosen randomly in $(0,1)$. Let $\|e(t)\| = \max\{\max_{1 \leq i \leq 10} |x_{i1}(t) - y_{i1}(t)|, \max_{1 \leq i \leq 10} |x_{i2}(t) - y_{i2}(t)|, \max_{1 \leq i \leq 10} |x_{i3}(t) - y_{i3}(t)|\}$,

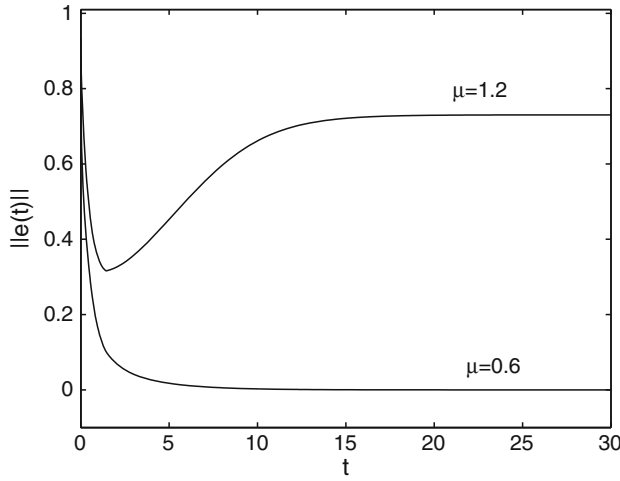


Fig. 1 Synchronization errors between (5) and (6) with directed connections for two values of μ in H with $\varepsilon = 0.5$, $A = B = A_{\text{direc}}$

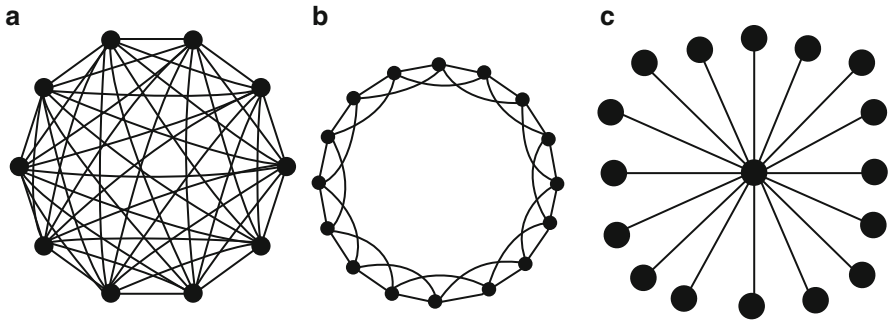


Fig. 2 (a) Globally coupled network; (b) nearest-neighbor coupled network; (c) star coupled network [3]

for $t \in [0, +\infty)$. Figure 1 plots the synchronization errors for different values of μ in H .

3.2 Undirected Connection

For this case, the considered topological structures are regular networks, including the globally coupled networks, nearest-neighbor coupled network and star coupled network, see Fig. 2.

The coupling connection of globally coupled network is expressed as follows, assume $A = B = A_{\text{glob}}$.

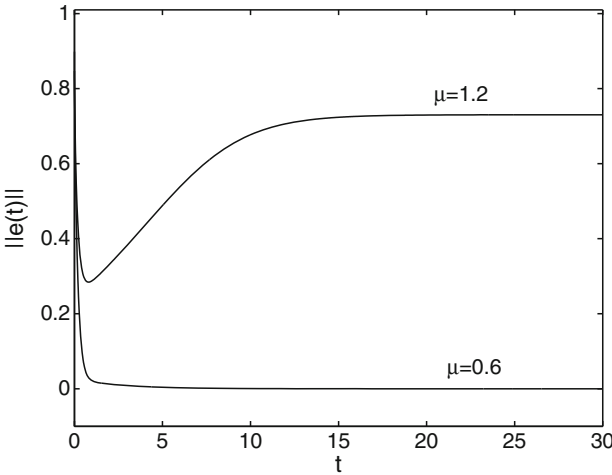


Fig. 3 Synchronization errors between (5) and (6) with global connections for two values of μ in H with $\varepsilon = 0.6$, $A = B = A_{\text{glob}}$

$$A_{\text{glob}} = \begin{bmatrix} -9 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -9 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -9 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -9 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -9 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -9 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -9 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -9 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -9 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -9 \end{bmatrix}. \tag{24}$$

Figure 3 plots the synchronization errors for different values of μ in H .

For the nearest-neighbor coupled network (the number of neighbors is 4), assume $A = B = A_{\text{near}}$.

$$A_{\text{near}} = \begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -4 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -4 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 \end{bmatrix}. \tag{25}$$

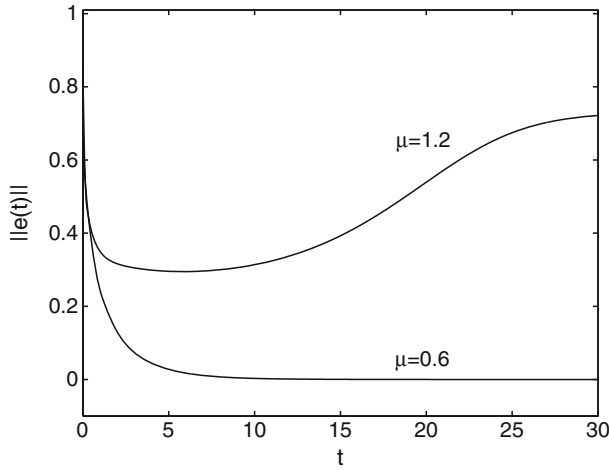


Fig. 4 Synchronization errors between (5) and (6) with nearest-neighbor connections for two values of μ in H with $\varepsilon = 0.6$, $A = B = A_{\text{near}}$

Figure 4 plots the synchronization errors for different values of μ in H .

Assume $A = B = A_{\text{star}}$ with star coupled network connection.

$$A_{\text{star}} = \begin{bmatrix} -9 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}. \quad (26)$$

Figure 5 plots the synchronization errors for different values of μ in H .

Since the coupling matrix is symmetric, we assume that its first eigenvalue is zero and the rest are negative. From our analysis, the real parts of the eigenvalues of $H + \varepsilon \lambda_k \Gamma$ are negative for arbitrary $\mu < 1$ in H . It immediately follows that the synchronization between network (5) and network (6) can be achieved. From our computations, we find that the values of μ in H play an important role in synchronous process and that the coupling strength ε seems not to have relation to synchronization.

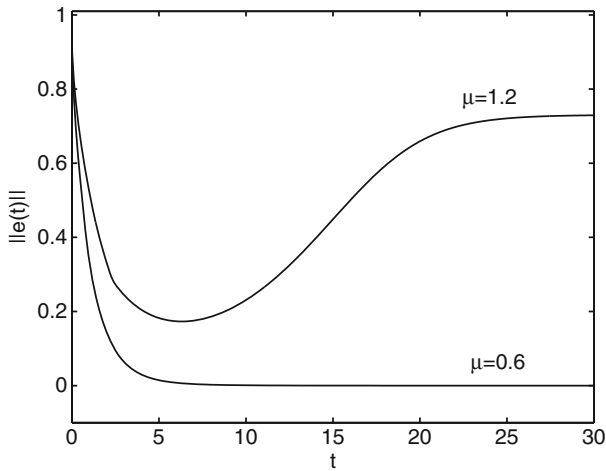


Fig. 5 Synchronization errors between (5) and (6) with star-coupled network connectivity for two values of μ in H with $\varepsilon = 0.6$, $A = B = A_{\text{star}}$

4 Conclusion

In this article, synchronization **between** two coupled complex networks but not that **inside** one network is studied. Here we choose an interesting master–slave action form to discuss this problem. We theoretically and numerically show that when driving–response networks have identical connection topologies, then synchronization between them can be achieved. Studying the synchronization between two networks with different dynamics ($F \neq G$) and topological structures ($A \neq B$) is our future work.

Acknowledgements This work was supported by the National Natural Science Foundation of China (grant no. 10872119), and Research Foundation (KYS075608083) of Hangzhou Dianzi University.

References

1. Watts DJ, Strogatz SH (1998) Collective dynamics of small-world networks. *Nature* 393: 440–442
2. Barabási AI, Albert R (1999) Emergence of scaling in random networks. *Science* 286:509–512
3. Wang XF, Chen GR (2002) Complex networks: topology, dynamics and synchronization. *Int J Bifurcat Chaos* 12:885–916
4. Li CP, Sun WG, Xu DL (2005) Synchronization of complex dynamical networks with nonlinear inner-coupling functions and time delays. *Prog Theor Phys* 114:749–761
5. Li CP, Sun WG, Kurths J (2006) Synchronization of complex dynamical networks with time delays. *Physica A* 361:24–34

6. Zhou CS, Motter AE, Kurths J (2006) Universality in the synchronization of weighted random networks. *Phys Rev Lett* 96:034101
7. Lü JH, Yu XH, Chen GR (2004) Chaos synchronization of general complex dynamical networks. *Physica A* 334:281–302
8. Li CP, Sun WG, Kurths J (2007) Synchronization between two coupled complex networks. *Phys Rev E* 76:046204
9. Jackson EA, Grosu I (1995) An open-plus-closed-loop (OPCL) control of complex dynamic systems. *Physica D* 85:1–9

Part IV

Fractional Modelling

Modeling Ultracapacitors as Fractional-Order Systems

Yang Wang, Tom T. Hartley, Carl F. Lorenzo, Jay L. Adams,
Joan E. Carletta, and Robert J. Veillette

Abstract Ultracapacitors display long-term transients lasting for many months. This paper shows that these long-term transients can be accurately represented using a fractional-order system model for the ultracapacitor impedance. Time-domain data is used to determine the impedance transfer-function coefficients. A circuit realization for the ultracapacitor is given which explicitly shows the fractional-order component. These long-term transient models will allow the development of improved ultracapacitor management systems.

1 Introduction

Ultracapacitors, also known as electrochemical double layer capacitors (EDLC) or supercapacitors, have certain advantages over other energy storage devices. The minute charge-separation distances that arise in the double layer structure as well as the large surface area of the porous activated carbon electrodes gives them extremely high capacitance [1]. They can provide energy densities from ten to several hundred times higher than those of conventional capacitors, and have power densities ten to twenty times higher than those of batteries [2]. In addition, the energy storage and release processes in ultracapacitors involve no chemical reactions; as a result, they have a much longer life and are therefore more environmentally friendly than batteries. As a result, ultracapacitors are becoming more widely used in energy storage systems [3–5].

The effective overall management of an ultracapacitor energy storage system, as well as the balancing of the individual ultracapacitor cells in a series-connected

Y. Wang, T.T. Hartley (✉), J.L. Adams, J.E. Carletta, and R.J. Veillette
The University of Akron, Akron, Ohio, USA 44325-3904
e-mail: yw24@uakron.edu; thartley@uakron.edu; jla36@uakron.edu; carlett@uakron.edu;
veillette@uakron.edu

C.F. Lorenzo
NASA Glenn Research Center, Cleveland, OH, USA 44135
e-mail: carl.f.lorenzo@grc.nasa.gov

string, depends on an accurate determination of the cell states of charge. If an ultracapacitor cell is modeled simply as a large capacitor in series with a small resistance, the capacitor charge can be determined from the instantaneous measurements of voltage and current at the terminals. However, an ultracapacitor cell behaves more as a distributed capacitance; so, the calculation of the true stored charge is considerably more complex. Even the amount of stored charge that should be considered to be available to do work in a given time interval is uncertain. The development of a suitable dynamic model of the ultracapacitor cell is essential to understanding the relationship between the stored charge and the terminal measurements, and hence to the effective ultracapacitor energy management.

A hybrid electric vehicle developed at the University of Akron for the Challenge X competition incorporates a bank of 143 series-connected ultracapacitor cells, each with a nominal capacitance of 3,500F and a voltage rating of 2.7 V [6]. A simple first-order RC model was identified for the bank using a time-domain least-squares method. In-vehicle system-level tests showed that the first-order model is sufficient for simulating fast charging and discharging; however, it is not as accurate for predicting longer term behavior [7]. Part of the motivation for the present work is to find a better model for the ultracapacitors, so as to better understand how the energy storage system will react as the vehicle is driven.

Much of the published work on ultracapacitor modeling uses the frequency-response methods such as electrochemical impedance spectroscopy (EIS) [8]. For example, a model with fourteen R , L , and C components is presented in [7]. Such methods are based on the direct measurement of the frequency response, and generally involve the use of expensive specialized instrumentation. Other ultracapacitor modeling work has assumed circuit models with specific nonlinear components. One such model, consisting of multiple interconnected RC branches, each having a voltage-dependent capacitance, is presented in [1]. Although the model reflects the actual physical structure of the ultracapacitor, it is difficult to implement in practice because it has a large number of parameters to identify. A simpler two-branch RC model is presented in [9]. A “fast” branch includes a voltage-dependent capacitance; a linear “slow” branch models the charge redistribution phenomenon. The parameters for the fast branch are determined by charging the ultracapacitor with a constant current for a short time interval; then, the parameters for the slow branch are determined by recording the terminal voltage over time with no further charging current. This method assumes a model with two widely separated time constants, and requires access to a controllable constant-current supply.

The distributed nature of the ultracapacitor cell capacitance can be considered as an infinite RC ladder circuit. Such a model is used in [10, 11]. Finite RC ladder circuit models and parallel-branch RC circuit models are likewise found in the literature. For example, a dynamic electro-thermal model of a supercapacitor is presented in [12]. A multivariable parameter optimization algorithm is used to determine the components of a fast branch, a medium branch, and a slow branch, as well as the components of the thermal model.

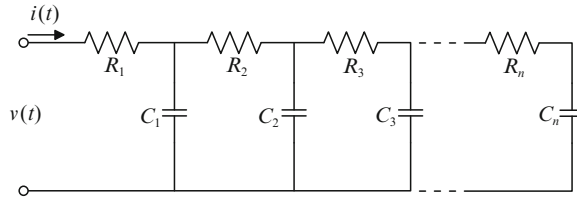


Fig. 1 Ultracapacitor modeled as an RC transmission line

2 Modeling and Identification

The present work develops a fractional-order dynamic model of an ultracapacitor cell using time-domain data. A least-squares algorithm is used for the identification of the transfer function coefficients from experimental data gathered using only ordinary laboratory instruments such as multimeters, oscilloscopes and constant voltage supplies. The identification process is done only with time domain data, and is based on time-dependent current profiles, with no initial assumptions made about the time constants.

The present model for the ultracapacitor is based on the idea, used in [11], that the ultracapacitor ultimately behaves as an infinite RC ladder, as depicted in Fig. 1. The sections of the ladder are ordered such that the shortest time constant is associated with the R and C closest to the terminals, and the time constants get longer and longer for sections farther from the terminals. This circuit reflects the behavior seen on the bench: when a cell is charged, the terminal voltage rises quickly, but once the charging current is cut off, the terminal voltage slowly drops, as charge redistributes itself from the capacitances closer to the terminals to those farther away. A similar circuit has also been shown to represent fractional-order systems [13].

3 Experiment and Results

The test circuit consisted of three Nesscap 3,500 F ultracapacitors in series, excited by a 5-V, 1,100-W power supply. Two MOSFET switches (one consisting of five parallel P-channel devices and the other consisting of two parallel N-channel devices) are employed to control the charging and discharging processes respectively.

A positive current was input to the ultracapacitor for 800 s. This current was not a constant, but resembled an RC-like exponential decay, and was obtained by closing the P-channel MOSFET until the ultracapacitor voltage reached 1.6 V, at which point the P-channel MOSFET was opened. The magnitude of the current decayed from roughly 26–1 A in 800 s, and had an area of 5,390 A-s. After about 2 months, the N-channel MOSFET was closed to discharge the ultracapacitor until the voltage was reduced to 0.5 V. Again the transient resembled an RC-like exponential decay. The magnitude of the current decayed from roughly 16–9 A in 130 s, and had an

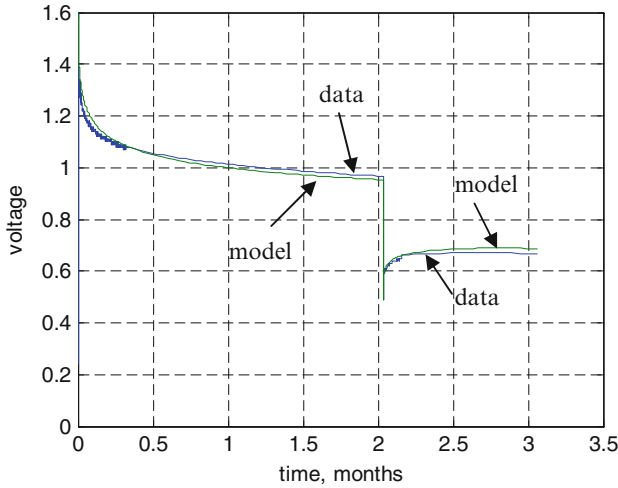


Fig. 2 Data and model voltage responses for the long-term ultracapacitor impulse response, time in 28-day months

area of 1,690 A-s. The current input durations were short relative to the relaxation transients of interest. As a consequence, the current inputs were approximated as impulses weighted by their respective areas. This approximation allows the transient response to be considered as the ultracapacitor's impulse response, which can then be Laplace transformed to give the ultracapacitor impedance transfer function. The data and resulting model time responses are shown in Fig. 2.

For this particular transient, the initial condition on the ultracapacitor was 0.24 V, and was obtained after the ultracapacitor had been at rest for 6 months. The time response was considered to be composed of a constant plus an appropriate R -function [14]. R -functions are convenient generalizations of the exponential functions and have the Laplace transform

$$L \{R_{q,v}(d, t)\} = \frac{s^v}{s^q - d} \quad (1)$$

The form chosen for the time response was

$$v(t) = a + b R_{0.33, -0.67}(-2.06, t)V. \quad (2)$$

where the dominant orders in this system, q and v , and the constant, d , were determined via optimization by trial and error, and the time units are in 28-day months.

Using least squares, the unknowns a and b were determined. The resulting time response was

$$v(t) = 0.480 + 1.00R_{0.33, -0.67}(-2.06, t)V \quad (3)$$

which must be added to the initial resting DC-voltage of 0.24 V. The associated Laplace transform is

$$V(s) = \frac{0.48}{s} + \frac{1}{s^{0.67} (s^{0.33} + 2.06)} = \frac{1.48 (s^{0.33} + 0.668)}{s (s^{0.33} + 2.06)}. \quad (4)$$

The impedance transfer function is obtained by dividing this Laplace transform by the area of the current impulse, which is 5,390 A-s = 0.00223 A-months,

$$\frac{V(s)}{I(s)} = Z(s) = \frac{664 (s^{0.33} + 0.668)}{s (s^{0.33} + 2.06)} \quad (5)$$

again with the time units in months. The model response is plotted with the measured data in Fig. 2, which shows good agreement of the model response with the data.

The model can be converted to other time units by using the Laplace transform identity, $f(ct) = [F(s/c)]/c$, which must be applied to both the transfer function and a corresponding input impulse. Converting the time units to seconds = (1/2419200) months, gives $c = 1/2419200$, and the transfer function is then

$$\frac{V(s)}{I(s)} = Z(s) = \frac{0.000274 (s^{0.33} + 0.00498)}{s (s^{0.33} + 0.0153)}. \quad (6)$$

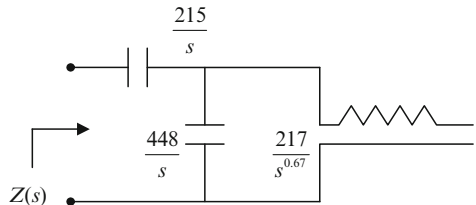
4 Circuit Realization

The impedance of Eq. 5 can be realized by the circuit shown in Fig. 3. This is readily seen by rewriting Eq. 5 as

$$Z(s) = \frac{215}{s} + \frac{448}{s + 2.06s^{0.67}}. \quad (7)$$

In Fig. 3, the first term is represented by the series capacitor, while the second term is represented by the parallel capacitor and fractional-order-line combination. This circuit provides additional physical insight into the time response of the ultracapacitor. An equivalent series resistor (ESR) does not result from these modeling assumptions, and could be included to improve the short-term model response.

Fig. 3 Circuit realization of the fractional-order ultracapacitor long-term model of Eq. 7, with time units in months



5 Conclusions

A long-term, fractional-order, linear, dynamic model for an ultracapacitor has been presented. The parameter identification is based on time-dependent current and voltage profiles, and as a result, the model provides a good fit to the transient behavior of the ultracapacitor. It is expected that improved long-term-accurate dynamic models will allow improved ultracapacitor management in electrical energy storage systems.

References

1. Belhachemi F, Rael S, Davat B (Oct. 2000) A physical based model of power electric double-layer supercapacitors. *IEEE Ind Appl Conf* 5:3069–3076
2. Mellor PH, Schofield N, Howe D (April 2000) Flywheel and supercapacitor peak power buffer technologies. *IEEE Seminar on Electric, Hybrid and Fuel Cell Vehicles*, pp 8/1–8/5
3. Moreno J, Ortuzar ME, Dixon JW (April 2006) Energy-management system for a hybrid electric vehicle, using ultracapacitors and neural networks. *IEEE Trans Ind Electron* 53: 614–623
4. Schupbach RM, Balda JC (Oct. 2003) The Role of ultracapacitors in an energy storage unit for vehicle power management. *Veh Technol Conf* 5:3236–3240
5. Dixon JW, Ortuzar ME (Aug. 2002) Ultracapacitors and DC-DC Converters in regenerative braking system. *IEEE Aero El Sys Mag* 17:16–21
6. Hicks J, Gruich R, Oldja A, Myers D, Hartley T, Veillette R, Husain I (2007) Ultracapacitor energy management and controller development for a series-parallel 2-by-2 hybrid electric vehicle. 2007 Vehicle Power and Propulsion Conference
7. Rafik R, Gualous H, Gallay R, Crausaz A, Berthon A (March 2007) Frequency, thermal and voltage supercapacitor characterization and modeling. *J Power Sources* 165(2):928–934
8. Buller S, Karden E, Kok D, De Doncker RW (Nov./Dec. 2002) Modeling the dynamic behavior of supercapacitors using impedances Spectroscopy. *IEEE Trans Ind Appl* 38(5):1622–1626
9. Faranda R, Gallina M, Son DT (May 2007) A new simplified model of double-layer capacitors. *International Conference on Clean Electrical Power*, pp 706–710
10. Nelms RM, Cahela DR, Tatarchuk BJ (April 2003) Modeling double-layer capacitor behavior using ladder circuits. *IEEE Trans Aero El Sys* 39(2), pp. 430–438
11. Miller JR (1995) Battery-capacitor power source for digital communication applications: simulations using advanced electrochemical capacitors. *Electrochem Soc Proc* 29–95: 246–254
12. Do KM (2004) A dynamic electro-thermal model of double layer supercapacitors for hev powertrain applications. M.S. Thesis, Department of Mechanical Engineering, The Ohio State University
13. Hartley TT, Lorenzo CF (Nov. 1998) Insights into the fractional-order initial value problem via semi-infinite systems. *NASA TM-1998-208407*
14. Hartley TT, Lorenzo CF (July 2002) Dynamics and control of initialized fractional-order systems. *Nonlinear Dynam* 29(1–4):201–233

IPMC Actuators Non Integer Order Models

**Riccardo Caponetto, Giovanni Dongola, Luigi Fortuna, Antonio Gallo,
and Salvatore Graziani**

Abstract In this chapter a new model for Ionic Polymer Metal Composites (IPMC) actuators, based on non-integer order models is proposed. IPMCs are very interesting polymer because of their capability to transform electrical energy into mechanical energy and vice-versa, making them particularly attractive for possible applications in different fields, such as robotics, aerospace, biomedicine, etc. An experimental setup has been realized to study the IPMCs behavior and an algorithm has been developed in Matlab environment in order to identify a fractional order model of IPMC actuators.

1 Introduction

In this work the authors propose new models for the electrical and electromechanical stages of Ionic Polymer Metal Composites (IPMC) actuators, based on non-integer order models.

IPMCs are innovative materials made of an ionic polymer membrane electroded on both sides with a noble metal. It is now well documented that IPMCs can exhibit large dynamic deformations if suitably electroded and forced by a time-varying voltage signal. Conversely, dynamic deformation of such ionic polymers produces dynamic electric fields across their electrodes. They can work either as low-voltage-activated motion actuators or as motion sensors [7, 8]. New models of the electrical and electromechanical stages of IPMC actuators allows us to estimate the IPMC actuator absorbed current and the relevant mechanical quantities (free deflection and/or blocked force). Knowledge of the absorbed current is a key factor in actuator design; indeed, it allows the power consumption for the device to be estimated and this is a fundamental aspect in applications such as autonomous robotic systems, where

R. Caponetto (✉), G. Dongola, L. Fortuna, A. Gallo, and S. Graziani
University of Catania, Engineering Faculty, D.I.E.E.S Viale A. Doria 6, 95125, Catania, Italy
e-mail: riccardo.caponetto@diees.unict.it; gdongola@diees.unict.it; lfortuna@diees.unict.it;
antonio@diees.unict.it; sgrazian@diees.unict.it

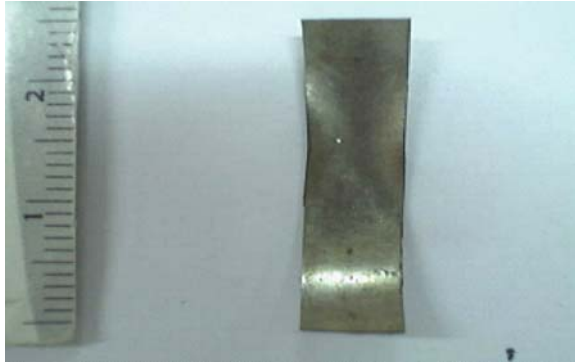


Fig. 1 A sample of IPMC

the designer must address the problem of power source availability. It is therefore of great importance to have a single model able to describe the relationships between applied voltage and absorbed current and between this current and the produced action [2]. For the specific application the ionic polymer used is Nafion (produced by Dupont and distributed by Sigma-Aldrich), while Platinum has been chemically deposited to form the electrodes [4]. The chemical process is applied to large sheets of Nafion. Subsequently some sheets undergo an ionic exchange process where hydrogen ions are replaced by sodium or lithium ions (this step is necessary in order to improve mechanical transduction performance). Each sheet is then cut into strips, using a surgical blade, to obtain the Device Under Test (DUT). A photo of samples used as DUTs is shown in Fig. 1.

2 Experimental Setup

When a voltage signal is applied across the thickness of the IPMC, mobile cations will move toward the cathode. Moreover, if an hydrated sample is considered, the cations will carry water molecules with them. The cathode area will expand while the anode area will shrink; consequently the polymer will bend toward the anode (Fig. 2a). Cations with a high hydration number will produce greater deformation than cations with a low hydration number [6]. For this reason, in motion-related applications, the hydrogen ion of the Nafion molecule is purposefully substituted, via an ion exchange process, with Na^+ , Li^+ , etc. Considering the beam parameters, the length L_{free} and the cross-sectional dimensions (thickness t and width w), we shall assume that the beam vibrates in the vertical plane (Fig. 2b).

The experimental setup is composed of a circuit to impose the voltage input signal to the membrane and a distance laser sensor to measure the tip deflection. The photo of the experimental setup is shown in Fig. 2c.

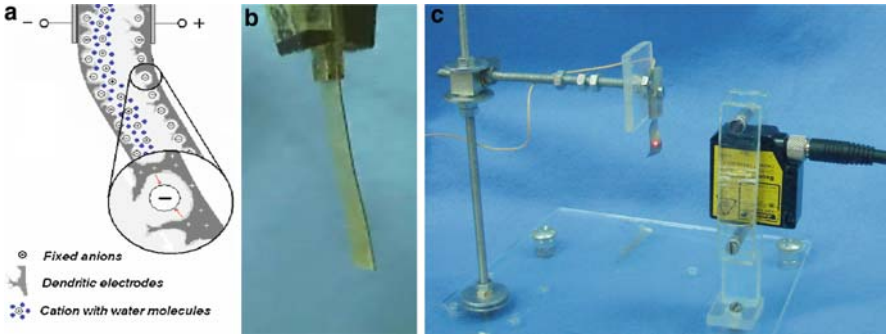


Fig. 2 (a) Chemical process of IPMC, (b) beam of IPMC, (c) photo of experimental setup

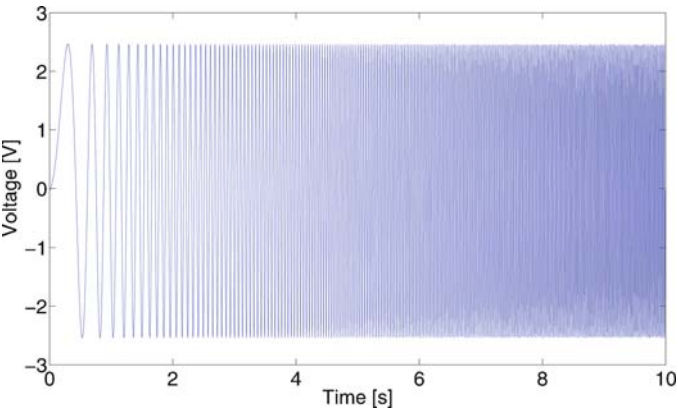


Fig. 3 Voltage input applied to the membrane

The deflection of the cantilever tip was measured with a commercially available distance laser sensor (Baumer Electrics OADM12U6430). Light from the laser diode was focused onto the end of the cantilever. The absorbed current is transduced by using a shunt resistor. The signals acquired by DAQ 6052E, that is, the voltage input imposed to the membrane, the current absorbed, and the deflection of the cantilever tip, measured with the laser sensor, are shown in Figs. 3–5, respectively.

The voltage input signal is a linear chirp signal from 500 mHz to 50 Hz. Using a sample frequency equal to 1,000 samples/s, 10,000 samples are obtained for a data acquisition campaign during 10 s. The output signal acquired, i.e. the deflection of the cantilever tip, shows clearly that the IPMC reaches the maximum deflection in the resonance condition.

Processing these data in Matlab Environment, the transfer functions voltage–current, current–deflection and voltage–deflection have been obtained, supposing that the system is linear, using the “tfestimate” Matlab function. Thus, the Bode diagram of the three functions, shown in Figs. 6, 7, and 8, have been estimated.

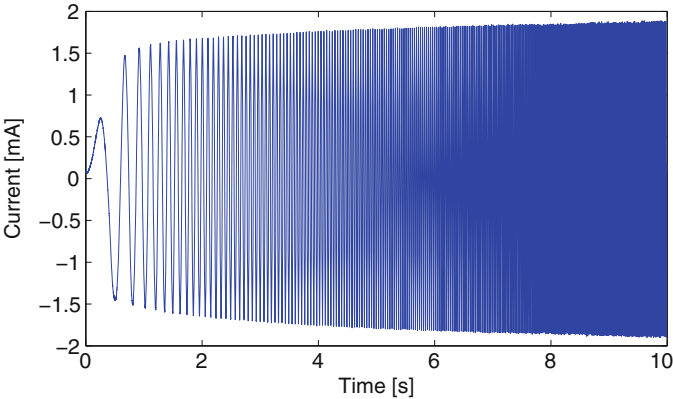


Fig. 4 Current absorbed by the membrane

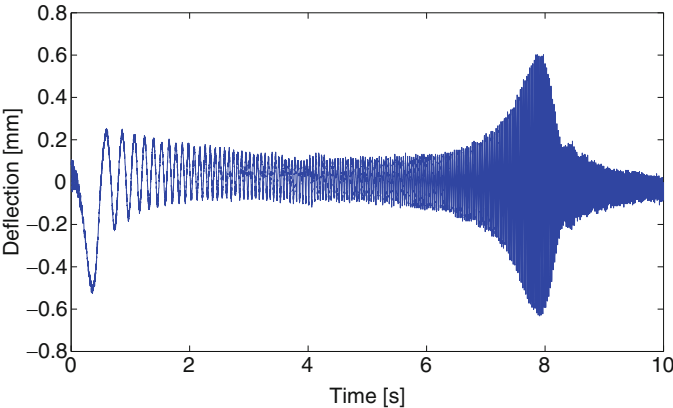


Fig. 5 Deflection of the cantilever tip measured with the laser sensor

The frequency analysis of the IPMCs behaviour is limited to the range frequency of 0.5–50 Hz because this is the range on which IPMCs work as actuators. By inspection of Bode diagrams it is clear that the three systems present a non integer order behaviour [1]. In fact the module Bode diagrams present a slope equal to $m \cdot 20$ db/decade, and the phase Bode diagrams present a phase lag equal to $n \cdot 90$, where m and n are real numbers. These Bode diagrams can be also identified by integer order models, but non integer order models allow to obtain comparable modeling performance by using a smaller set of parameters [2, 3].

Therefore, it was decided to identify the models of the three systems with non-integer order models. Since in this case the values of fractional exponents need to be estimated along with the corresponding transfer function zeros and poles values, the identification problem is nonlinear and an adequate optimization procedure needs to be used.

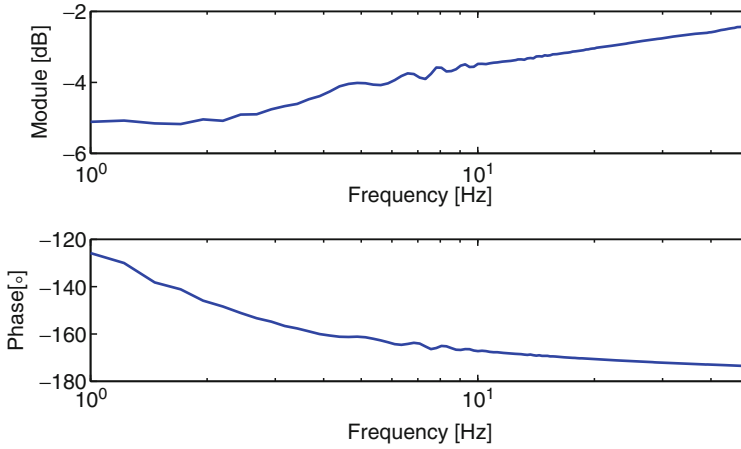


Fig. 6 Bode diagrams of the system voltage–current deduced from experimental data

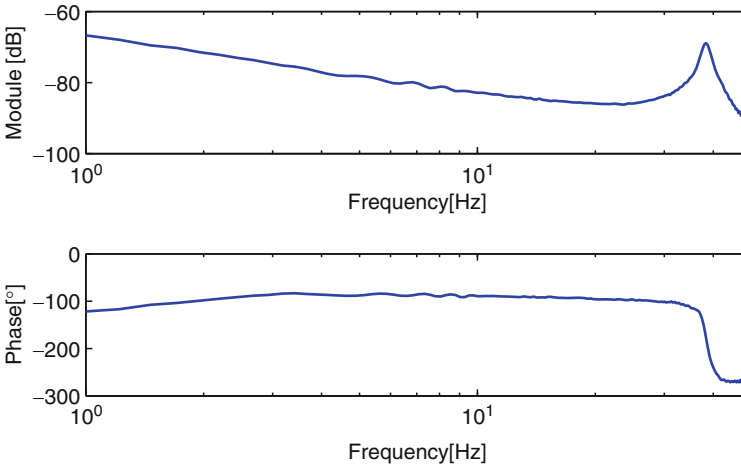


Fig. 7 Bode diagrams of the system current–deflection deduced from experimental data

3 Experimental Results

Applying the Marquardt algorithm [5] to the available data, the models obtained for the three transfer function, voltage–current, current–deflection and voltage–deflection are shown in Eqs. 1, 2, and 3, respectively.

$$\frac{I(s)}{V(s)} = 0.5s^{0.09} \frac{\left(\frac{s}{0.01} + 1\right)^{1.2}}{\left(\frac{s}{1.5} + 1\right)^{1.2}} \quad (1)$$

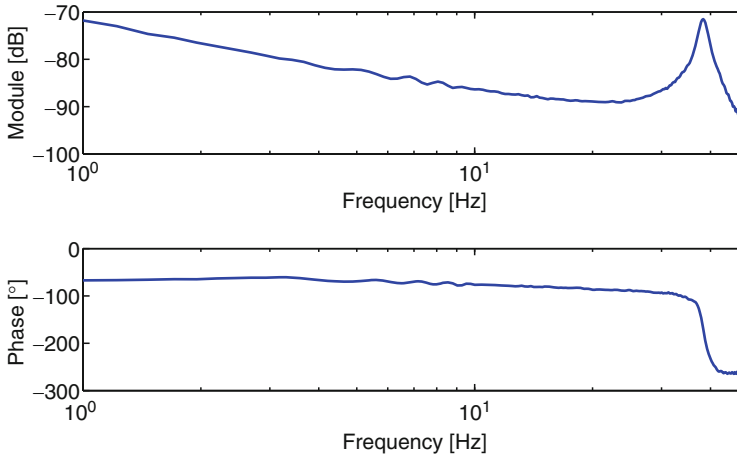


Fig. 8 Bode diagrams of the system voltage-deflection deduced from experimental data

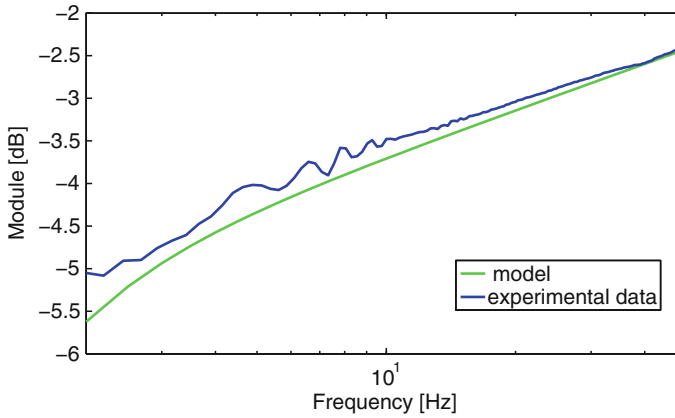


Fig. 9 Module comparison between predicted and measured voltage-current transfer function

$$\frac{D(s)}{I(s)} = \frac{680}{s^{0.876}(s^2 + 3.85s + 5880)^{1.15}} \frac{\left(\frac{s}{1.5} + 1\right)^{1.2}}{\left(\frac{s}{0.01} + 1\right)^{1.2}} \quad (2)$$

$$\frac{D(s)}{V(s)} = \frac{340}{s^{0.756}(s^2 + 3.85s + 5880)^{1.15}} \quad (3)$$

A comparison between the transfer functions as predicted by model and corresponding acquired data is shown in Figs. 9–14. More specifically, the Module and Phase Bode diagrams are shown, respectively. The graph shown in the reported

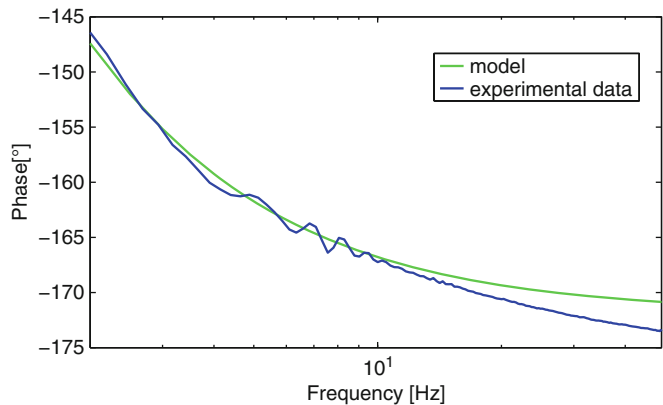


Fig. 10 Phase comparison between predicted and measured voltage-current transfer function

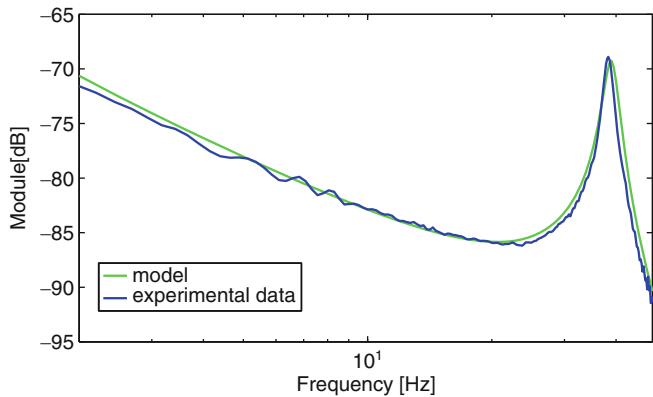


Fig. 11 Module comparison between predicted and measured current-deflection transfer function

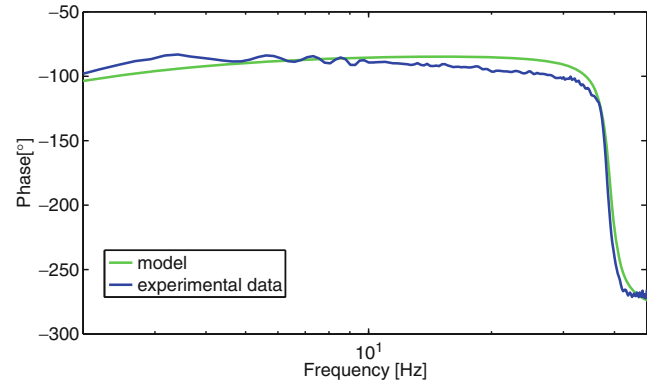


Fig. 12 Phase comparison between predicted and measured current-deflection transfer function

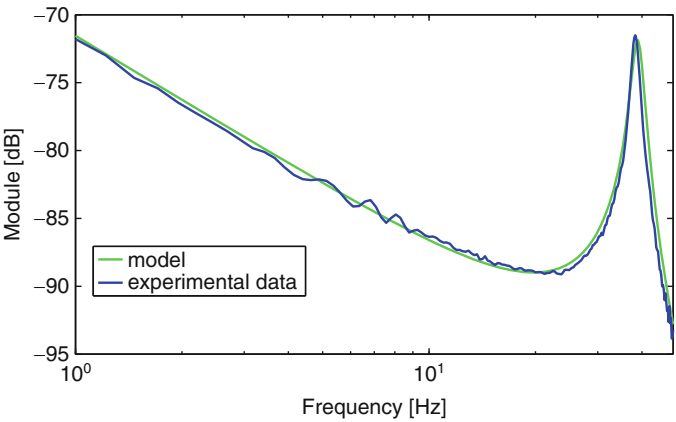


Fig. 13 Module comparison between predicted and measured voltage-deflection transfer function

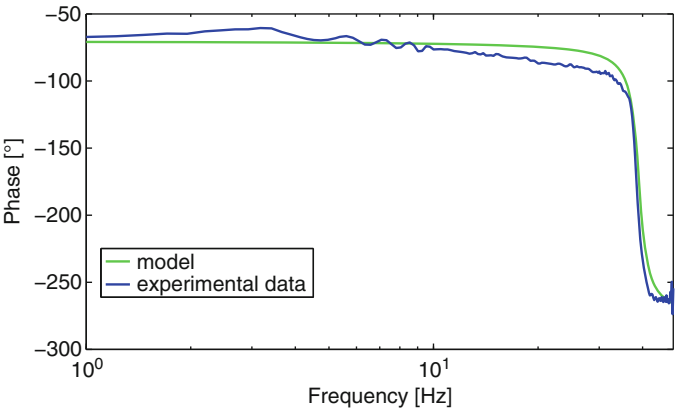


Fig. 14 Phase comparison between predicted and measured voltage-deflection transfer function

figures is referred to a 117 *Nafion* IPMC with Sodium as counter ion and 25 mm long, 3 mm wide, and 200 μm tick. Results show a good prediction of the frequency response.

Another important comment on the three transfer functions (1), (2), and (3) is that in the transfer function voltage–current and current–deflection there are any dynamics that mutually compensate them and do not appear in the total transfer function voltage–deflection.

In order to validate the obtained models also in the time domain, the estimated value of the deflection of the cantilever tip by the non integer order model is plotted in Fig. 15. This follows faithfully the values measured with the laser sensor.

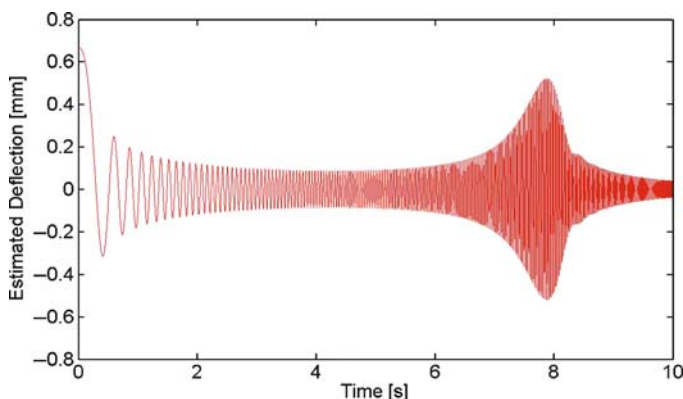


Fig. 15 Comparison in the time domain between the deflection measured with the laser sensor, and the value estimated by the non integer order model

4 Conclusion

In this chapter new models for the electrical and electromechanical stages of IPMC actuators, based on non-integer order models has been presented. In particular three fractional order models have been developed after that an experimental setup has been realized and the Marquardt algorithm has been developed in Matlab Environment. Next step is to define the link between the parameters that characterize the IPMC and the parameters of the model identified. This is a very valuable aspect for designers who intend to investigate the effects of changes in the geometry of IPMC based actuators. Work is in progress in the laboratories of the DIEES of the University of Catania to perform this analysis.

References

1. Arena P, Caponetto R, Fortuna L, Porto D (2000) Non linear non integer order systems - an introduction. Series A – vol 38, Nonlinear Science. World Scientific, Singapore
2. Bonomo C, Fortuna L, Giannone P, Graziani S, Strazzeri S (2007) A nonlinear model for ionic polymer metal composites as actuators. *Smart Mater Struct* 16:1–12
3. Caponetto R, Dongola G, Fortuna L, Graziani S, Strazzeri S (2008) A fractional model for IPMC actuators. *IEEE International Instrumentation and Measurement Technology Conference*. Canada
4. Kim KJ, Shahinpoor M (2003) Ionic polymermetal composites: II. Manufacturing techniques. *Smart Mater Struct* 12:65–79
5. Marquardt DW (1963) An algorithm for least-squares estimation of nonlinear parameters. *J Soc Ind Appl Math* 11–2:431–441
6. Nemat-Nasser S (2002) Micromechanics of actuation of ionic polymer-metal composites. *J Appl Phys* 92:2899–2915

7. Shahinpoor M, Kim KJ (2001) Ionic polymermetal composites: I. Fundamentals. *Smart Mater Struct* 10:819–833
8. Shainpoor M, Bar-Cohen Y, Simpson JO, Smith J (1998) Ionic polymer-metal composites (IPMC) as biomimetic sensors, actuators, and artificial muscle. *Smart Mater Struct* 7: R15–R30

On the Implementation of a Limited Frequency Band Integrator and Application to Energetic Material Ignition Prediction

Jocelyn Sabatier, Mathieu Merveillaut, Alain Oustaloup, Cyril Gruau, and Hervé Trumel

Abstract This paper describes an algorithm for the implementation a fractional order integrator. This algorithm is based on a fractional integrator transfer function approximation by a recursive distribution of poles and zeros. This algorithm is then used to predict the ignition of energetic materials. For these materials, the ignition under low velocity impact can be predicted through the fractional integration of a local function of pressure and of shear strain rate inside the material. It is shown that the proposed algorithm allows obtaining accurate results with a low computational complexity.

1 Introduction

This paper presents an application of fractional differentiation in the field of energetic materials (solid propellants and explosives) modelling. These materials may react violently under unwanted or malevolent impacts, and, for safety purposes, the impact velocity threshold at which ignition occurs must be predicted accurately. An adequate model exists in the literature [6, 7] and needs the computation of the fractional integral of a function $\varphi(t)$ of the shear strain rate and the pressure in the material.

The fractional integral of $\varphi(t)$ is obtained in this paper by filtering $\varphi(t)$ with a fractional order operator transmittance approximation, owing to the infinite dimension of the operator. The approximation is done in a user-defined frequency band, and is based on a recursive distribution of poles and zeros. Such an approximation

J. Sabatier (✉), M. Merveillaut, and A. Oustaloup
IMS - UMR 5218 CNRS, Université de Bordeaux, Bat A4, 351 cours de la Libération – F-33405
TALENCE Cedex, France
e-mail: jocelyn.sabatier@ims-bordeaux.fr; mathieu.merveillaut@ims-bordeaux.fr;
alain.oustaloup@ims-bordeaux.fr

C. Gruau and H. Trumel
CEA, DAM, Le Ripault, F-37260 Monts, France
e-mail: cyril.gruau@cea.fr; herve.trumel@cea.fr

appeared for the first time in the literature in the sixties in two different works probably done in parallel by both authors [8, 18]. Some years later, this method was used for similar implementations of fractional integrators [15]. Thereafter A. Oustaloup developed again this synthesis method [22] (without any a priori knowledge of the previous works in view of the limited distribution and the information media of the time), applied it several times, and extended it to complex order fractional integration [23].

In this paper, the above mentioned filtering method is applied to signals $\varphi(t)$ provided by a finite element simulation of the impact of a projectile on a heavily confined energetic material sample. The results are compared with a reference signal, also obtained numerically, but using the mathematic definition of the fractional integral of $\varphi(t)$. Few studies exist in the literature, that allow evaluating numerical errors associated with a pole and zero recursive distribution based fractional integration method. Also, no method exists to select the synthesis parameters imposed by the method. In the paper an analysis is proposed to fill this gap.

2 A Model to Predict Energetic Materials Ignition

Under specific dynamic solicitations (for instance impacts of heavy projectiles within a velocity range varying from 10 to 100 m/s), the deformation of some energetic materials lead to strong local temperature increase (the processes are macroscopically adiabatic due to the time scale considered).

The class of energetic materials considered in this study obeys Arrhenius-like chemical reaction kinetics, whereas the reaction is strongly exothermic. For fast deformation processes, the mechanical dissipation overcomes small scale thermal conduction, thus resulting in local heating. When a sufficiently high temperature is reached, rapid decomposition occurs, accompanied with chemical heat release. This process is known as “thermal explosion”.

This is described in a global fashion in the model put forward by [6, 7], who propose the following non-ignition condition:

$$\int_0^t (t - \tau)^{-n} \varphi(\tau) d\tau < \Omega. \quad (1)$$

In Eq. 1, the function $\varphi(t)$ is given by:

$$\varphi(t) = \left(\frac{\sigma_A(t)}{E} \right)^{\frac{2(n-1)}{3}} \dot{\epsilon}(t) \quad (2)$$

where

- n is a fractional number close to 0.77 depending on the nature of the energetic material (thermal conductivity, specific heat, chemical kinetics and heat release rate)
- $\sigma_A(t)$, E , $\dot{\epsilon}(t)$ denote respectively the macroscopic pressure, the Young's modulus and the norm of the shear strain rate inside the material
- Ω denotes the maximal level of integral (1) that ensures material non ignition; in other words, it represents the ignition threshold

To summarise, Eq. 1 is the mathematical translation of the following condition: there is no ignition of the considered energetic material if, in any point of the material, the fractional integral of order $\gamma = 1 - n \approx 0.2$ of the function $\varphi(t)$ remains less than a given value Ω , given that eqn. (1) can be rewritten as

$$I = \int_0^t (t - \tau)^{-(1-\gamma)} \varphi(\tau) d\tau < \Omega. \quad (3)$$

Further details on fractional differentiation can be found in [19,21,24,27].

3 A Testbench for Ignition Prediction Model Validation

In this work, the function $\varphi(t)$ is provided by a finite element numerical simulation of the mechanical structure in which the energetic material is inserted. For this purpose, the structure is meshed and the function $\varphi(t)$ is computed at each mesh node. Since the integral I of Eq. 3 is computed at each node, a numerical method for fractional order integration is required.

Figure 1 gives a representation of the 0.5 mm step mesh and the reference number of each node.

Figure 2 presents an example of function $\varphi(t)$ at node 1,314. In the sequel, only this node is considered. As shown in Figs. 2 and 3 zones can be distinguished:

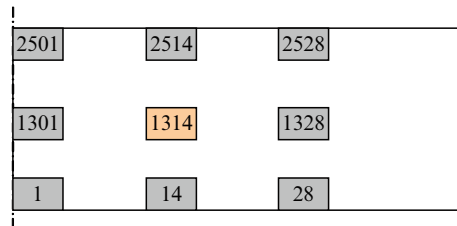


Fig. 1 Representation of the 0.5 mm step mesh and node numbering

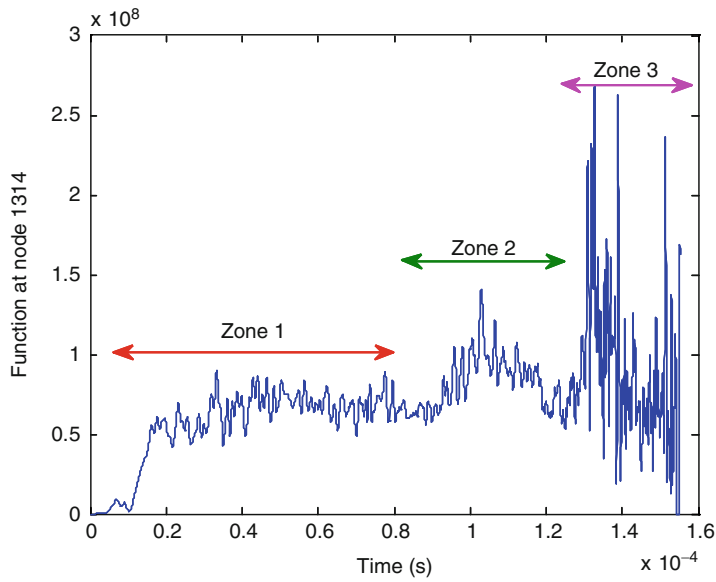


Fig. 2 An example of function $\varphi(t)$ obtained at node 1314

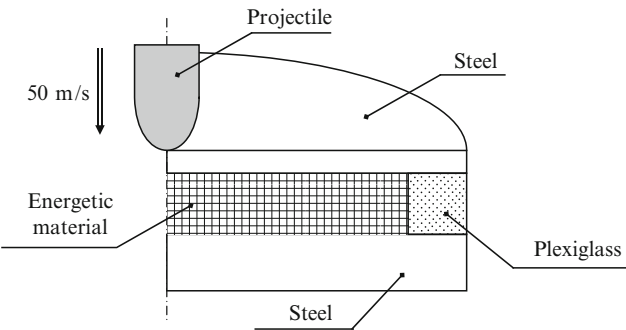


Fig. 3 Description of the mechanical structure in which the energetic material is inserted

- Zone 1: zone where $\sigma_A(t)$ is dominating with respect to $\dot{\epsilon}(t)$
- Zone 2: zone where $\dot{\epsilon}(t)$ is dominating with respect to $\sigma_A(t)$
- Zone 3: zone of numerical instability

The signal main characteristics are the following:

- Maximal time length: 1 ms,
- Variable sampling period with a minimal period of 1 ns.

In order to validate the model given by Eq. 3, the finite element software Abaqus Explicit, used to produce function $\varphi(t)$ and also to validate the fractional order integration algorithm for Eq. 3 computation, a testbench was constructed. As shown

in the representation of Fig. 3, the energetic material is confined between two steel plates and a plexiglas washer. This structure is impacted by a hemispherical nosed rigid projectile at a velocity of 50 m/s. Axisymmetry is assumed for numerical simplicity.

4 Fractional Integration Algorithm

4.1 Fractional Integration Numerical Method Specifications

Several methods for fractional integration can now be found in the literature [2, 3] (including some dedicated to fractional order system simulation):

- Approximation by a time varying system [4, 25]
- Variable change based methods [17, 31]
- Adams method generalisation [11]
- Variable step methods [1]
- Cubic order interpolation based methods [16]

For the considered application, fractional integration through filtering method is used. As illustrated by Fig. 4, the fractional integral of function $\varphi(t)$ corresponds to the output of a fractional order integrator filter $I^\gamma(s)$ whose input is function $\varphi(t)$.

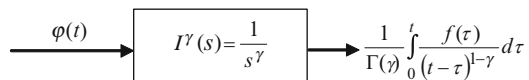
The fractional order operator is of infinite dimension. Thus, for practical implementation, the fractional order integrator filter must be approximated by an integer order transfer function. Several methods can then be used:

- Fractional order integrator discretisation (method resulting from Grunwald-Letnikov definition, used in [12] improved in [10, 28–30] and also [24] through “Short Memory Principle” introduction)
- Signal modelling methods (Pade, Prony or Schanck methods) [5]

To solve the ignition prediction problem, the method based on poles and zeros recursive distribution in a finite frequency band was selected. This choice is the result of an analysis of the specifications imposed by the problem. The fractional order integration algorithm must indeed:

- Allow (quasi) real-time computation (only a low number of operations at each node is required).
- Not require to store a large number of samples of function $\varphi(t)$ (given the large number of nodes, it is impossible to store all the function $\varphi(t)$ history to compute the fractional integral by post computation).

Fig. 4 Fractional order integration of $\varphi(t)$ through filtering



- Allow variable sampling period (a variable step method is used to produce functions $\varphi(t)$).
- Be implemented easily and the user be guided to select algorithm parameters.
- Allow error estimations.

4.2 Algorithm Description

To obtain a fractional order integrator approximation, a method consists, in a first step, in limiting the frequency band on which the fractional order behaviour is wished. The following approximation can thus be adopted:

$$I^\gamma(s) \approx I_a^\gamma(s) = C_0 \left(\frac{1 + \frac{s}{\omega_h}}{1 + \frac{s}{\omega_b}} \right)^\gamma. \quad (4)$$

The transfer function (4) however remains fractional. To obtain an approximation by a transfer function only involving integer powers of Laplace variable s , a method consists in the substitution of:

- A series of stairs to the gain asymptotic diagram (to get a gain diagram whose slope is not a multiple of -20 dB/decade)
- A series of crenels to the phase asymptotic diagram (to get a phase diagram which is not a multiple of -90°)

Such an idea is illustrated by Fig. 5 and leads to approximate the transmittance $I_a^\gamma(s)$ by

$$I_a^\gamma(s) = \left(\frac{1 + \frac{s}{\omega_h}}{1 + \frac{s}{\omega_b}} \right)^\gamma \approx I_N^\gamma(s) = \frac{\prod_{k=1}^N \left(1 + \frac{s}{\omega'_k} \right)}{\prod_{k=1}^N \left(1 + \frac{s}{\omega_k} \right)} \quad (5)$$

where corner frequencies ω_k and ω'_k are recursively distributed:

$$\omega'_{i+1} = \alpha\eta\omega'_i \quad \omega_{i+1} = \alpha\eta\omega_i \quad \omega'_1 = \alpha\omega_1 \quad (6)$$

with

$$\alpha\eta = 10^{\frac{\log(\omega_h/\omega_b)}{N}} \quad \alpha = 10^{\gamma \log(\alpha\eta)} \quad \omega_1 = \omega_b \sqrt{\eta}. \quad (7)$$

This recursivity is illustrated by Fig. 5, that presents a comparison of transmittances $I^\gamma(s)$, $I_a^\gamma(s)$ and $I_N^\gamma(s)$ in asymptotic Bode diagrams.

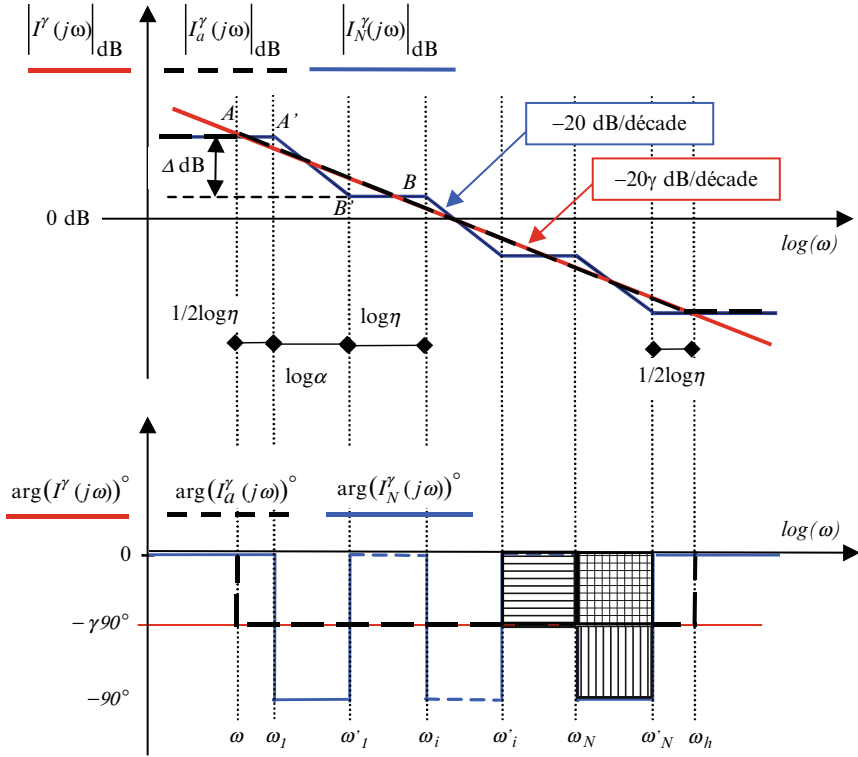


Fig. 5 $I^\gamma(s)$, $I_a^\gamma(s)$ and $I_N^\gamma(s)$ Bode diagram comparison

5 Application to Ignition Prediction of Energetic Materials

5.1 Implementation Description

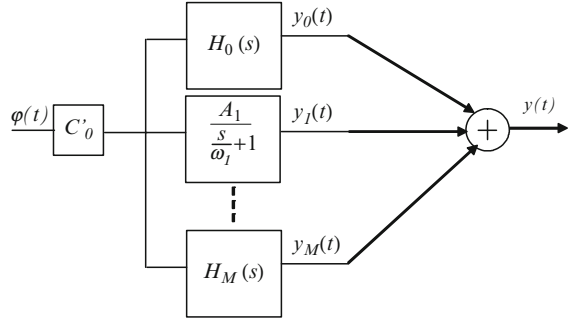
Approximation $I_N^\gamma(s)$ of relation (5) is now applied to fractional integration of function $\varphi(t)$. A partial fraction expansion of $I_N^\gamma(s)$ is computed in order to reduce:

- Numerical errors accumulation (only one past sample is used in the recurrence relation of each term)
- Requested storage memory

and leads to

$$I_N^\gamma(p) = C'_0 \frac{\prod_{k=1}^N \left(\frac{s}{\omega'_k} + 1 \right)}{\prod_{k=1}^N \left(\frac{s}{\omega_k} + 1 \right)} = C'_0 \left(A_0 + \sum_{k=1}^N \frac{A_k}{\frac{s}{\omega_k} + 1} \right) \quad (8)$$

Fig. 6 Algorithm implementation



with $A_0 = 1$ and

$$A_i = \frac{\prod_{k=1, k \neq i}^N \left(\frac{-\omega_i}{\omega'_k} + 1 \right)}{\prod_{k=1, k \neq i}^N \left(\frac{-\omega_i}{\omega_k} + 1 \right)} \quad i \in \{1 \dots N\}. \quad (9)$$

As illustrated by Fig. 6, the fractional integration of function $\varphi(t)$ thus consists in filtering the signal $\varphi(t)$ by a set of first order transmittances given by

$$C'_0 \frac{A_k}{\left(\frac{s}{\omega_k} + 1 \right)} = \frac{Y_k(s)}{U(s)} = H_k(s). \quad (10)$$

In order to reach the best compromise between accuracy and memory saving, the discretisation of filters $H_k(s)$ is carried out using the Euler's formula

$$\frac{d}{dt} \approx \frac{1 - z^{-1}}{h} \quad (11)$$

in which z^{-1} denotes the delay operator and h denotes the sampling period. For reactive material ignition prediction, Tustin and Al-Alaoui formulas were also implemented. However it was found that these schemes lead to an increased computation complexity for limited accuracy improvement.

5.2 Some Rules of Thumb

The filter synthesis method previously described imposes to select three parameters: corner frequencies ω_b and ω_h , and number of cells N .

Various tests allowed obtaining the following rule of thumb to choose these parameters (see also [9] and [14]). A good compromise between accuracy and computational complexity is obtained with:

$$N \approx 1.5 \log \left(\frac{\omega_b}{\omega_h} \right). \quad (12)$$

If T_{fin} denotes the signal duration, then the parameter ω_b must be

$$\omega_b = \frac{2\pi}{10^3 \times T_{fin}}. \quad (13)$$

If ΔT_{min} denotes the minimal time step, then the parameter ω_h must be

$$\omega_h = \frac{2\pi}{\Delta T_{min}}. \quad (14)$$

A justification of these last three relations using error bounds is provided in [26]. A more explicit justification (based on H_∞ type error bounds computation [20]) will be presented elsewhere due to the paper size limitation.

5.3 Reference Signal Computation

Given that the function $\varphi(t)$ to be integrated is sampled with a variable sampling period, it is supposed that $\varphi(t)$ is defined by

$$\varphi(t) = \sum_{k=0}^M \varphi_k (H(t - t_k) - H(t - t_{k+1})) \quad (15)$$

in which $H(t)$ denotes the Heaviside function such that $H(t) = 0$ for $t \leq 0$ and $H(t) = 1$ for $t > 0$.

The γ -order integral of $\varphi(t)$ is thus defined by

$$I^\gamma \{\varphi(t)\} = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{\sum_{k=0}^M \varphi_k (H(\tau - t_k) - H(\tau - t_{k+1}))}{(t - \tau)^{1-\gamma}} d\tau \quad (16)$$

and thus

$$I^\gamma \{\varphi(t)\} = \sum_{k=0}^M \varphi_k \frac{1}{\Gamma(\gamma + 1)} ((t - t_k)^\gamma H(t - t_k) - (t - t_{k+1})^\gamma H(t - t_{k+1})) \quad (17)$$

given that

$$\int_0^t \frac{H(\tau - t_k)}{(t - \tau)^{1-\gamma}} d\tau = \int_{t_k}^t \frac{d\tau}{(t - \tau)^{1-\gamma}} = \frac{1}{\gamma} (t - t_k)^\gamma. \quad (18)$$

The signal thus obtained, denoted by $y_{ref}(t)$, will be used as a reference one to analyze the accuracy of the algorithm previously described. Also note that the computation of $y_{ref}(t)$ is really time and memory consuming. Equation 17 thus cannot be used for the quasi-real time ignition prediction problem (the algorithm of Sect. 5.1 is much better).

5.4 Application and Result Analysis

The signal provided by the finite element software at node 1314 is now integrated. For this signal $T_{fin} = 1.6 \cdot 10^{-4}$ and $\Delta T_{min} = 2 \cdot 10^{-9}$, Eqs. 13 and 14 lead to

$$\omega_b = 1,25 \text{ rad/s}, \quad \omega_h = 3,14 \cdot 10^9 \text{ rad/s}. \quad (19)$$

If $N = 15$ is chosen and given that $\gamma = 0, 2$, the parameters of Eq. 11 are given by

$$\alpha\eta = 4,23 \quad \alpha = 11,33. \quad (20)$$

Figure 7 presents a comparison of the reference signal $y_{ref}(t)$ with the one resulting from the fractional order integration algorithm previously described. The comparison highlights that the numerical method used produces highly accurate results.

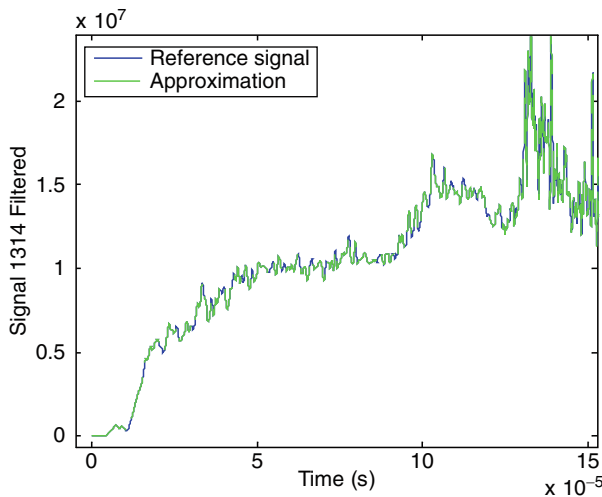
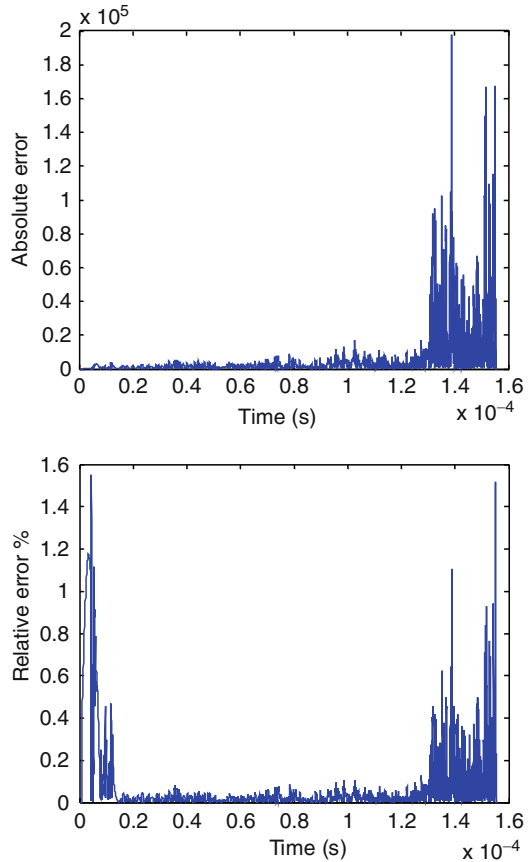


Fig. 7 Comparison of the reference signal with the signal obtained by the implemented algorithm

Fig. 8 Absolute and relative errors between the reference signal and the signal obtained by the implemented algorithm



To evaluate the algorithm accuracy, Fig. 8 presents the absolute and relative errors between the reference signal and the signal obtained through filtering.

6 Conclusion

In this paper, fractional order integration is applied to energetic material ignition prediction. According to a model recently published, ignition occurs if the fractional order integral of a function $\varphi(t)$ of pressure and shear strain rate reaches a given threshold. In the present application, the pressure and shear strain rate are computed by a finite element software which simulates the mechanical behaviour of a testbench, in order to validate the ignition model and the fractional order integration method.

The computation of the fractional order integral of function $\varphi(t)$ is here based on its filtering by a transfer function is an approximation in the frequency domain

of a fractional order integrator (limited frequency band integrator). It is based on a recursive distribution of poles and zeros. A discretisation scheme is then applied to the resulting transfer function.

This method was chosen because it is one among the few methods that simultaneously:

- Allows low computational complexity
- Does not require to store a large number of samples of the function $\varphi(t)$ (it is impossible to store all the function $\varphi(t)$ history to compute the fractional integral by post computation)
- Allows variable sampling period (a variable step method is used to produce functions $\varphi(t)$)
- Can be implemented easily and proposes methods to select implementation parameters,
- Provides approximation error bounds

This study has demonstrated that the fractional integration method used provides very accurate results while remaining fairly simple to implement and requiring quite limited memory and computational time. It was also shown that other limited frequency band approximations or discretisation scheme that can be used do not yield much improved accuracy but greatly increase a lot computational complexity. The practical interest of the presented method has been recently demonstrated in [13].

References

1. Adolfsson K, Enelund M, Larsson S (2004) Adaptive discretization of fractional order viscoelasticity using sparse time history. *Comput Method Appl Mech Eng* 193:4567–4590
2. Aoun M, Malti R, Levron F, Oustaloup A (2003) Numerical simulations of fractional systems. First Symposium on fractional derivatives and their applications at ASME-DETC Conference (International Design Engineering Technical Conferences). Chicago, USA
3. Aoun M, Malti R, Levron F, Oustaloup A (2004) Numerical simulation of fractional systems: an avoerview of existing methods and improvements. *Nonlinear Dynam* 38(1–4):117–131
4. Arena P, Fortuna L, Porto D (1998) Fractional integration via time-varying systems. International Symposium on Nonlinear Theory and its Applications (NOLTA'98). Crans-Montona, Switzerland, pp 1125–1127
5. Barbosa RS, Tenreiro Machado JA, Ferreira I (2004) Least squares design of digital fractional order operators. First IFAC workshop on Fractional differentiation and its applications, FDA'04. Bordeaux, France
6. Browning RV (1995) Microstructural model of mechanical initiation of energetic materials. In: Schmidt SC (eds) *Proceedings of Shock Compression of Condensed Matter*. AIP Conference Proceedings 370, New York
7. Browning RV, Scammon RJ (2001) Microstructural model of ignition for time varying loading conditions. In: Furnish MDE, Thadhani NN, Horie, Y (eds) *Proceedings of Shock Compression of Condensed Matter*. AIP Conference Proceedings 620, New York
8. Carlson GE, Halijak CA (1961) Simulation of the fractional derivative operator $\sqrt{s}$ and the fractional integral operator $1/\sqrt{s}$. In: *Proceedings of the Central State Simulation Council Meeting*. Kansas State University Bulletin 45:7.
9. Carmona Ph, Coutin L (1998) Fractional Brownian motion and the Markov property. *Elect Comm Probab* 3:95–107

10. Chen Y, Moore KL (2002) Discretization schemes for fractional-order differentiators and integrators. *IEEE Trans Circ Sys: Fund Theor Appl* 49(3):363–367
11. Diethelm K, Ford NJ, Freed AD (2004) Detiled error analysis for a fractional Adams method. *Numer Algorithms* 36:31–52
12. Gorenflo R (1997) Fractional calculus: some numerical methods. In: Carpinteri A, Mainardi F (eds) *Fractal and Fractional Calculus in Continuous Mechanisms*. Springer, New York, pp 277–290
13. Gruau C, Picart D, Belmas R, Bouton E, Delmaire-Sizes F, Sabatier J, Trumel H (2009) Ignition of a confined high explosive under low velocity impact. *Int J Impact Engng* 36(4):537–550
14. Guglielmi M (2006) $1/f^{\alpha}$ signal synthesis with precision control. *Signal Process* 86(10):2548–2553
15. Ichise M, Nagayanagi Y, Kojima T (1971) An analog simulation of non integer order transfer functions for analysis of electrode processes. *J Electroanal Chem Interfacial Electrochem* 33:253–265
16. Kumar P, Agrawal OP (2005) A cubic scheme for numerical solution of fractional differential equations. In: *Proceedings of the Euromech Conference, ENOC'2005*. Eindhoven, the Netherlands
17. Ma C, Hory Y (2003) Geometric interpretation of discrete fractional order controllers based on sampling time scaling property and experimental verification of fractional $1/s^{\alpha}$ systems' robustness. Symposium on fractional derivatives and their applications at ASME-DETC Conference (International Design Engineering Technical Conferences). Chicago, USA
18. Manabe S (1961) The non integer integral and its application to control systems. *ETJ Japan* 6(3–4):83–87
19. Miller KS, Ross B (1993) An introduction to the fractional calculus and fractional differential equations. Wiley-Interscience, New York
20. Moze M, Sabatier J, Oustaloup A (2008) On bounded real lemma for fractional systems. IFAC world Congress. Seoul, South Korea
21. Oldham KB, Spanier J (1974) The fractional calculus: theory and applications of differentiation and integration to arbitrary order. Academic, New York and London
22. Oustaloup A (1975) Etude et réalisation d'un système d'asservissement d'ordre $3/2$ de la fréquence d'un laser à colorant continu. Ph.D. Thesis, Univ. Bordeaux I, France
23. Oustaloup A (1995) La dérivation non entière, théorie, synthèse et applications, Editions Hermès
24. Podlubny I (1999) Fractional differential equations. *Mathematics in Sciences and Engineering*, no.198. Academic Press, San Diego
25. Rutman RS, Gorenflo R (1995) Simulation and inversion of fractional integration: a systems theory approach. In: *Proceedings of the International Workshop on Inverse Problems*, 17–19 January 1995, HoChiMinh City. Publications of the HoChiMinh City Mathematical Society, Vol 2, pp 141–148
26. Sabatier J, Merveillaut M, Malti R, Oustaloup A (2008) On a representation of fractional order systems: interests for the initial condition problem – 3rd IFAC Workshop on “Fractional Differentiation and its Applications” (FDA'08). Ankara, Turkey, 5–7 November, 2008
27. Samko SG, Kilbas AA, Marichev OI (1993) Fractional integrals and derivatives. Gordon and Breach Science Publishers, London
28. Teinreiro Machado JA (1997) Analysis and design of fractional order control systems. *J Syst Anal – Model Simul* 27:107–122
29. Valério D, Sa da Costa J (2002) Time domain implementations of non-integer order controllers. In: *Proceedings of Controlo'2002*. Portuguese Conference on Automatic Control. Portugal, pp 353–358
30. Valério D, Sa da Costa J (2003) Digital implementation of non-integer control and its application to a two link robotic arm. In: *Proceeding of the European Control Conference, ECC'2003*. Cambridge, UK
31. Wahi P, Chatterjee A (2003) Averaging for oscillations with light fractional order damping. First Symposium on fractional derivatives and their applications at ASME-DETC Conference (International Design Engineering Technical Conferences). Chicago, USA

Fractional Order Model of Beam Heating Process and Its Experimental Verification

Andrzej Dzieliński and Dominik Sierociuk

Abstract In the paper the application of fractional order calculus to the modelling of a beam heating process is discussed. The original process description in the form of the partial differential equation (Heat Transfer Equation) is transformed into a fractional order partial differential equation when the heat-flux is treated as the system input and the temperature is the system output. Using the Laplace transform, the transfer function of the beam heating system and its frequency response are obtained from the time-domain description. The theoretical results are verified with the experimental setup, using the thermoelectric (Peltier) module. The experimental results match the theoretical ones with high degree of accuracy.

1 Introduction

In many thermal problems the temperature of the body is related to the heat flux. One of the examples of such a problem is the temperature of electrical windings which strongly depends on the heat flux generated by the currents (see e.g. [8]). In such a case the dynamical relation between the temperature and the heat flux is described by the Partial Differential Equation (Heat Diffusion Equation) [2]. In the paper, following the approach from [8], the fractional order model of the heat diffusion process is given and its properties are discussed. This and other applications of fractional order calculus to modelling of thermal processes is investigated e.g. in [1]. On the other hand, the control of such thermal plants have been discussed in [5, 6] where the fractional order PID-type controller has been employed to the

A. Dzieliński

Institute of Control and Industrial Electronics, Warsaw University of Technology, Koszykowa 75,
00-662 Warsaw, Poland
e-mail: adziel@isep.pw.edu.pl

D. Sierociuk (✉)

Institute of Control and Industrial Electronics, Warsaw University of Technology, Koszykowa 75,
00-662 Warsaw, Poland
e-mail: dsieroci@isep.pw.edu.pl

temperature control problem. In this paper the use of fractional order calculus to modelling of the beam heating using a thermoelectric module also known as Peltier module is presented.

The fractional calculus is a generalization of a traditional integer order integral and differential calculus for real or even complex orders. One of these generalizations results is the following Riemann–Liouville definition of differ-integral [4, 7]:

Definition 1. Riemann–Liouville definition of fractional order differ-integral

$${}_a D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_a^t (t-u)^{-\alpha-1} f(u) du & \alpha < 0 \\ \frac{d^m}{dt^m} \left[\frac{1}{\Gamma(m-\alpha)} \int_a^t (t-u)^{m-\alpha-1} f(u) du \right] & \alpha > 0, \end{cases}$$

where

$$m-1 \leq \alpha < m$$

and $\alpha \in \mathbb{R}$ is a fractional order of the differ-integral of the function $f(t)$. ■

The Laplace transform of the fractional order differ-integral is given as follows:

$$\mathcal{L} [{}_0 D_t^\alpha f(t)] = \begin{cases} s^\alpha F(s) & \alpha < 0 \\ s^\alpha F(s) - \sum_{k=0}^{j-1} s^k {}_0 D_x^{\alpha-k-1} f(0) & \alpha > 0, \end{cases}$$

where $j-1 < \alpha \leq j$, and $j \in \mathbb{N}$

1.1 Transfer Function $e^{(Ts)^\alpha}$ and Its Bode Diagram

Let us assume the following transfer function

$$G(s) = e^{(Ts)^\alpha}, \quad (1)$$

for which the spectral transfer function is given by

$$G(j\omega) = e^{(Tj\omega)^\alpha} = e^{(T\omega)^\alpha (\cos \frac{\pi}{2} \alpha + j \sin \frac{\pi}{2} \alpha)}. \quad (2)$$

The magnitude of the transfer function is given as follows

$$A(\omega) = e^{(T\omega)^\alpha \cos \frac{\pi}{2} \alpha}, \quad (3)$$

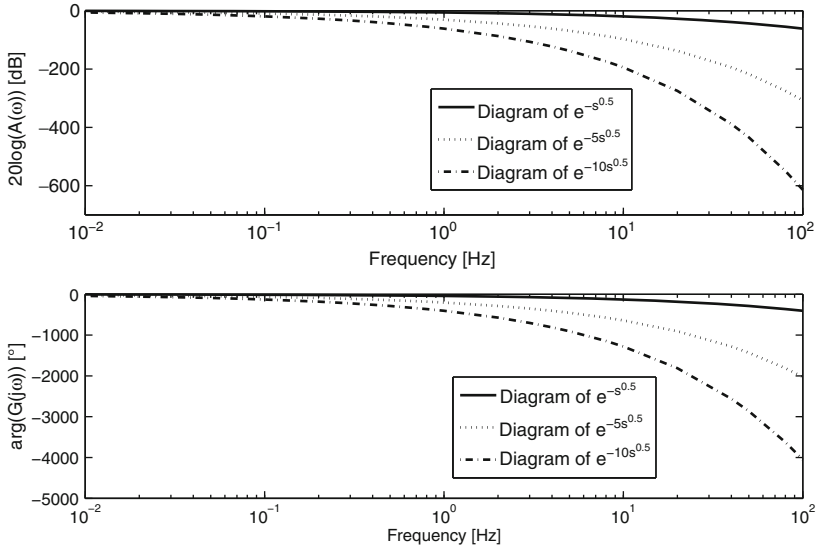


Fig. 1 Bode diagrams of $e^{-(Ts)^{0.5}}$ systems for $T = 1, 5, 10$

which gives

$$M(\omega) = 20 \log(e^{(T\omega)^\alpha \cos \frac{\pi}{2}\alpha}). \quad (4)$$

The phase properties are given as follows:

$$\varphi(\omega) = (T\omega)^\alpha \sin \frac{\pi}{2}\alpha.$$

The Bode diagram of this transfer function for different values of T and $\alpha = 0.5$ is presented in Fig. 1.

The interpretation of the transfer function $e^{-(Ts)^{0.5}}$ is not as easy as the interpretation of the $e^{-(Ts)}$ function. Looking at the frequency response it can be noted that this is not a pure delay system where the phase shift changes exponentially, but there is also a non-zero effect in the magnitude.

2 Mathematical Description of Heating Process

The heating process of a semi-infinite beam can be described by the following partial differential Eq. [2]

$$\frac{\partial^2}{\partial \lambda^2} T(t, \lambda) = a^2 \frac{\partial}{\partial t} T(t, \lambda), \quad (5)$$

with the following boundary conditions:

$$T(0, \lambda) = 0, \quad T(t, 0) = u(t),$$

where $T(t, \lambda)$ is a temperature of the beam at time instant t and coordinate λ , and a is a parameter which depends on beam parameters like heat conductivity and density.

Let us assume the following equation where the boundary conditions are the same as in (5):

$$\frac{\partial}{\partial \lambda} T(t, \lambda) = a \frac{\partial^{0.5}}{\partial t^{0.5}} T(t, \lambda). \quad (6)$$

By applying the derivative $\frac{\partial}{\partial \lambda}$ to both sides of the equation the following relation is obtained:

$$\frac{\partial^2}{\partial \lambda^2} T(t, \lambda) = a \frac{\partial^{0.5}}{\partial t^{0.5}} \frac{\partial}{\partial \lambda} T(t, \lambda). \quad (7)$$

Using again Eq. (6) we achieve

$$\frac{\partial^2}{\partial \lambda^2} T(t, \lambda) = a^2 \frac{\partial^{0.5}}{\partial t^{0.5}} \frac{\partial^{0.5}}{\partial t^{0.5}} T(t, \lambda). \quad (8)$$

This finally gives the traditional heat transfer partial differential Eq. (5).

Using the following notation

$$H(t, \lambda) = \frac{\partial}{\partial \lambda} T(t, \lambda), \quad (9)$$

where $H(t, \lambda)$ is the heat flux at time t and length coordinate λ , the following equation is obtained:

$$H(t, \lambda) = a \frac{\partial^{0.5}}{\partial t^{0.5}} T(t, \lambda). \quad (10)$$

Hence by first order differentiation with respect to λ of both side of previous equation and using Eq. (9) the following fractional order partial differential equation, describing heat flux transfer is achieved:

$$\frac{\partial}{\partial \lambda} H(t, \lambda) = a \frac{\partial^{0.5}}{\partial t^{0.5}} H(t, \lambda). \quad (11)$$

Applying the Laplace transformation with respect to t to this equation we obtain

$$\frac{\partial}{\partial \lambda} H(s, \lambda) = as^{0.5} H(s, \lambda) - {}_0 D_t^{-0.5} H(0, \lambda). \quad (12)$$

The solution of this equation (for $H(0, \lambda) = 0$) is given as follows

$$H(s, \lambda) = e^{as^{0.5}\lambda} H(s, 0), \quad (13)$$

from where the following relation describing the heat transfer with respect to the heat flux is obtained

$$T(s, \lambda) = \frac{1}{as^{0.5}} e^{as^{0.5}\lambda} H(s, 0). \quad (14)$$

3 Experimental Verification of Heating Process Model

The heat distribution process modelling by fractional order PDEs and their respective counterparts in frequency domain has been verified by the experiments with real physical thermal system. The results obtained from the model proposed have been compared with those obtained from the experiment.

3.1 Experimental Setup

The experimental setup contains:

1. dSPACE DS1103 PPC card with a PC
2. Electronic interface with OPA 549 power amplifier
3. Thermoelectric (Peltier) module SCTB NORD TM-127-1.0-3.9-MS
4. Six temperature sensors LM35DH

and its idea is presented in Fig. 2. The placement of the temperature sensors is depicted in Fig. 3

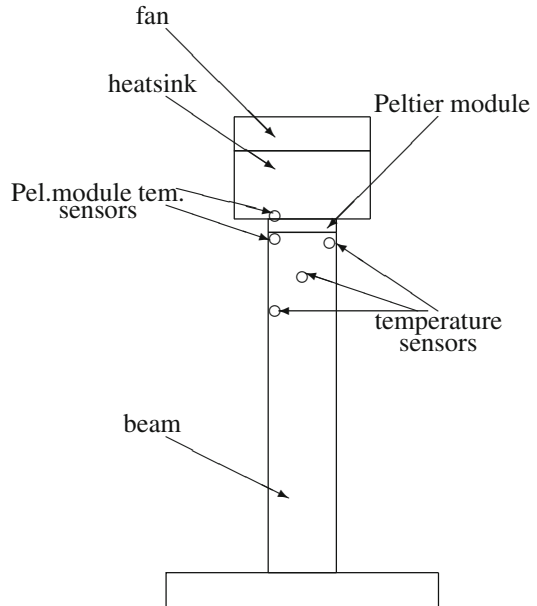


Fig. 2 The experimental setup

Fig. 3 Sensors placement on the beam

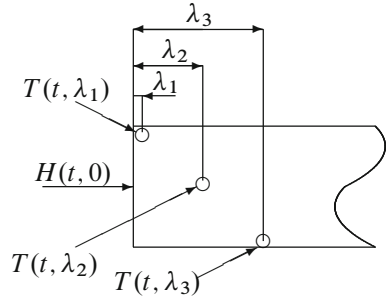
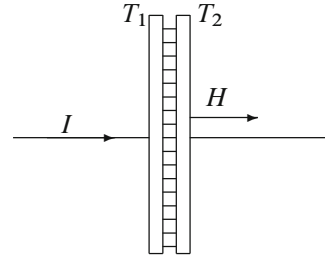


Fig. 4 Peltier module diagram



3.2 Thermoelectric Module – Peltier Module

In the experiment the TM-127-1.0-3.9-MS Peltier module, produced by SCTB NORD, is used. The main parameters of the module are as follows: $I_{max} = 3.9A$, $U_{max} = 15.5V$, $P_{max} = 34.0W$, $\Delta T_{max} = 71.0^\circ C$.

Peltier module can be described by the following Eq. [3]

$$H = \alpha I T_2 + \frac{1}{2} k R I^2 - k K_p (T_2 - T_1), \quad (15)$$

where H is the value of heat flux, k is the direction of the flux, α , R , K_p are the Peltier module parameters, I is the current applied to the module, T_1 and T_2 are the temperatures of the both sides of the module. A simplified diagram of the Peltier module used is given in Fig. 4.

Based on the module characteristics provided by the producer the following parameters are obtained: $R = 4.8879 \Omega$, $\alpha = 0.0565$, $K_p = 0.4978$. From Eq. (15) the following relation for obtaining the module current for desired value of heat flux is then achieved:

$$I = \frac{-k\alpha T_2 + \sqrt{(\alpha T_2)^2 + 2R(K_p(T_2 - T_1) - H)}}{kR}. \quad (16)$$

3.3 Modelling Results

The transfer function based on Eq. (14) is given as follows (the additional parameters are used for modelling unknown relations eg. current–heat flux):

$$G(\lambda, s) = \frac{T(s, \lambda)}{H(s, 0)} = \frac{T_1}{s^{0.5}} e^{\lambda(T_2 s)^{0.5}}. \quad (17)$$

For desired values of $\lambda = \{\lambda'_1, \lambda'_2, \lambda'_3\}$ corresponding to three sensors mounted on the beam, the set of transfer functions is obtained. Values of λ'_i are regularized for $\lambda'_1 = 1$.

The transfer function (17) was derived with the assumption that the heat is not emitted to the outside of the beam. This does not happen in a real plant, so we have to adjust the transfer functions by replacing the fractional (0.5) order integrator by the fractional (0.5) order inertia unit. In such a case the transfer function has the following form:

$$G(\lambda, s) = \frac{T(s, \lambda)}{H(s, 0)} = \frac{T_1}{(T_3 s)^{0.5} + 1} e^{\lambda(T_2 s)^{0.5}}. \quad (18)$$

For $\lambda = \lambda_1$ the following parameters of the transfer function were achieved using Bode diagram matching:

$$T_1 = 1.758, \quad T_2 = 12.875, \quad T_3 = 88.799.$$

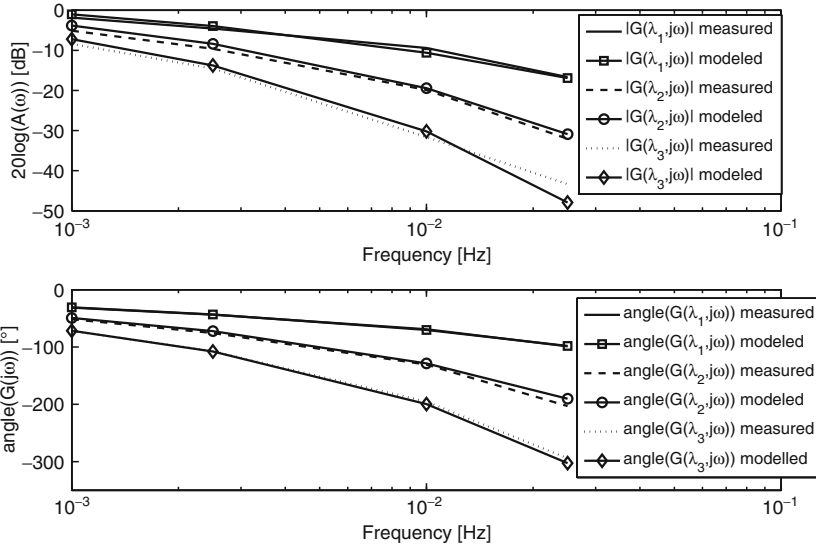


Fig. 5 Bode diagrams of measurements and modelling results

For other values of λ the same parameters were used. The λ values used are: $\lambda'_1 = 1$, $\lambda'_2 = 2.6$, $\lambda'_3 = 4.55$. The results of modeling are presented in Fig. 5. It may be observed that the measured data for real thermal plant match quite accurately the values obtained from the model in the range of frequencies from $f = 10^{-3}\text{Hz}$ to $f = 0.25 \cdot 10^{-1}\text{Hz}$. This is especially the case for $\lambda = \lambda_1$ (see top plots of magnitude and angle in Fig. 5). However, for other values of λ the accuracy of modelling is also very good (see middle and bottom magnitude and angle plots in Fig. 5).

4 Conclusion

In the paper a fractional order model of heat transfer process has been presented. The model was derived from the partial differential equation describing the heat transfer process. It was shown that when one treats the heat flux as the input variable and the temperature as the output one the equation becomes of fractional (0.5) order. Using Laplace transform this equation yields the fractional order transfer function. The frequency response of such a system has been investigated and it has been compared to the one obtained in real experiment. In order to obtain the heat-flux – temperature relation in the real system a thermoelectric element (Peltier module) has been used in the laboratory setup. The theoretical results match the experimental ones with high degree of accuracy.

References

1. Das S (2007) Fractional calculus for system identification and controls. Springer, Berlin
2. Mikusiński J (1983) Operational Calculus. PWN-Polish Scientific Publishers, Warszawa
3. Mitrani D, Tome J, Salazar J, Turo A, Garcia M, Chavez J (2005) Methodology for extracting thermoelectric module parameters. IEEE Trans Instrum Meas 54(4):1548–1552
4. Oldham KB, Spanier (1974) The fractional calculus. Academic, New York
5. Petráš I, Vinagre B (2002) Practical application of digital fractional-order controller to temperature control. Acta Montanistica Slovaca 7(2):131–137
6. Petráš I, Vinagre B, Dorčák L, Feliu V (2002) Fractional digital control of a heat solid: experimental results. In: Proceedings of International Carpathian Control Conference ICCCC'02. Malenovice, Czech Republic, pp 365–370
7. Podlubny I (1999) Fractional differential equations. Academic, San Diego
8. Poinot T, Jemni A, Trigeassou JC (2002) Solution of inverse heat problems in electrical machines with noninteger models. In: Proceedings of International Conference on Systems, Man and Cybernetics, Hammamet, Tunisia, vol 6, p 6

Analytical Design Method for Fractional Order Controller Using Fractional Reference Model

Badreddine Boudjehem, Djali Boudjehem, and Hicham Tebbikh

Abstract This paper propose a simple tuning method for a fractional order controller based on gain and phase margin specifications. Model reference design technique is used to design a fractional controller for first order plus time delay FOPTD. Fractional integral transfer function is adopted as a reference model to ensure that the closed loop system is robust to gain variations and the step responses exhibit an iso-damping property. Simple analytical formulae are derived to determine fractional controller parameters. New relation that relates directly gain and phase margins to fractional order is obtained. An illustrative example is presented. Simulation results show that the desired specifications are fulfilled and the fractional controller is robust in comparison with classical controller.

1 Introduction

In recent years, fractional calculus and its applications in many areas in science and engineering have attracted more intention of research communities. This is due to the fact that the theoretical aspects are well established. In control system, fractional order controllers are successfully used to enhance the performance of the feedback control loop. Podlubny proposed an extended PID controller namely fractional PID ($PI^\lambda D^\mu$) where the order of integration and derivation λ and μ respectively can be real numbers [14]. Further research activities are running in order to develop new design methods and tuning rules, for fractional-order controllers. Some of them are based on an extension of the classical control theory [4–6, 10–12, 15].

B. Boudjehem (✉)

Département d'électronique, Université de Skikda, Route El Hadeik BP 26 Skikda 21000 Algérie,
Laboratoire d'Automatique et Informatique de Guelma (LAIG), Université de Guelma, BP 401
Guelma 24000 Algérie
e-mail: b.boudjehem@yahoo.fr

D. Boudjehem and H. Tebbikh

LAIG, Universté de Guelma, BP 401 Guelma 24000 Algérie
e-mail: dj.Boudjehem@yahoo.fr; Tebbikh@yahoo.com

Gain and phase margins (GPM) have always played an important role to ensure systems robustness, and they serve as a performance measure [1, 7]. Therefore, these frequency specifications are widely used for classical controller design. In contrast, gain and phase margins are still few considered in the desired specifications of fractional order control systems. In [8] an optimal fractional order PID controller based on specified gain and phase margins with a minimum ISE criteria has been designed by using a differential evolution algorithm.

The aim of this paper is to propose a simple analytical tuning method based on gain and phase margin specifications, using fractional model. We obtain a nonlinear equation relates directly the phase margin and gain margin to the fractional order of the controller. The paper is organized as follows: Sect. 2 gives an overview on fractional order systems. Section 3 presents the proposed design method. New formulae are derived to determine controller parameters. An illustrative example are given in Sect. 4. Finally, conclusions are stated in Sect. 5.

2 Fractional Order Systems

The mathematical definition of fractional derivatives and integrals has been the subject of several different approaches [13]. One of the most used definition of the fractional integration is the Riemann–Liouville definition

$$D^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau \quad (1)$$

where $\Gamma(\cdot)$ is the Gamma function and α is the order of the integration.

In the control theory, the Laplace transform method is a useful tool for both the system analysis and the controller synthesis. The Laplace transform of the fractional integral given by Riemann–Liouville, under zero initial conditions for order α is

$$D^{-\alpha} f(t) = s^{-\alpha} F(s) \quad (2)$$

where $F(s)$ is the normal Laplace transformation $f(t)$.

Fractional order system is a system described by a fractional differential equation:

$$\sum_{i=1}^N a_i D^{\alpha_i} y(t) = \sum_{j=1}^M b_j D^{\alpha_j} e(t) \quad (3)$$

where D is the derivative operation, (α_i, α_j) are the orders of differintegration and (a_i, b_j) are constant coefficients.

An ideal fractional order system was proposed by Bode (see [3]), whose open loop transfer function is an irrational function of the type

$$H(s) = \left(\frac{\omega_c}{s} \right)^{\alpha} \quad (4)$$

where α is the fractional order and ω_c is the gain crossover frequency.

Both time and frequency analysis of this fractional order system were the object of several works (see [2, 9]). It is characterized in Bode plot by a constant slope magnitude and constant phase given by -20α and $-\alpha\frac{\pi}{2}$, respectively.

3 The Proposed Design Method

The proposed design method is achieved in two steps. The first one is to determine the controller transfer function in such way that the open loop transfer function except time delay, must take a form of fractional order integral. The second one is to adjust the remaind controller parameters. The stability of the system is guaranteed by the desired gain and phase margins. In this work we are limited to first order systems.

A large number of industrial plants can approximately be modeled by first order plus time delay (FOPTD) transfer function as follows:

$$G(s) = \frac{K}{\tau s + 1} e^{-Ls} \quad (5)$$

where s is the Laplace variable, L is the time delay, τ is the time constant and K is the static gain that can be varied ($K \in [K_{min}, K_{max}]$). The unity output feedback configuration is depicted in Fig. 1.

Owing to obtain an open loop transfer function as fractional transfer function except time delay the controller transfer function must has the form:

$$C(s) = K_p \frac{1 + Ts}{s^\alpha} \quad (6)$$

Indeed, with $T = \tau$, the new open loop is given by

$$C(s)G(s) = K_p K \frac{e^{-Ls}}{s^\alpha} \quad (7)$$

It seems that the controller given by Eq. 6 can also be expressed as

$$C(s) = K_p \left(\frac{1}{s^\alpha} + Ts^{1-\alpha} \right) \quad (8)$$

which is a particular case of $PD^\beta I^\alpha$, where $\beta = 1 - \alpha$.

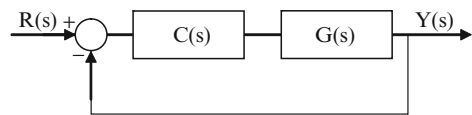


Fig. 1 Closed loop control system

The tuning objective is to determine the remainder parameters K_p , and α satisfying the desired gain and phase margins A_m and ϕ_m , respectively.

The gain and the phase expressions of (7) are given respectively by

$$|C(j\omega)G(j\omega)| = \frac{K_p K}{\omega^\alpha} \quad (9)$$

$$\phi = -L\omega - \alpha \frac{\pi}{2} \quad (10)$$

The control system specifications are given in terms of gain and phase margins. They imply that

$$|C(j\omega_p)G(j\omega_p)| = -\frac{1}{A_m} \quad (11)$$

$$\phi_m = \pi + \angle C(j\omega_g) G(j\omega_g) \quad (12)$$

where ω_g and ω_p are the gain and phase crossover frequencies of the open-loop system, respectively.

By substituting, we obtain the gain relations

$$\frac{K K_p}{\omega_g^\alpha} = 1 \quad (13)$$

$$\frac{\omega_p^\alpha}{K K_p} = A_m \quad (14)$$

and the phase relations

$$\phi_m = \pi - L\omega_g - \alpha \frac{\pi}{2} \quad (15)$$

$$\pi = \alpha \frac{\pi}{2} + L\omega_p \quad (16)$$

Using Eqs. 13–16, we obtain the new nonlinear relation:

$$A_m = \left(\frac{\pi - \alpha \frac{\pi}{2}}{\pi - \phi_m - \alpha \frac{\pi}{2}} \right)^\alpha \quad (17)$$

Using a classical numerical method we determine the fractional order α for each desired gain and phase margin values (A_m, ϕ_m).

Using Eqs. 15 and 16, we obtain the gain and the phase crossover frequencies respectively

$$\omega_g = \frac{\pi - \phi_m - \alpha \frac{\pi}{2}}{L} \quad (18)$$

$$\omega_p = \frac{\pi - \alpha \frac{\pi}{2}}{L} \quad (19)$$

Table 1 Result of different gain and phase margin specifications with $L = 10$ s

α	ω_g (rad/s)	ω_p (rad/s)	ϕ_m^* (deg)	A_m^*	ϕ_m (deg)	A_m
1.05	0.056	0.149	55.2	3.006	55	3
1.13	0.040	0.136	54.9	3.981	55	4
1.08	0.039	0.144	60.1	3.996	60	4
1.14	0.030	0.135	59.7	5.168	60	5

The proportional gain is given by

$$K_p = \frac{\omega_p^\alpha}{KA_m} = \frac{\omega_g^\alpha}{K} \quad (20)$$

Equation 17 shows that the specified gain and the phase margins depend only on the fractional order α . Therefore, any variation of the gain K has no effect to the closed loop control system performance. So that, the system is robust to gain variations.

From Eq. 20, we note that the value of the gain $K_p = \frac{\omega_g^\alpha}{K}$ is the same as the one proposed by Bode [3]. In addition, the relations (17), (18), (19) and (20) may be considered as direct new rules.

Different gain margins and phase margins are specified for the FOPTD process in Table 1.

It can see that the gain and the phase margins obtained by the proposed method (marked by *) are very close to the specified ones. For example if the desired specification (A_m, ϕ_m) are (3, 55), using Eqs. 17–20, we found that $\alpha = 1.05$, $\omega_p = 0.056$, $\omega_g = 0.1492$, $\phi_m^* = 55$ and $A_m^* = 3.0062$. The advantage of the method are using simple analytical tuning rules based only on two frequency specification (GPM), without used any optimization algorithm. Therefore, we obtain a results with acceptable errors.

4 Illustrative Example

In this section, an illustrative example of the proposed method is given. Consider the Ph dynamic model of real sugar cane juice neutralization process given by the approximated FOPTD transfer function [10]:

$$G(s) = 0.55 \frac{e^{-10s}}{1 + 62s} \quad (21)$$

In this paper we consider only that the gain can be changed with a variation range of $K \in [0.15, 0.94]$ where the nominal gain value is $K = 0.55$. The specifications for the controlled system are the gain margin $A_m = 4$, the phase margin $\phi_m = 55$ deg and robustness to gain variations.

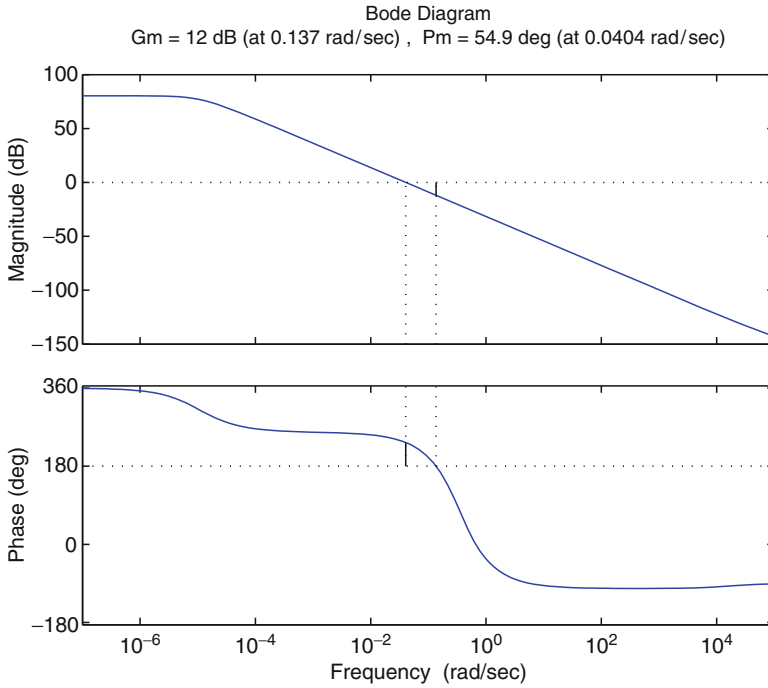


Fig. 2 Bode diagram for the nominal controlled system ($K = 0.55$)

The obtained fractional controller parameters are $K_p = 0.0404$, $T = 62$ s and $\alpha = 1.13$. Therefore, the transfer function of the fractional controller is

$$C(s) = 0.0474 \frac{1 + 62s}{s^{1.134}} \quad (22)$$

that can be expressed as:

$$(s) = 0.0474 \frac{1}{s} \left(\frac{1}{s^{0.134}} + 62s^{0.87} \right) \quad (23)$$

For the numerical simulation, the delay time is approximated by 2^{nd} order padé approximation.

Figure 2 presents the Bode diagram of the nominal controlled system. This figure shows clearly that the specified gain and phase margins are close to those obtained by the proposed tuning method ($A_m = 12\text{ dB} \equiv 3.98$, $\phi_m = 54.9$). Figure 3 illustrates the step responses for different values of K . This figure shows that the fractional controller designed by the proposed method permits to have a time response with iso-overshoot (iso-damping) for different values of gain K . Therefore, the fractional controller is robust to gain variations.

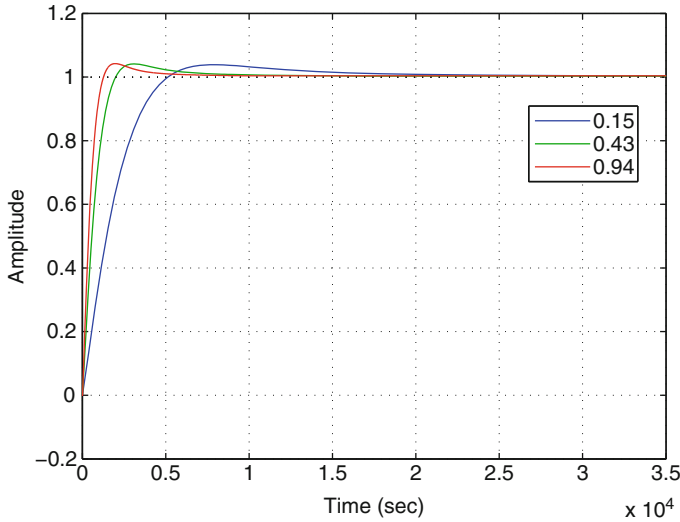


Fig. 3 Step response of the controlled system for different values of K

In order to compare the results of the proposed method with other methods, the system is controlled by a conventional PI. The PI controller parameters are obtained with the same gain and phase margin specifications (GPM) using Ho-Hang-Cao method [7]. The PI transfer function obtained is:

$$C(s) = 4.262 \frac{1 + 36.585s}{36.585s} \quad (24)$$

Figure 4 presents the Bode diagram of the nominal system with PI controller. This figure shows clearly that the specified gain and phase margins are approximately close to those obtained with conventional PI controller ($A_m = 11.9$ db $\equiv 3.93$, $\phi_m = 53.8$).

Figure 5 illustrates the step responses for different values of K . This figure shows clearly that the time responses have different overshoot values. Therefore, the system is not robust to gain variations.

5 Conclusions

In this paper, a simple tuning method for fractional controller has been presented to achieve user-specified gain and phase margins using fractional integral transfer function as a reference model. Analytical formulae were derived to determine fractional controller parameters. A nonlinear relation that relates directly gain and

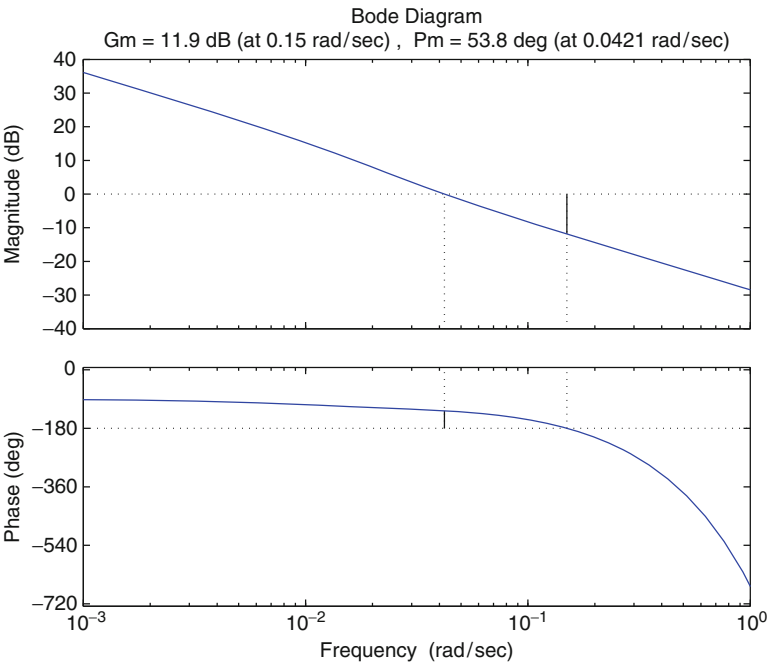


Fig. 4 Bode diagram for nominal system with PI controller ($K = 0.55$)

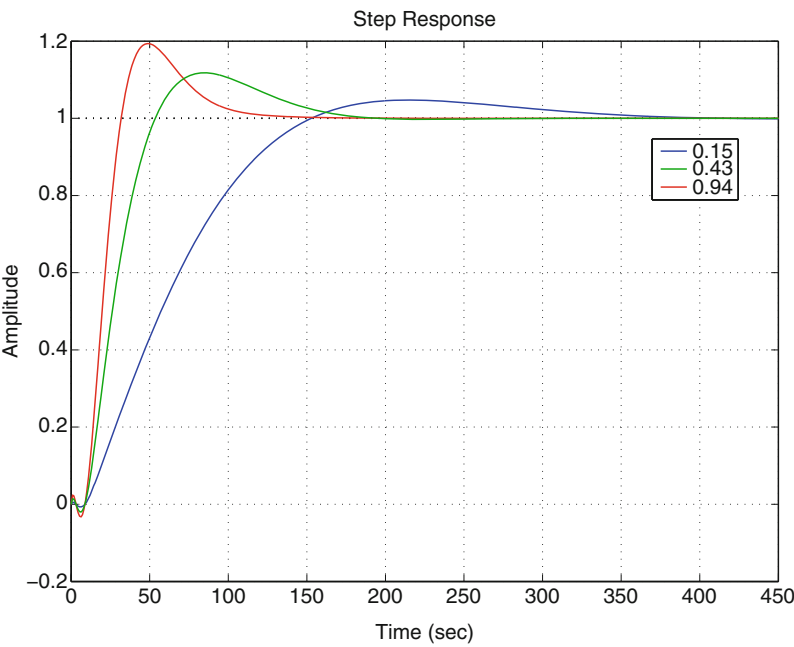


Fig. 5 Step response of the controlled system with PI controller for different values of K

phase margins to fractional order was obtained. The simulation results have shown that the desired specifications are fulfilled. In addition, the step responses exhibit an iso-damping property.

References

1. Aström K, Hägglund T (1995) PID controllers: theory, design, and tuning. Instrument Society of America, North Carolina
2. Barbosa RS, Machado JAT, Ferreira IM (2004) Tuning of PID controllers based on Bode's ideal transfer function. *Nonlinear Dynam* 138:305–321
3. Bode, HW (1945) Network analysis and feedback amplifier design. van Nostrand, New York
4. Calderón AJ, Vinagre BM, Feliu V (2006) Fractional order control strategies for power electronic buck converters. *Signal process* 86:2803–2819
5. Cao J, Cao BG (2006) Design of fractional order controller based on particle swarm optimisation. *Int J Control Autom syst* 4:775–781
6. Dorcak L, Petras I, Kostial I et al (2001) State-space controller desing for the fractional-order regulated system. *International Carpathian Control Conference*, Krynica, Poland, 15–20
7. Ho WK, Hang CC, Cao LS (1995) Tuning of PID controllers based on gain and phase margin specifications. *Automatica* 31:497–502
8. Leu JF, Tsay SY, Hwang C (2002) Design of optimal fractional-order PID controllers. *J Chin Inst Chem Eng* 33:193–202
9. Manabe S (1960) The non-integer integral and its application to control systems. *ETJ of Japan* 6:83–87
10. Monje CA, Calderón AJ, Vinagre BM et al (2004) On fractional PI^{λ} controllers: some tuning rules for robustness to plant uncertainties. *Nonlinear Dynam* 8:369–381
11. Monje CA, Vinagre BM, Felieu V et al (2008) Tuning and autotuning of fractional order controllers for industry application. *Control Eng Pract* 7:798–812
12. Natarj PSV, Tharewal S (2007) On fractional-order QFT controllers. *J Dyn Sys Meas Contr* 129:212–218
13. Oldham K, Spanier J (1974) The fractional calculus: theory and application of differentiation and integration to arbitrary order. Wiley, New York
14. Podlubny I (1999) Fractional-order systems and PID-controllers. *IEEE Trans Autom Contr* 44:208–213
15. Valério D, Costa JS (2006) Tuning of fractional PID controllers with ziegler-nichols-type rules. *Signal Process* 86:2771–2784

On Observability of Nonlinear Discrete-Time Fractional-Order Control Systems

Dorota Mozyrska* and Zbigniew Bartosiewicz†

Abstract Nonlinear discrete-time control systems of arbitrary fractional order are studied. This class extends the standard discrete-time systems. Various concepts of observability defined for standard discrete-time systems are transferred to the systems of fractional order. Observation space of the system is defined and used to characterize the indistinguishability relation and different observability properties. Two-dimensional nonlinear system with one output function that depends only on one state variable is studied in detail. In particular, it is studied how the orders of the state equations influence the rank condition that implies local observability.

1 Introduction

A dynamical system with output is observable if any two distinct initial states can be distinguished by observing the output. If no restrictions on the initial states are imposed, this property has a global character and is hard to be checked for nonlinear systems. Thus rather local versions of observability are studied and used in applications. For standard continuous-time or discrete-time nonlinear control systems, described, respectively, by differential or difference equations, it is shown that the indistinguishability relation is characterized by a family of real functions defined on the state space. They are obtained either by taking appropriate Lie derivatives of the output functions or by computing their compositions with the maps that define the dynamics of the system. Then various versions of local observability are characterized by local properties of this family of functions (see e.g. [2–4]).

D. Mozyrska (✉) and Z. Bartosiewicz
Białystok Technical University, Białystok, Poland
e-mail: admoz@w.tkb.pl; bartos@pb.edu.pl

* Partially supported by the Białystok Technical University grant W/WI/7/07.

† Partially supported by the Białystok Technical University grant S/WI/1/08.

Recently there has been a growing interest in dynamical and control systems described by differential or difference equations of fractional order (see e.g. [7, 8, 10, 11]). The class of such systems extends the set of standard models of the real-life phenomena. It is known that derivatives of the fractional order appear in models of many technical devices. In this paper we study nonlinear discrete-time control systems with output whose dynamics is governed by a difference equation of the form: $(D^\alpha x)(k+1) = f(x(k), u(k))$, where D^α denotes the backward difference of order α , which may be any real number. When $\alpha = 0$, D^α becomes the identity operator and we obtain the standard form of the discrete-time control system. If $\alpha \neq 0$, the equation contains the backward shifts of the state x , but the standard initial condition $x(0) = x_0$ assures existence and uniqueness of forward trajectories for all instances $t \geq 0$. Thus indistinguishability of states may be defined in the same manner as for the standard case when $\alpha = 0$. We construct a family of functions that characterizes the indistinguishability relation. Then we study different concepts of observability: global observability, local observability, local injectivity and its stronger version defined by a rank condition (known as Hermann–Krener condition for standard continuous-time systems). We show that the properties are related to each other in the same fashion as for standard discrete-time systems with $\alpha = 0$.

One of the interesting issues concerning the system of fractional order is to what extent the order α affects qualitative properties of the system. We attack different observability properties in the case of two state variables. We show that some properties are stable with respect to variation of α . We also give several examples that exhibit some irregular behavior. If α is a natural number, D^α may be interpreted as the nabla derivative of order α on the time scale $\mathbb{Z}$ consisting of integer numbers. By a time scale we mean a nonempty closed subset of the set $\mathbb{R}$ of all real numbers. The standard cases of time scales are $\mathbb{R}$, $\mathbb{Z}$, or $c\mathbb{Z}$, $c > 0$, but more complicated time scales can be also used. The differential calculus on time scales allows to unify continuous and discrete analysis. For more information we refer the reader to [1, 5].

2 Fractional Discrete-Type Calculus

In this paper, as a definition of the fractional discrete derivative, the Grünwald–Letnikov definition, [9, 10], is used. Let us recall that the generalized binomial coefficient $\binom{\alpha}{j}$, where $\alpha \in \mathbb{R}$, is defined as follows:

$$\binom{\alpha}{j} = \begin{cases} 1, & j = 0; \\ \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}, & j = 1, 2, \dots \end{cases} \quad (1)$$

Let c be a positive real number. By $c\mathbb{Z}$ we denote the set of all numbers cj , where $j \in \mathbb{Z}$. It is an example of time scale. Let $x : c\mathbb{Z} \rightarrow \mathbb{R}^n$, where $n \in \mathbb{N}$, and let $\alpha \in \mathbb{R}$.

Definition 1. The Grünwald–Letnikov (backward) difference of fractional order α of the function $x(\cdot)$ at $t \in c\mathbb{Z}$ is given by

$$D^\alpha x(t) = \frac{1}{c^\alpha} \sum_{j=0}^{t/c} (-1)^j \binom{\alpha}{j} x(t - jc). \quad (2)$$

From now on we shall assume that $c = 1$. For $\alpha = 1$ formula (2) gives the backward difference of the first order (the nabla derivative on the time scale $\mathbb{T} = c\mathbb{Z}$ at $t \in \mathbb{T}$). For $\alpha = -1$ we have that $D^{-1}x(t) = c \sum_{j=0}^{t/c} x(t - jc) = \int_0^{t+c} x(\tau) \Delta\tau$, so it is the delta-integration on the time scale $c\mathbb{Z}$. For $n = 0$ we have $D^0 f(t) = f(t)$. In many applications authors require that $0 < \alpha \leq 1$.

A standard discrete-time state-space nonlinear system on $\mathbb{R}^n$ can be assumed to have the form

$$x(k+1) = f(x(k), u(k)), \quad (3)$$

$$y(k) = h(x(k)), \quad (4)$$

where $k \in \mathbb{Z}$, x is the state vector and $x(k) \in \mathbb{R}^n$, u is the system input and $u(k) \in \mathbb{R}^m$, y is the system output, $y(k) \in \mathbb{R}^r$. As $D^1 x(k+1) = x(k+1) - x(k)$, Eq. 3 can be written as: $D^1 x(k+1) = f(x(k), u(k)) - x(k)$. Then $x(k+1) = D^1 x(k+1) + x(k)$. In particular, when a dynamics is linear, i.e. $f(x(k), u(k)) = Ax(k) + Bu(k)$, where A, B are matrices of appropriate dimensions, then the system takes on the form $D^1 x(k+1) = (A - I)x(k) + Bu(k)$. The first order difference can be generalized to the difference of any non-integer order according to Definition 1.

Definition 2. The discrete-time nonlinear fractional-order control system is given by

$$D^\alpha x(k+1) = f(x(k), u(k)) \quad (5)$$

$$y(k) = h(x(k)) \quad (6)$$

or equivalently

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) - \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x(k-j+1), \\ y(k) &= h(x(k)). \end{aligned}$$

When $\alpha = 0$, then we get the standard discrete-time state-space nonlinear system (3). As $x(k) \in \mathbb{R}^n$, one could set different orders of the derivative for different variables. And for that we consider the following generalization of Definition 2 [11].

Definition 3. The generalized discrete nonlinear fractional-order control system (denoted by Γ) is expressed by

$$\begin{aligned} D^A x(k+1) &= f(x(k), u(k)) \\ y(k) &= h(x(k)) \end{aligned} \quad (7)$$

or equivalently

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \\ &\quad - \sum_{j=1}^{k+1} (-1)^j \Lambda_j x(k-j+1), \\ y(k) &= h(x(k)), \end{aligned} \quad (8)$$

where $D^A x(k+1) = \begin{bmatrix} D^{\alpha_1} x_1(k+1) \\ \vdots \\ D^{\alpha_n} x_n(k+1) \end{bmatrix}$, while $\Lambda_j = \text{diag} \left[\binom{\alpha_1}{j}, \dots, \binom{\alpha_n}{j} \right]$ is a diagonal matrix and $x(k+1) = (x_1(k+1), x_2(k+1), \dots, x_n(k+1))' \in \mathbb{R}^n$.

We assume that functions f and h are analytic with respect to x . Denote by $J_0(m)$ the set of all sequences $U = (u(0), u(1), \dots)$, where $u(k) \in \mathbb{R}^m$. Define inductively the sequence of mappings (see also [2]): $F_k : \mathbb{R}^n \times J_0(m) \rightarrow \mathbb{R}^n$ by $F_0(x, U) = x$, $F_{k+1}(x, U) = f(F_k(x, U), u(k)) - \sum_{j=1}^{k+1} (-1)^j \Lambda_j F_{k-j+1}(x, U)$. Let define $\gamma(k, p, U) = F_k(p, U)$. Then γ is the state forward trajectory of the dynamics of the system Γ (see (7) or (8)), i.e. a solution $x(\cdot)$, which is uniquely defined by the initial state $x(0) = p$ and the control sequence U . Then we also have uniquely defined output forward trajectory $\eta(\cdot, p, U)$, i.e. the output trajectory corresponding to the initial condition $x(0) = p$ and the control sequence U . When it is evaluated at time k , it has the form $y(k) = \eta(k, p, U) = h(\gamma(k, p, U)) = h(F_k(p, U))$.

As various notions of observability properties are directly connected with indistinguishability relation between initial points, we first give the definition of this concept. It is similar to indistinguishability studied for usual nonlinear control systems, see [2–4].

Definition 4. We say that $p, q \in \mathbb{R}^n$ are indistinguishable with respect to Γ if $\eta(k, p, U) = \eta(k, q, U)$ for every control sequence U and any nonnegative integer k . Otherwise p, q are distinguishable.

Then we have the following.

Proposition 1. States $p, q \in \mathbb{R}^n$ are indistinguishable with respect to Γ if and only if $h(F_k(p, U)) = h(F_k(q, U))$ for every control sequence U and any nonnegative integer k .

3 Observability Concepts

The main tool in investigations of observability is the idea of the observation space of a given control system. Let Γ be a discrete-time nonlinear fractional-order control systems given by (8). Let $H(\Gamma)$ denote the set of all functions of the form $h_i \circ$

$F_k(\cdot, U)$, where $i = 1, \dots, r$, $k \in \mathbb{Z}_+ \cup \{0\}$, h_i is the i th component of h , U is a control sequence. Hence $H(\Gamma)$ consists of compositions of h_i with maps $F_k(\cdot, U) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $k \in \mathbb{N} \cup \{0\}$. When $k = 0$, there is no composition and $h_i \circ F_0(\cdot, U)$ gives h_i . As f and h are analytic with respect to x , $H(\Gamma)$ is a set of global analytic functions on $\mathbb{R}^n$.

In statements about a rank condition (known as Hermann–Krener condition for the standard continuous-time case) there is used the space $dH(\Gamma)(p)$, $p \in \mathbb{R}^n$. For that it is worthy to know the following, obvious, but important description.

Proposition 2. $dH(\Gamma)(p) = \left\{ \sum_{j=1}^n \frac{\partial h_i}{\partial x_j}(F_k(p, U)) \nabla F_{k,j}(p, U) dx \right\}$, where $k \geq 0$ and $i = 1, \dots, r$, and U is arbitrary control sequence.

Definition 5. The linear subspace of the space of analytic functions on $\mathbb{R}^n$ spanned by $H(\Gamma)$ is called the *observation space of the system* Γ .

Proposition 3. Two states $p, q \in \mathbb{R}^n$ are indistinguishable with respect to Γ if and only if for all $\varphi \in H(\Gamma)$: $\varphi(p) = \varphi(q)$.

Proof. From Proposition 1 two states $p, q \in \mathbb{R}^n$ are indistinguishable if for every control sequence U and any nonnegative integer k we have $h(F_k(p, U)) = h(F_k(q, U))$. This is equivalent to $h_i(F_k(p, U)) = h_i(F_k(q, U))$ for every $i = 1, \dots, r$, $k \in \mathbb{N} \cup \{0\}$ and every control sequence U . As each function $\varphi \in H(\Gamma)$ has the form $\varphi = h_i \circ F_k(\cdot, U)$, then for all $\varphi \in H(\Gamma)$ holds that $\varphi(p) = \varphi(q)$. This is valid also for linear combinations of such functions.

Definition 6. We say that Γ is *globally observable* if for any two different points $p, q \in \mathbb{R}^n$, p and q are distinguishable. Γ is *locally observable at* $p \in \mathbb{R}^n$ if there is a neighborhood V of p such that for any $q \in V \setminus \{p\}$, p and q are distinguishable. The system Γ is *locally observable*, if it is locally observable at each point of $\mathbb{R}^n$. We say that Γ is *locally injective at* p if there is a neighborhood V of p such that for every $q_1, q_2 \in V$: $\forall \varphi \in H(\Gamma)$, $\varphi(q_1) = \varphi(q_2) \Rightarrow q_1 = q_2$.

Sometimes it is impossible to calculate the whole space $H(\Gamma)$ or we can only check some properties of the system after a few steps of composition with functions F_k . Then we consider also compositions up to some level $s \in \mathbb{N} \cup \{0\}$ and for that we define the following set of functions: $H_s(\Gamma) := \{h_i(F_k(\cdot, U)) : i = 1, \dots, r; k = 0, \dots, s, U \text{ is a control sequence}\}$. With these spaces we connect the following group of properties:

Definition 7. We say that Γ is *globally observable in s steps*, s – nonnegative integer, if for any $p, q \in \mathbb{R}^n$, $p \neq q$, we have that p and q are distinguishable by functions from the set $H_s(\Gamma) := \{h_i(F_k(\cdot, U)) : i = 1, \dots, r, k \in \{0, \dots, s\}, U \in J_0(m)\}$.

We say that Γ is *locally observable at* $p \in \mathbb{R}^n$ *in s steps*, s – nonnegative integer, if there is a neighborhood V of p such that for any $q \in V \setminus \{p\}$, p and q are distinguishable by functions from the set $H_s(\Gamma)$.

Remark 1. If there exists $s \geq 0$ such that Γ is globally (locally) observable in s steps then Γ is globally (locally) observable.

The next proposition was formulated in [2] for usual discrete-time systems and it works for the set of functions corresponding to a system, so it does not depend on a class of a system but on an observation space. Originally the results for spaces of functions in stronger version of local observability was proved in [6].

Proposition 4. *Consider the following conditions:*

1. $\dim dH(\Gamma)(p) = n$
2. $H(\Gamma)$ is injective at p
3. Γ is locally observable at p

Then $1 \Rightarrow 2 \Rightarrow 3$ and none of these implications can be reversed.

When system Γ is linear, i.e. $f(x(k), u(k)) = Ax(k) + Bu(k)$, $h(x(k)) = Cx(k)$, where A, B, C are matrices of appropriate dimensions, then global observability is equivalent to local injectivity and local observability and does not depend on a point p . But it can depend on the number of steps s . For linear Γ defined on $\mathbb{R}^n$ it is enough to consider only $H_{n-1}(\Gamma)$, hence one can ask about observability in $s = n - 1$ steps. Taking this idea we ask when Γ is globally observable or when it is locally injective using as a tool only functions from the set $H_{n-1}(\Gamma)$.

Now we consider $n = 2$ with special case of one-dimensional output function $h(p) = h(x_1)$, where $p = (x_1, x_2)$.

Proposition 5. *Let Γ has the following form:*

$$\begin{aligned} D^{\alpha_1} x_1(k+1) &= f_1(x(k), u(k)) \\ D^{\alpha_2} x_2(k+1) &= f_2(x(k), u(k)) \\ y(k) &= h(x_1(k)). \end{aligned}$$

Let $p = (p_1, p_2) \in \mathbb{R}^2$. If there is a control sequence U such that

$$h'(p_1)h'(F_1(p, U))\frac{\partial f_1}{\partial x_2}(p, U) \neq 0, \quad (9)$$

then Γ is locally injective at p . Additionally this property does not depend on α_2 .

Proof. Firstly let us notice that $H_1(\Gamma)(x) = \{h(x_1), h(F_1(x, U)), U \in J_0(m)\}$ and $dh(F_1(p, U)) = h'(F_1(p, U))\nabla F_{1,1}(p)dx$ and also $dh(F_1(p, U)) = h'(f_1(p, U) + \alpha_1 p_1)((\frac{\partial f_1}{\partial x_1}(p) + \alpha_1)dx_1 + \frac{\partial f_1}{\partial x_2}(p)dx_2)$. Then $dH_1(\Gamma)(p) = \{dh(p_1), dh(f_1(p, U) + \alpha_1 p_1)\} = \{h'(p_1)dx_1, h'(F_1(p, U))\nabla F_{1,1}(p)dx\}$, for all $U \in J_0(m)$. Nextly we notice that, $\dim dH_1(\Gamma)(p) = 2$ if and only if

$$\begin{vmatrix} h'(p_1) & 0 \\ h'(F_{1,1})\frac{\partial F_{1,1}}{\partial x_1}(p, U) & h'(F_{1,1})\frac{\partial F_{1,1}}{\partial x_2}(p, U) \end{vmatrix} = h'(p_1)h'(F_1(p, U))\frac{\partial f_1}{\partial x_2}(p, U) \neq 0.$$

Corollary 1. Let Γ be as in Proposition 5. If $h'(p) \neq 0$, $h'(f_1(p) + \alpha_1 p_1) \neq 0$ and $\frac{\partial f_1}{\partial x_2}(p) \neq 0$, then Γ is locally injective at p .

Corollary 2. Let $p = (0, 0)$ be the equilibrium point of the function $f_1(x, u)$ for all $u \in \mathbb{R}^m$. If there is U such that: $h'(p) \frac{\partial f_1}{\partial x_2}(p, U) \neq 0$ then Γ is locally injective at p .

Proposition 5 can be easily generalized in the following way.

Proposition 6. Let Γ have the following form:

$$\begin{aligned} D^\alpha x(k+1) &= f(x(k), u(k)) \\ y(k) &= h(x_1(k)). \end{aligned}$$

If $\dim\{h'(p_1)dx_1, h'(F_1(p, U))\nabla F_{1,1}(p)dx, h'(F_{n-1}(p, U))\nabla F_{n-1,1}(p)dx, \dots\} = n$, then Γ is locally injective at $p = (p_1, \dots, p_n)$.

4 Fractional Order and Observability for Two-Dimensional Systems

We are also interested in investigations if observability (global or local) depends on the fractional order α of the system when we use the same function f .

Example 1. Let Γ be one-dimensional and be given by

$$\Gamma : D^\alpha x(k+1) = x^2(k)u(k), \quad y(k) = x^2(k).$$

Then $H(\Gamma) = \{x^2, x^2(x + \alpha)^2, \frac{x^2}{4}(2x^3 + 4\alpha x^2 + 2\alpha x(\alpha - 1) - \alpha(\alpha + 1))^2, \dots\}$. Then for $\alpha = 0$, Γ is not globally observable. But it is locally observable at every point $p \in \mathbb{R}$. While for $\alpha \neq 0$, Γ is globally observable in $s = 1$ steps, because the set $H_1(\Gamma) = \{h, h(F_1)\} = \{x^2, x^2(x + \alpha)^2\}$ distinguishes every two states.

Example 2. Let us consider system Γ given by (8) but with $f(x, u) \equiv 0$. Then $F_0(x, U) = x$, $F_1(x, U) = \Lambda_1 x = (\alpha_1 x_1, \dots, \alpha_n x_n)'$, $F_2(x, U) = (\Lambda_1^2 - \Lambda_2)x, \dots$. When $\Lambda_1 = 0$ (it means that for all k : $\alpha_k = 0$) then $H(\Gamma) = \{h_i, i = 1, \dots, p\}$. And then observability in $s = 0$ is the same as in any s steps, because then $\Lambda_j = 0$ for all $j \in \mathbb{N}$. Next for $\Lambda_1^2 - \Lambda_2 = 0$, (when $\Lambda_1 \neq 0$) we have that for all k : $\alpha_k = -1$. But then next $F_k(x, U)$ may be not equal zero for $k \geq 3$. For $h(x(k)) = Cx(k)$, where C is a matrix $r \times n$: $\dim dH(\Gamma)(p) = \{C \frac{\partial F_k}{\partial x}(p, U)dx, k \geq 0\} = \dim \text{span}\{C \frac{\partial F_k}{\partial x}, k \geq 0\}$ and $\frac{\partial F_k}{\partial x}$ are constant.

Proposition 7. Let Γ_0 be a system in the following form:

$$\Gamma_0 : \begin{aligned} x(k+1) &= f(x(k), u(k)) \\ y(k) &= h(x_1(k)). \end{aligned}$$

while Γ_α be a generalized fractional-order system with a sequence Λ of fractional-orders:

$$\Gamma_\alpha : \begin{cases} D^\Lambda x(k+1) = f(x(k), u(k)) \\ y(k) = h(x_1(k)), \end{cases}$$

where $x = (x_1, x_2)'$ and $h : \mathbb{R}^2 \rightarrow \mathbb{R}$. If $\dim dH_1(\Gamma_0)(p) = 2$ then there is a value $\alpha_g > 0$ such that for all α_1, α_2 , satisfying $|\alpha_i| < \alpha_g, i = 1, 2$, we have that $\dim dH_1(\Gamma_\alpha)(p) = 2$.

Proof. From Proposition 5: $dH_1(\Gamma_0)(p) = 2$ if there exists control $u(0)$ such that

$$h'(p_1)h'(f_1(p, u(0)))\frac{\partial f_1}{\partial x_2}(p, u(0)) \neq 0.$$

Then $h'(p_1)\frac{\partial f_1}{\partial x_2}(p, u(0)) \neq 0$ and there is neighborhood V of point p such that for all q we have $h'(q_1)\frac{\partial f_1}{\partial x_2}(q, u(0)) \neq 0$. Let $q \in K(p, \varepsilon) \subset V$. Next as $h'(f_1(p, u(0))) \neq 0$, then there is ε_1 such that for $z \in W \subset \mathbb{R}^2$ satisfying $\|z - f(p)\| < \varepsilon_1$ the following holds: $h'(z_1) \neq 0$. Let $z = f(p) + \Lambda_1 p = (f_1(p) + \alpha_1 p_1, f_2(p) + \alpha_2 p_2)$. Then $\|z - f(p)\| = \|\Lambda_0 p\|$. Let $|\alpha_i| < \frac{\varepsilon_1}{\varepsilon}$. This implies that $\|z - f(p)\| < \frac{\varepsilon_1}{\varepsilon}\|p\| < \varepsilon_1$. It gives that $f(p) + \Lambda_0 p \in W$ and for that $h'(f_1(p, u(0) + \alpha_1 p_1)) \neq 0$. Hence for $\alpha_i, i = 1, 2$, being small enough we have the thesis.

References

1. Agarwal RP, Bohner M (1999) Basic calculus on time scales and some of its applications. *Results Math* 35(1–2):3–22
2. Bartosiewicz Z (1998) Local observability of discrete-time analytic systems. *Proceedings of CONTROLO'98, Coimbra, Portugal, 9–11 September 1998*
3. Bartosiewicz Z (1999) Real analytic geometry and local observability. In: *Differential geometry and control (Boulder, CO, 1997)*. *Proc Sympos Pure Math* 64:65–72. Amer Math Soc, Providence, RI
4. Bartosiewicz Z, Mozyrska D (1997) Algebraic criteria for stable local observability of analytic systems on $\mathbb{R}^n$. In: *Proceedings of European Control Conference ECC-97, Brussels, Belgium*
5. Bohner M, Peterson A (2001) *Dynamic equations on time scales. An introduction with applications*
6. Hermann R, Krener AJ (1977) Nonlinear controllability and observability. *IEEE Trans AC*(22):728–740
7. Kaczorek T (2007) Reachability and controllability to zero of cone fractional linear systems. *Archiv Contr Sci* 17(3):357–367
8. Kaczorek T (2008) Reachability of fractional positive continuous-time linear systems. Technical report, Białystok Technical University
9. Oldham KB, Spanier J (1974) *The fractional calculus*. Academic, New York
10. Podlubny I (1999) *Fractional differential systems*. Academic, San Diego
11. Sierociuk D, Dzieliński A (2006) Fractional Kalman filter algorithm for the states, parameters and order of fractional system estimation. *Int J Appl Math Comput Sci* 16(1):129–140

Chaotic Fractional Order Delayed Cellular Neural Network

Vedat Çelik and Yakup Demir

Abstract This paper deals with the fractional order model of the two-cell autonomous Delayed Cellular Neural Network which exhibits chaotic behavior. Numerical simulation results demonstrate that the chaos can be observed in fractional order Delayed Cellular Neural Network for fractional order $q \geq 0.1$. Also the τ delay time values for which the chaos occurs in q system order, is quantitatively defined using largest Lyapunov exponents.

1 Introduction

Fractional calculus was first introduced approximately 300 years ago, and has been the subject of many studies to date [1, 2]. However, recently an important research was not observed in literature due to the limiting computing power about the fractional order systems and the dynamics introduced by them. As given in [3] point out research carried out in recent years shows that the fractional order differentials equations are an effective tool for describing complex dynamics and many physical and engineering systems can be modeled efficiently using them. The fact that the physical and engineering systems can be modeled efficiently by the fractional order modeling requires us to investigate the dynamics that might exist in fractional order systems.

Chaos, which is observed in nonlinear systems and which can not be predicted beforehand has recently been a focus of interest of the researchers. As parallel to these studies, finding out the existence of the chaotic dynamics in fractional order systems has become an important research topic. The conditions for the chaos to surface in a continuous-time system are: existence of nonlinear element in the system, sensitivity dependency to initial conditions and the order of the system to be

V. Çelik (✉) and Y. Demir

Firat University, Engineering Faculty, Electrical & Electronics Eng. Dept., 23119 Elazig, Turkey
e-mail: vedat23@hotmail.com; ydemir@firat.edu.tr

three or higher [4]. Almost all of the natural and man-made systems are nonlinear nature [5]. This shows how high the possibility for us to face chaotic behaviors.

For a continuous time system, sensitivity dependency to the initial conditions which is the most important condition of chaos is not observed in integer order non-linear systems where the order of the system is smaller than three. However chaotic behavior is observed both in the fractional order models of existing chaotic systems and in newly defined fractional order systems despite the fact that the order of the system is smaller than three [6–9]. In addition to this, it is not possible to define a limit about the order of the system for continues-time fractional order systems as in continues-time integer order systems.

None of the aforementioned fractional order chaotic systems are delayed time systems. Despite it is possible for chaotic behaviors to exist in the fractional order models of integer order delayed time chaotic systems, no studies found in the literature about this.

For this reason, the fractional order model of a chaotic integer order, two-cell autonomous Delayed Cellular Neural Network (*DCNN*) given in [10] is provided and the existence of chaos in this model, and for which system order $2q$ and delay time τ values the chaotic behavior becomes apparent will be shown.

The rest of this article is organized as follows. Fractional order operators and its approximation are given in Sect. 2. The chaotic integer order *DCNN* given in [10] and its fractional order model are provided in Sect. 3. In Sect. 4, using numerical simulations, it is shown that the chaos is observed in fractional order delay time system for the fractional order between $0.1 \leq q < 1$. Additionally the τ time delay values, where the chaotic behavior is occurred in q system order, is determined quantitatively using the largest Lyapunov exponents. In Sect. 5 the conclusion is given.

2 Fractional Operators and its Approximation

In general, fractional calculus describes the fractional order integro-differential operator:

$${}_a D_t^q = \begin{cases} \frac{d^q}{dt^q} & R(q) > 0 \\ 1 & R(q) = 0 \\ \int_a^t (d\tau)^{-q} & R(q) < 0 \end{cases} \quad (1)$$

Although there exist many mathematical definitions for fractional order derivative and integral, one of the most general differintegral definition is Riemann-Liouville definition [1].

$$\frac{d^q f}{dt^q} = \frac{1}{\Gamma(-q)} \int_0^t \frac{f(\tau)}{(t-\tau)^{q+1}} d\tau, \quad q < 0 \quad (2)$$

Here $\Gamma(\cdot)$ is Gamma function and q is a fractional value. For $q > 0$, the integer n^{th} derivative can be obtained by the fractional $(q - n)^{th}$ integral.

$$\frac{d^q f}{dt^q} = \frac{d^n}{dt^n} \left[\frac{d^{q-n} f}{dt^{q-n}} \right], \quad q > 0 \quad (3)$$

Laplace transformation is an important tool for the basic engineering analysis of the linear systems. The Laplace transformation of (3) for all q values where n is an integer and $n - 1 < q < n$ is obtained as follows [2]:

$$L \left\{ \frac{d^q f(t)}{dt^q} \right\} = s^q L \{f(t)\} - \sum_{k=0}^{n-1} s^k \left[\frac{d^{q-1-k} f(t)}{dt^{q-1-k}} \right]_{t=0} \quad (4)$$

If the initial conditions are zero, then (4) can be reduced to following:

$$L \left\{ \frac{d^q f(t)}{dt^q} \right\} = s^q L \{f(t)\} \quad (5)$$

In this case, the transfer function of a fractional order integrator with order q in Laplace domain is:

$$F(s) = \frac{1}{s^q} \quad (6)$$

It is not always easy to obtain the time responses of the systems which contain the fractional order differintegral expression using the aforementioned standard definitions for wide time intervals. For certain frequency intervals, one of the ways of dealing with this problem is obtain the integer order linear transfer functions approximated to the frequency response of the fractional order operators. In [11], an algorithm is proposed for approximating the fractional order integral operator to linear transfer functions. The algorithm can be described in short as the process of approximating the fractional order operators with desired frequency intervals and magnitude error to the linear transfer functions by using their Bode based frequency responses for $s = j\omega$. Table 1 obtained in [6] gives the approximated transfer functions with integer order for $q = 0.1-0.9$ with step time 0.1 of fractional order operator were obtained in an error of 2 dB at $\omega = 0.01-100$ rad/s frequency interval. Figure 1 shows the Bode diagrams of the fractional order integral operator and approximated linear transfer function for $q = 0.9$. Bode diagrams are obtained for the interval of $\omega = 0.01-100$ rad/s. It can be seen that this is the best approximation for this interval.

In the simulations explained in Section 4 of this paper, we use the integer order approximate transfer function in Table 1 obtained in [6] instead of fractional order integral operator in cost of a small error. As seen from Fig. 1, this error is in acceptable range.

Table 1 The largest Lyapunov exponent for order of system q and time delay τ

q	τ	Largest Lyapunov Exponent ($\lambda_{\max}$)
0.9	0.98	0.000042
0.8	1.05	0.0024
0.7	0.93	0.00029
0.6	0.82	0.000049
0.5	0.57	0.029
0.4	0.4	0.0127
0.3	0.3	0.0916
0.2	0.1	0.0255
0.1	0.012	0.0214

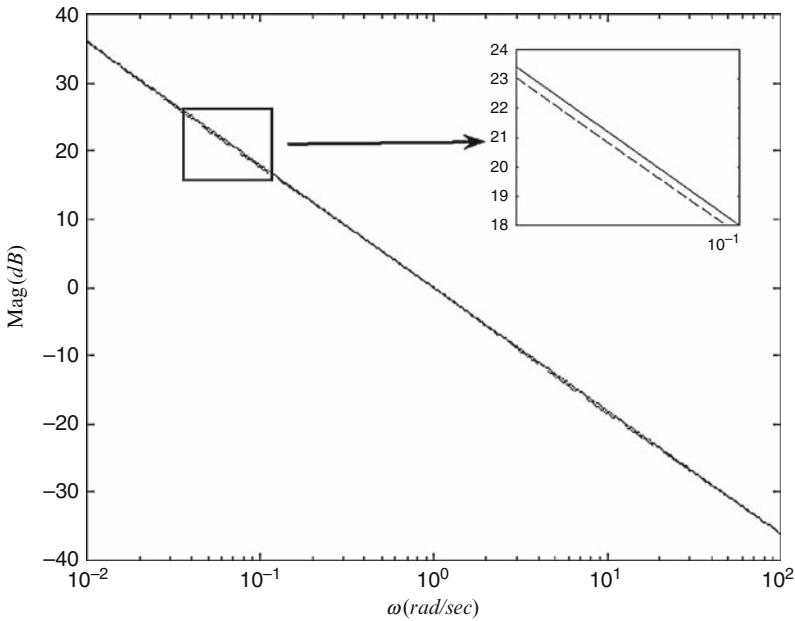


Fig. 1 Bode plot of approximate (*dash*) and actual (*solid*) function

3 Fractional Order Delayed Cellular Neural Network

The *DCNN* which exhibits chaotic behaviors given in [10], has an autonomous form with two-cell. This form is identified by following equations:

$$\dot{x}_i(t) = -x_i(t) + A^0 y_i(t) + A^\tau y_i(t - \tau); \quad i = 1, 2 \quad (7)$$

$$y_i(t) = \frac{1}{2}(|x_i(t) + 1| - |x_i(t) - 1|); \quad i = 1, 2 \quad (8)$$

$$A^0 = \begin{bmatrix} a_{11}^0 & a_{12}^0 \\ a_{21}^0 & a_{22}^0 \end{bmatrix}; \quad A^\tau = \begin{bmatrix} a_{11}^\tau & a_{12}^\tau \\ a_{21}^\tau & a_{22}^\tau \end{bmatrix}; \quad a_{11}^0 = a_{22}^0; a_{11}^\tau = a_{22}^\tau \quad (9)$$

Here, $x_i(t)$ is the state variable for each cell, $y_i(t)$ is the cell output, $y_i(t - \tau)$ is delayed cell output, A^0 is cloning-template and A^τ is delay cloning-template. The cloning-template and delay cloning-template of the integer order *DCNN* are taken as following:

$$A^0 = \begin{bmatrix} 1 + \frac{\pi}{4} & 20 \\ 0.1 & 1 + \frac{\pi}{4} \end{bmatrix}; \quad A^\tau = \begin{bmatrix} -\sqrt{2}\frac{\pi}{4}1.3 & 0.1 \\ 0.1 & -\sqrt{2}\frac{\pi}{4}1.3 \end{bmatrix} \quad (10)$$

For the delay time values of $\tau = 0.84, 0.85, 0.9$ and 0.95 , the system exhibit chaotic behaviors [10].

Fractional order model of the aforementioned two cells integer order autonomous chaotic *DCNN* is expressed with following equation.

$$\begin{bmatrix} \frac{d^{q_1} x_1(t)}{dt^{q_1}} \\ \frac{d^{q_2} x_2(t)}{dt^{q_2}} \end{bmatrix} = \begin{bmatrix} -x_1(t) \\ -x_2(t) \end{bmatrix} + A^0 \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + A^\tau \begin{bmatrix} y_1(t - \tau) \\ y_2(t - \tau) \end{bmatrix} \quad (11)$$

The output function $y_i(t)$ be the same as Eq. 8. Here q_1 and q_2 are the fractional order of the first and the second cells respectively, $x_i(t)$ is the state variable for each cell, $y_i(t)$ is the cell output, $y_i(t - \tau)$ is delayed cell output, A^0 is cloning-template and A^τ is delay cloning-template.

4 Chaos in the Fractional Order Delayed Cellular Neural Network

In this section, the existence of the chaos in the fractional order two-cell autonomous *DCNN* given in Eq. 11 is showed with the simulation results obtained by MATLAB/Simulink environment. For this, the cloning-template and delay cloning-template of the fractional order *DCNN* is taken as in Eq. 10. Moreover the fractional order of two-cell is equally chosen and $q_1 = q_2 = q$ is accepted. In this case, chaotic behaviors are observed in the fractional order system according to some values of the order q and the delay time τ . In Figs. 2 and 3 the state space diagrams of the fractional order delayed system for $q = 0.9 - 0.6$ and $q = 0.4 - 0.1$ intervals, respectively; where for τ delay time values the chaotic behaviors occurs.

Note that, for the chaos to occur in the system, the values of the delay time τ decreases as the fractional order q decreases. Additionally, as q decreases, the chaotic behaviors exhibiting in the system are observed in a narrow region.

In addition to this, Fig. 4 shows a section from the variation of obtained state variable x_1 versus time for $q = 0.3, 0.5$ and 0.8 in τ delay time values in which chaos observed. According to the obtained results, the change rate of state variable with respect to time increases as q decreases.

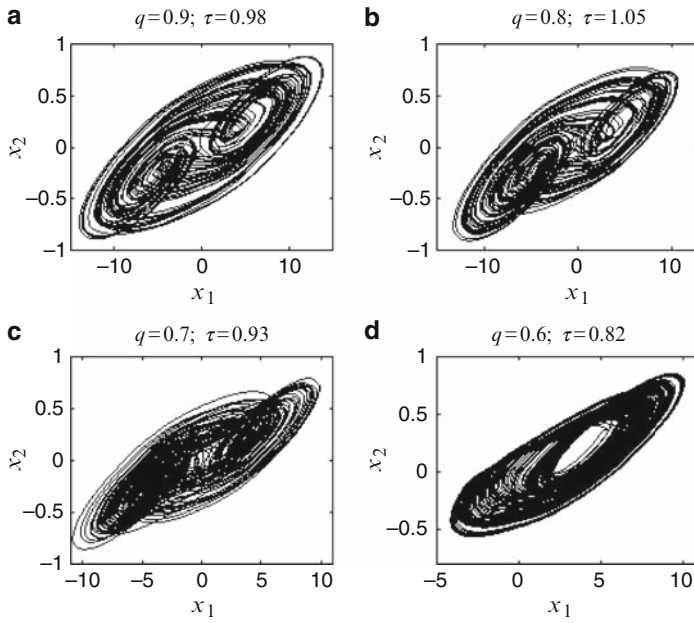


Fig. 2 State space diagram of fractional order DCNN for (a) $q = 0.9$, (b) $q = 0.8$, (c) $q = 0.7$ and (d) $q = 0.6$

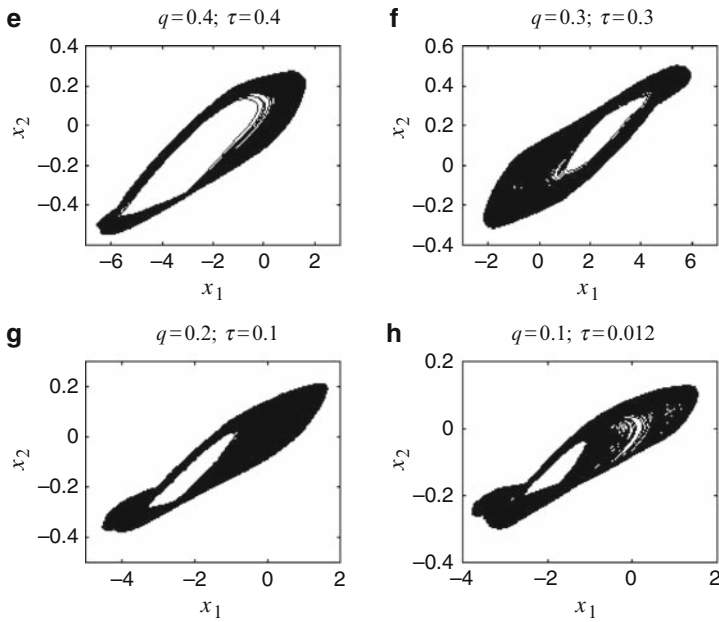


Fig. 3 State space diagram of fractional order DCNN for (e) $q = 0.4$, (f) $q = 0.3$, (g) $q = 0.2$ and (h) $q = 0.1$

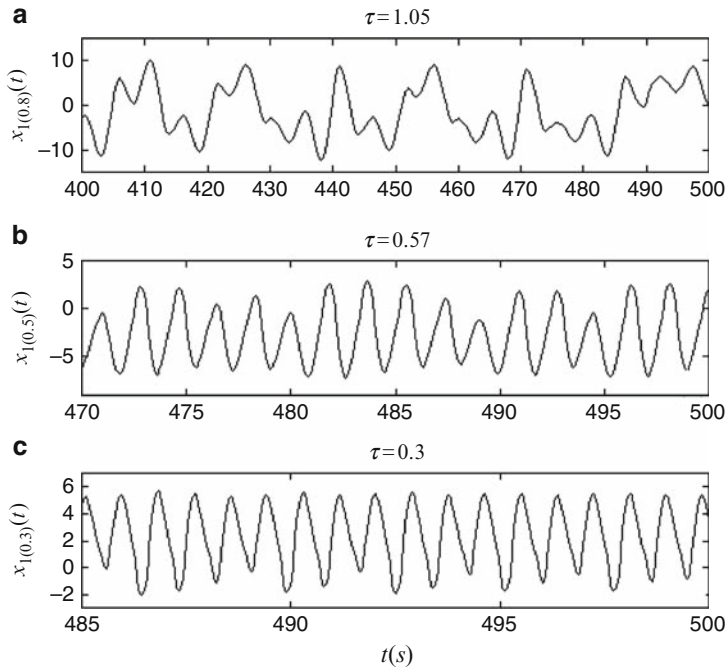


Fig. 4 Time response of complex trajectories of fractional order model for (a) $q = 0.8$, (b) $q = 0.5$ and (c) $q = 0.3$

Based on the time responses of the system for two nearest initial conditions, it is quantitatively possible to detect the existence of chaotic behaviors in the system, using largest Lyapunov exponents values [12]. Obtained largest Lyapunov exponents values are given in Table 1.

5 Conclusions

In this paper, the fractional order model of *DCNN* which exhibits chaotic behaviors is obtained. For certain τ delay time values, the existence of chaos in fractional order *DCNN* for $0.1 \leq q < 1$ interval is shown. The simulation results demonstrate that the τ values in which the system exhibits chaotic behavior decreases as the fractional order q decreases, and these behaviors occur in a narrow region. Furthermore it is observed that as q decreases the variation rate of state variable with respect to time also decreases.

References

1. Oldham K, Spainer J (1974) Fractional calculus. Academic, New York
2. Podlubny I (1999) Fractional differential equations. Academic, New York
3. Oustaloup A, Levron F, Mathieu B, Nanot FM (2000) Frequency-band complex noninteger differentiator: characterization and synthesis. *IEEE Trans CAS-I* 47:25–39
4. Moon FC (1992) Chaotic and fractal dynamics. Wiley, New York
5. Schweppe FC (1973) Uncertain dynamic systems. Prentice-Hall, New Jersey
6. Hartley TT, Lorenzo CF, Qammer HK (1995) Chaos in a fractional order Chua's system. *IEEE Trans CAS-I*, 42:485–490
7. Arena P, Fortuna L, Porto D (2000) Chaotic behavior in noninteger-order cellular neural networks. *Phys Rev E* 61:776–781
8. Ahmad WM, Sprott JC (2003) Chaos in fractional-order autonomous nonlinear systems. *Chaos Soliton Fract* 16:339–351
9. Li C, Peng G (2004) Chaos in Chen's system with a fractional order. *Chaos Soliton Fract* 22:443–450
10. Gilli M (1993) Strange attractors in delayed cellular neural networks. *IEEE Trans CAS-I*, 40:849–853
11. Charef A, Sun HH, Tsao YY, Onaral B (1992) Fractal system as represented by singularity function. *IEEE Trans Autom Control* 37:1465–1470
12. Rosenstein MT, Collins JJ, De Luca CJ (1993) A practical method for calculating largest Lyapunov exponents from small data sets. *Physica D* 65:117–134

Fractional Wavelet Transform for the Quantitative Spectral Analysis of Two-Component System

Murat Kanbur, Ibrahim Narin, Esra Özdemir, Erdal Dinç,
and Dumitru Baleanu

Abstract The fractional wavelet transform (FWT) combined with zero crossing technique was applied to the absorption spectra for the quantitative resolution of a binary mixture consisting of trimethoprim and sulfachloropyridazine sodium. The absorption spectra of trimethoprim and sulfachloropyridazine sodium were processed by FWT method. In the following step, the FWT spectra were obtained by plotting the fractional wavelet coefficients versus the wavelength. Classical second derivative method was applied to the fractional wavelet signals in the wavelet domain. A calibration graphs for each substance were obtained by measuring the FWT-amplitudes at an appropriate wavelength corresponding to a zero crossing point in the derivative spectra of the FWT-signals. The amounts of trimethoprim and sulfachloropyridazine sodium in tablets.

M. Kanbur

Erciyes University, Faculty of Veterinary, Department of Pharmacology and Toxicology,
38090, Kocasinan, Kayseri, Turkey
e-mail: kanburm@hotmail.com

I. Narin

Erciyes University, Faculty of Pharmacy, Department of Analytical Chemistry, 38039,
Kayseri, Turkey
e-mail: narin@erciyes.edu.tr

E. Özdemir

Erciyes University, Institute of Health Department of Analytical Chemistry Sciences, 38039,
Kayseri, Turkey
e-mail: ozdemir.es@hotmail.com

E. Dinç (✉)

Department of Analytical Chemistry, Faculty of Pharmacy, Ankara, University,
0610 Tandoğan, Ankara, Turkey
e-mail: dinc@pharmacy.ankara.edu.tr

D. Baleanu

Department of Mathematics and Computer Sciences, Faculty of Art and Sciences,
Çankaya University, 06530 Balgat, Ankara, Turkey
and

National Institute for Laser, Plasma and Radiation, Physics, Institute of Space Sciences,
Magurele-Bucharest, P.O. Box: MG-23, R 76911, Romania
e-mail: dumitru@cankaya.edu.tr

1 Introduction

In analytical chemistry, the development of economic, accurate and precise methods is very important phenomena for the routine analysis and the quality control of the combined commercial products containing two or more active substances without any chemical separation treatment. Some analytical conventional methods such as derivative spectrophotometry and chromatographic methods based on preliminary separation treatments have been used for the resolution of the above analytical problems. In all cases, these conventional methods in the analytical quantitative applications may not give desirable results. Finding new method to treat the complex systems within analytical chemistry is still an open problem in this area.

Particularly, the wavelet transforms methods and their applications in the analytical chemistry have increased the potential power of the spectrophotometric techniques for the quantitative resolution of the overlapping spectra and other spectral problems.

Recently, a new transform wavelet transform based on the fractional B-splines was initiated [1–4].

The mathematical idea of fractional derivatives has represented the subject of interest for various branches of science [5–7]. As it is already known the splines play a significant role on the early development of the theory of the wavelet transform [8–14].

The generalization of the spline constructions was proposed in [11], namely new wavelet bases with a continuous order parameter was obtained.

The new fractional splines have all properties of the polynomial splines with the exception of compact support when the order α is non-integer.

The main advantage of this construction is that we can build the wavelet bases parameterized by the continuously-varying regularity parameter α .

In this study, the simultaneous quantitative resolution of the binary mixture of trimethoprim (TMP) and sulfachloropyridazine sodium (SCP) was performed by applying the second derivative approach to the fractional wavelet signals. This methodology was named as FWT-DS² method.

The experimental results obtained from FWT-D² approach were compared with those obtained by the classical second derivative spectrophotometry (direct D²-method).

2 Signal Processing Method

The fractional calculus gained a lot of importance in several fields of science and engineering. For example the wavelet method and its generalization were extensively used during the last decades in the signal processing analysis of complex systems.

2.1 B-spline

In this subsection we define briefly the notion of *B-spline*. A **B-spline** is a generalization of the Bezier curve. Let a vector known as the knot is defined $T = \{t_0, t_1, \dots, t_m\}$ where T denotes a non-decreasing sequence with $t_i \in [0, 1]$, and define control points P_0, P_n . Let us define degree as $p = m - n - 1$. The knots $t_{p+1}, \dots, t_{m-p-1}$ are called internal knots. If we define the basis functional as

$$N_{i,0}(t) = \begin{cases} 1, & \text{if } t_i \leq t < t_{i+1} \text{ and } t_i < t_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

and

$$N_{i,p}(t) = \frac{t - t_i}{t_{i+p} - t_i} N_{i,p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}} N_{i+1,p-1}(t), \quad (2)$$

then the curve defined by

$$C(t) = \sum_{i=0}^n P_i N_{i,p}(t) \quad (3)$$

is a B-spline.

2.2 Fractional B-spline

The fractional B-spline is defined as follows

$$\beta_+^\alpha(x) = \frac{\Delta_+^{\alpha+1} x_+^\alpha}{\Gamma(\alpha+1)} = \frac{\sum_{k=0}^{+\infty} (-1)^k \binom{\alpha+1}{k} (x-k)_+^\alpha}{\Gamma(\alpha+1)}, \quad (4)$$

where Euler's Gamma function is defined as follows

$$\Gamma(\alpha+1) = \int_0^{+\infty} x^\alpha e^{-x} dx \quad (5)$$

and

$$(x-k)_+^\alpha = \max(x-k, 0)^\alpha \quad (6)$$

The forward fractional finite difference operator of order α is defined as

$$\Delta_+^\alpha f(x) = \sum_{k=0}^{+\infty} (-1)^k \binom{\alpha}{k} f(x-k), \quad (7)$$

where

$$\binom{\alpha}{k} = -\frac{\Gamma(\alpha + 1)}{\Gamma(k + 1)\Gamma(\alpha - k + 1)} \quad (8)$$

The above defined B-splines fulfill the convolution property

$$\beta_+^{\alpha_1} * \beta_+^{\alpha_2} = \beta_+^{\alpha_1 + \alpha_2}. \quad (9)$$

The centered fractional B-splines of degree α are given by

$$\beta_*(x) = \frac{1}{\Gamma(\alpha + 1)} \sum_{k \in \mathbb{Z}} (-1)^k \binom{\alpha+1}{k} |x - k|_*^\alpha, \quad (10)$$

where $|x|_*^\alpha$ has the following form

$$|x|_*^\alpha = \begin{cases} \frac{|x|^\alpha}{-2 \sin(\frac{\pi}{2}\alpha)}, & \alpha \text{ not even} \\ \frac{x^{2n} \log x}{(-1)^{1+n}\pi}, & \alpha \text{ even} \end{cases} \quad (11)$$

2.3 Fractional B-spline Wavelets

The definition of the fractional B-spline wavelets is below

$$\psi_+^\alpha\left(\frac{x}{2}\right) = \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{2^\alpha} \sum_{l \in \mathbb{Z}} \binom{\alpha+1}{l} \beta_*^{2\alpha+1}(l + k - 1) \beta_+^\alpha(x - k) \quad (12)$$

The fractional splines wavelets obey the following

$$\int_{-\infty}^{+\infty} X^n \psi_+^\alpha(x) dx = 0, \quad (13)$$

and the Fourier transform fulfills the following relations

$$\hat{\psi}_+^\alpha(\varpi) = C(j\varpi)^{\alpha+1}, \text{ as } \varpi \rightarrow 0 \quad (14)$$

and

$$\hat{\psi}_*^\alpha(\varpi) = C(j\varpi)^{\alpha+1}, \text{ as } \varpi \rightarrow 0, \quad (15)$$

where $\hat{\psi}_*^\alpha(\varpi)$ is symmetric. We observe that the fractional spline wavelets behaves like fractional derivative operator.

3 Experimental

3.1 Apparatus and Software

The absorption spectra were recorded by using a Shimadzu UV-160 double beam UV-visible spectrophotometer connected to a computer loaded with Shimadzu UVPC software and a LEXMARK-E320 printer. The data treatment was performed in a Pentium 42.8 GHz (512 Mb RAM) computer by using the Microsoft EXCEL and MATLAB 7.0 software.

3.2 Preparation of Standard Solutions

Stock solutions of TMP and SCP were separately prepared by dissolving 25 mg of each of drugs in 100 mL calibrated flask by using a solvent system consisting of 0.1 M HCl and methanol (50:50, v/v). Calibration solutions in the range of 2–12 for TMP and 4–24 $\mu\text{g/mL}$ for SCP were obtained from the above stock solutions.

3.3 Commercial Veterinary Products

In our study, a veterinary preparation (COSUMIX PLUS, water soluble powder, containing 100 mg SCP and 20 mg TMP per g, Novartis Pharm. Ind., Istanbul-Turkey) was investigated by the developed analytical methods.

SCP and TMP standards were obtained from Novartis Pharm. Ind., Istanbul-Turkey.

4 Method Applications

4.1 Fractional Wavelet Transform of the Absorption Spectra

The absorption spectra of the calibration solutions were recorded in the spectral region from 200 to 404.7 nm as shown in Fig. 1. The analogue procedure was applied to the solutions of the synthetic mixtures and commercial veterinary samples. As it can be seen from Fig. 1, the simultaneous quantitative resolution of SCP and TMP is not possible by using the conventional spectral methods due to the overlapping spectra of the analyzed compounds.

In the presented experimental conditions, we planned to use the derivative spectrophotometry after the application of Fractional wavelet transform (FWT) to the original absorption spectra.

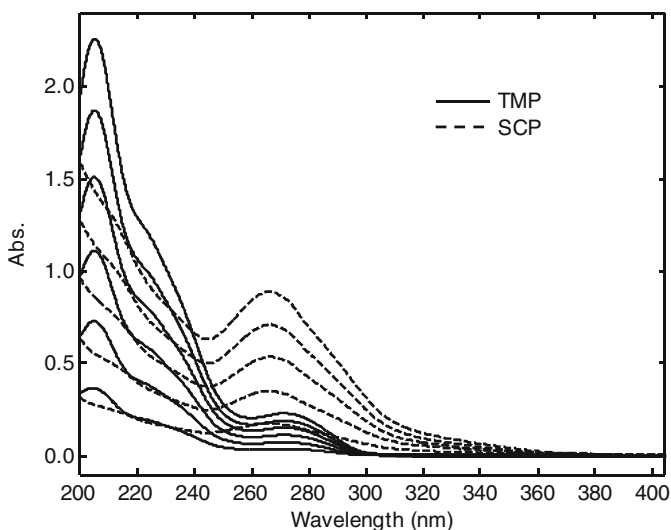


Fig. 1 Zero-order spectra of TMP (—) and SCP (----) in the range of 2–12 and 4–24 $\mu\text{g/mL}$, respectively in 0.1 M HCl and methanol (50:50, v/v)

In the signal analysis procedure, FWT was applied to the spectral signals corresponding to the absorbance data vectors having 2,048 points in the spectral range of 200.0–404.7 nm.

In this signal analysis, several parameters α and depths of the decomposition (J) were tested for optimization of the fractional signal processing and $\alpha = 0.25$ and $J = 2$ were found to be the optimal one. The type of B-splines was considered to be causal orthonormal.

Under the above optimized conditions, FWT was applied to the absorbance data vectors corresponding to the absorption spectra of the calibration solutions in the range of 2–12 $\mu\text{g/mL}$ for TMP and 4–24 $\mu\text{g/mL}$ for SCP (see Fig. 1). In the next step, FWT-spectra were obtained as shown in Fig. 2.

The same signal processing method was subjected to the absorbance data vectors of the synthetic mixtures and commercial samples containing TMP and SCP substances.

4.2 Calibration Graphs

For the determination of TMP and SCP in samples, the calibration solutions were prepared in the linear concentration range of 2–12 $\mu\text{g/mL}$ for TMP and 4–24 $\mu\text{g/mL}$ for SCP, respectively. The absorption spectra of the calibration solutions were recorded. The obtained original absorption spectra were used for the proposed methods and their calibration graphs.

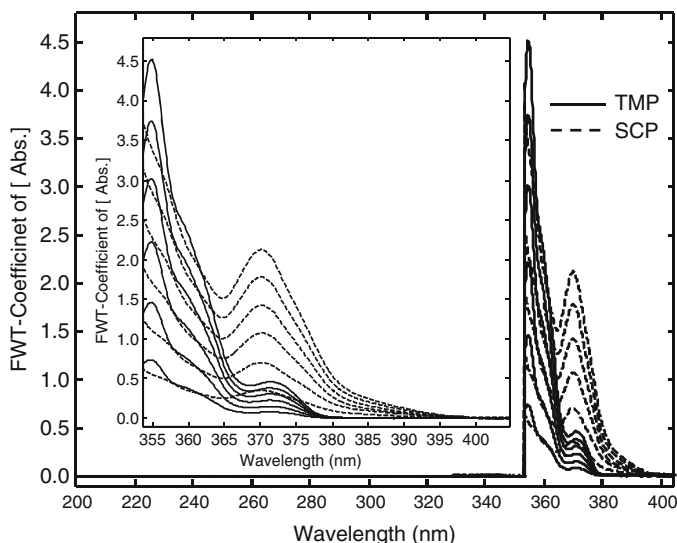


Fig. 2 FWT spectra of TMP (—) and SCP (----) in the range of 2–12 and 4–24 $\mu\text{g/mL}$, respectively in 0.1 M HCl and methanol (50:50, v/v)

Second derivative transform with the intervals of $\Delta\lambda = 6$ nm was simultaneously applied to the original absorption spectra and their FWT-spectra shown in Figs. 1 and 2, respectively. The application of the second derivative procedure to the original absorption spectra and their FWT-spectra was named as DS^2 and FWT-DS^2 methods.

Second derivative of the FWT-spectra were performed with the intervals of $\Delta\lambda = 6$ without using any smoothing procedure and scale factor (see Fig. 3). However, in the DS^2 method, the second derivative spectra of the original absorption spectra were obtained by using the intervals of $\Delta\lambda = 6$ nm and scale factor = 6. In addition, their second derivative spectra were smoothed with the intervals of $\Delta\lambda = 6$ nm as indicated in Figure 4.

In the application of the direct FWT-DS^2 method, two calibration graphs for TMP and SCP were obtained by measuring the FWT-intensities at 364.0 nm (corresponding to a zero-crossing point for SCP) and at 367.1 nm (corresponding to a zero-crossing point for TMP) in the FWT spectral range of 353.6–390.0 nm, respectively (see Fig. 3).

For the application of DS^2 method to the original absorption spectra, two calibration graphs for TMP and SCP were obtained by measuring the second derivative amplitudes at 284.1 nm (corresponding to a zero-crossing point for SCP) and 256.7 nm (corresponding to a zero-crossing point for TMP) in the second derivative spectra (see Fig. 4).

Linear regression analysis and their statistical parameters were summarized in Table 1. When the calculated statistical parameters were observed for direct DS^2 -method and FTW-DS^2 -method, the correlation coefficients were found more

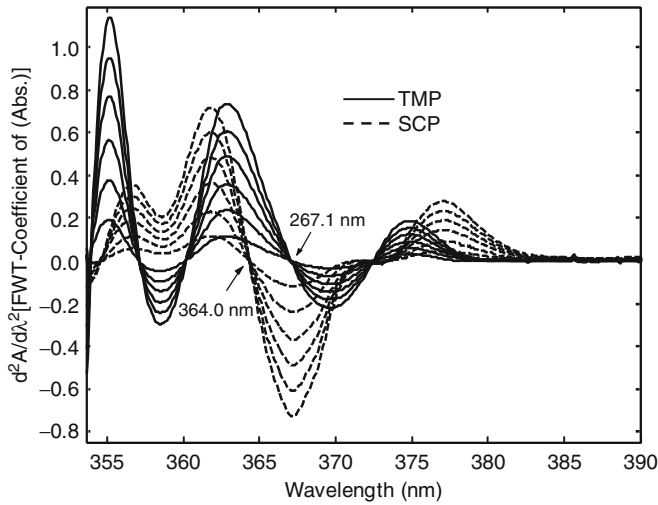


Fig. 3 Second derivative of the FWT spectra of TMP (—) and SCP (----) in the calibration range of 2–12 and 4–24 $\mu\text{g/mL}$, respectively

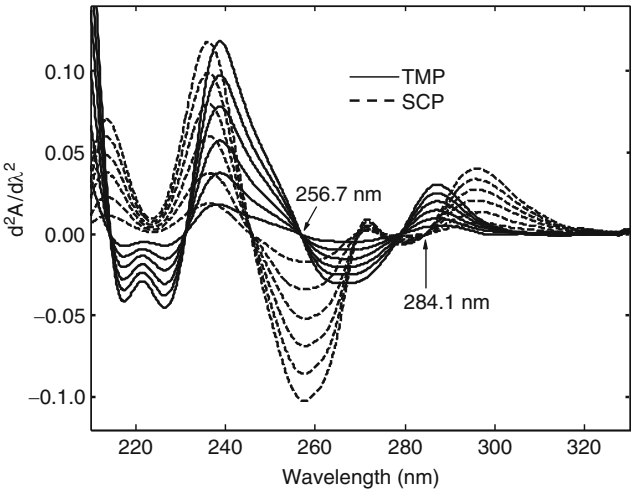


Fig. 4 Second derivative of the zero-order spectra of TMP (—) and SCP (----) in the calibration range of 2–12 and 4–24 $\mu\text{g/mL}$, respectively

than 0.999 for both FWT-DS² and direct DS² approaches. The limit of detection (LOD; signal-to-noise (S/N) ratio is 3:1) and the limit of quantitation (LOQ; S/N is 10:1) were calculated by using the standard deviation of slopes of the calibration equations as presented in Table 1.

Table 1 Calibration parameters obtained by the linear regression analysis

Parameter	FWT-DS ² method		DS ² method	
	TMP	SCP	TMP	SCP
λ (nm)	364.0	284.1	284.1	367.1
Range (μg/mL)	2.0–12.0	4.0–24.0	2.0–12.0	4.0–24.0
m	-1.08×10^{-2}	-2.69×10^{-4}	-2.69×10^{-4}	2.61×10^{-3}
n	5.24×10^{-2}	2.05×10^{-3}	2.05×10^{-3}	-3.05×10^{-2}
r	0.9999	1.0000	1.0000	0.9999
SE(m)	3.04×10^{-3}	7.94×10^{-5}	7.94×10^{-5}	3.61×10^{-3}
SE(n)	3.91×10^{-4}	1.02×10^{-5}	1.02×10^{-5}	2.32×10^{-4}
SE(r)	3.27×10^{-3}	8.53×10^{-5}	8.53×10^{-5}	3.88×10^{-3}
LOD (μg/mL)	0.43	0.28	0.28	0.87
LOQ (μg/mL)	1.42	0.95	0.95	2.90

M = Slope of the linear regression equation.
n = Intercept of the linear regression equation.
r = Correlation coefficient of the linear regression equation.
SE(m) = Standard error of the slope.
SE(n) = Standard error of the intercept.
SE(r) = Standard error of the correlation coefficient.
LOD = Limit of detection and LOQ = Limit of quantitation.

4.3 Method Validation

In this study, two proposed approaches (FWT-DS² method and direct DS² method) were validated by analyzing the synthetic mixtures of TMP and SCP.

To control the validity of the methods, the simultaneous resolution of the synthetic mixtures containing various concentrations of TMP and SCP was performed by FWT-DS² and direct DS² methods. The means recoveries and the relative standard deviations of the methods were summarized in Table 2. Their numerical values were found satisfactory to validate the accuracy and precision of the proposed two methods.

4.4 Analysis of the Commercial Veterinary Formulation

The determination results obtained by applying FWT-DS² and DS² to the commercial veterinary formulation were depicted in Table 3. It was observed that the results of the proposed methods were very close to each other as well as to the label value of commercial veterinary formulation.

In addition to this, the statistic parameters indicate that the applied methods are suitable to determine the two compounds in a commercial veterinary formulation.

In the application of the proposed methods to the commercial formulation, no interference was reported during the analysis.

Table 2 Recovery results obtained by applying the methods to the synthetic mixtures

Recovery (%)						
No.	Mixture (µg/mL)		FWT-DS ²		DS ²	
	TMP	SCP	SCP	TMP	SCP	TMP
1	2.0	20.0	101.9	97.5	94.8	98.2
2	4.0	20.0	101.9	97.8	98.0	98.2
3	6.0	20.0	100.2	97.9	104.7	98.9
4	8.0	20.0	100.1	99.2	99.8	96.8
5	10.0	20.0	101.4	98.3	97.6	97.7
6	12.0	20.0	102.7	97.5	98.2	96.8
7	4.0	4.0	101.2	96.5	97.1	93.3
8	4.0	8.0	98.9	96.9	100.6	96.0
9	4.0	12.0	100.5	97.3	104.5	97.8
10	4.0	16.0	97.2	97.5	101.2	97.4
11	4.0	20.0	99.7	97.3	105.7	98.0
12	4.0	24.0	100.5	97.6	106.3	98.4
		Mean	100.5	97.6	100.7	97.3
		SD	1.49	0.67	3.78	1.49
		RSD	1.48	0.68	3.76	1.53

SD = Standard deviation and RSD = Relative standard deviation.

Table 3 Determination results of TMP and SCP in veterinary formulation

	mg/g			
	FWT-DS ² -method		DS ² -method	
	TMP	SCP	TMP	SCP
	20.2	99.2	21.5	97.2
	19.8	97.4	22.4	97.6
	19.7	100.5	22.4	100.8
	20.4	101.5	21.3	101.4
	20.2	100.6	21.6	100.5
Mean	20.1	99.8	21.8	99.5
SD	0.28	1.60	0.53	1.93
RSD	1.39	1.60	2.43	1.94
SE	0.12	0.72	0.24	0.86
CL	0.24	1.40	0.46	1.69

SE = Standard error, CL = Confidence limit ($p = 0.05$).

5 Conclusions

Fractional calculus represents a powerful tool applied successfully in several fields of science and engineering.

In this study a new method was developed based on the simultaneous use of fractional wavelet analysis and zero-crossing technique. This approach was applied to the quantitative resolution of TMP and SCP compounds in their binary

mixtures without using any chemical separation procedure. To compare the results of FWT-DS² method, the direct DS² method was subjected to the analysis of binary mixture containing related compounds. By analyzing the obtained results we conclude that our new proposed method gives better results than those obtained by the classical derivative method.

The analyzed binary mixture represents a complex system and the fractional wavelet transform was found suitable to describe its quantitative analysis.

References

1. Blu T, Unser M (2000) The fractional spline wavelet transform: definition and implementation. Proceedings of the Twenty-Fifth IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP'00). Istanbul, Turkey, 5–9 June, vol. I, pp 512–515
2. Blu T, Unser M (2002) Wavelets, fractals, and radial basis functions. *IEEE Trans Signal Process* 50(3):543–553
3. Unser M, Blu T (1999) Construction of fractional spline wavelet bases. In: *proc.SPIE Wavelets Applications in Signal and Image Processing VII*, Denver, CO, 3813, pp 422–431
4. Unser M, Blu T (2000) Fractional splines and wavelets. *SIAM Rev* 42(1):43–67
5. Miller KS, Ross B (1993) An introduction to the fractional integrals and derivatives-theory and applications. Gordon and Breach, Longhorne, PA
6. Oldham KB, Spanier J (1974) The fractional calculus. Academic, New York
7. Podlubny I (1999) Fractional differential equations. Academic, San Diego
8. Daubechies I (1992) Ten lectures on wavelets. Society for Industrial and Applied Mathematics, Philadelphia, 1992
9. Walczak B (2000) Wavelets in chemistry. Elsevier, Amsterdam, The Netherlands
10. Dinç E, Baleanu D (2004) Multicomponent quantitative resolution of binary mixtures by using continuous wavelet transform. *J AOAC Int* 87(2):360–365
11. Dinç E, Baleanu D (2006) A new fractional wavelet approach for simultaneous determination of sodium and sulbactam sodium in a binary mixture. *Spectr Acta Part* 63(3):631–638
12. Dinç E, Baleanu D (2004) Application of the wavelet method for the simultaneous quantitative determination of benazepril and hydrochlorothiazide in their mixtures. *J AOAC Int* 87(4): 834–841
13. Dinç E, Baleanu D (2007) A review on the wavelet transform applications in analytical chemistry. In: Tas K, Tenreiro Machado JA, Baleanu D (eds) *Mathematical Methods in Engineering*. Springer, pp 265–285
14. Dinç E, Baleanu D, Üstündağ Ö (2003) An approach to quantitative two-component analysis of a mixture containing hydrochlorothiazide and spironolactone in tablets by one-dimensional continuous Daubechies and biorthogonal wavelet analysis of UV-spectra. *Spectr Lett* 36: 341–355

Fractional Wavelet Transform and Chemometric Calibrations for the Simultaneous Determination of Amlodipine and Valsartan in Their Complex Mixture

Mustafa Çelebier, Sacide Altınöz, and Erdal Dinç

Abstract Fractional wavelet transform (FWT) was applied to the original absorption spectra of amlodipine–valsartan mixture. The determination of amlodipine and valsartan in their commercial tablets in the presence of inactive ingredients was carried out by using a full-spectrum multivariate calibration method, partial least squares (PLS). Two PLS algorithms (PLS1 and PLS2) based on different pre-processing methodologies were used for a comparison between the predicted concentrations of amlodipine and valsartan in samples. The experimental calibration matrix corresponding to the concentrations levels between 1.08–17.27 $\mu\text{g/mL}$ for amlodipine and 3.00–35.00 $\mu\text{g/mL}$ for valsartan. A recovery study with the synthetic mixtures was carried out and the obtained results were highly satisfactory. The predicted concentration results obtained by applying the proposed PLS calibrations to real samples were satisfactory in all cases.

1 Introduction

Amlodipine (AMD) and valsartan (VAL) is a combination that used to treat high blood pressure (hypertension). In this antihypertensive combination, AMD and VAL are named as calcium channel and angiotensin II receptor blockers, respectively. The routine analysis and quality control of the above drug combination before and after commercial formulation is very important task for the analytical chemistry in drug industry due to the related regulations and human health.

M. Çelebier and S. Altınöz

Department of Analytical Chemistry, Faculty of Pharmacy, Hacettepe University,
06100, Ankara, Turkey
e-mail: celebier@hacettepe.edu.tr

E. Dinç (✉)

Department of Analytical Chemistry, Faculty of Pharmacy, Ankara University, 06100 Tandoğan,
Ankara, Turkey
e-mail: dinc@pharmacy.ankara.edu.tr

A few applications of the classical spectrophotometric [1, 2] and chromatographic [3] methods for the simultaneous analysis of amlodipine and valsartan were reported in previous studies.

For the analytical purposes, derivative spectrophotometry and its modified versions has been used extensively in fast quantitative analysis of mixtures. In some cases, these spectral methods may not lead desirable analytical results due to the strong spectral overlapping characteristics of compounds, decreasing signal intensity with worsening signal-to-noise ratio (S/N) in higher derivative orders.

On the other hand, chromatographic methods and their hyphenated versions require a prior separation step and other tedious analytical process during analysis for searching optimal separation conditions.

Taking into account the disadvantageous of the above classical analytical methods, analytical chemists need to develop new analytical techniques, approaches or methods to overcome the drawbacks of the above mentioned methods for the analytical problems. For these reasons, classical analytical methods in combination with many mathematical algorithms or statistical tools have been used for solving the above problems [4–23].

In this study, FWT method was applied to the spectral data vectors of AMD, VAL and two-component mixture in order to increase the peak amplitudes with data reduction. In addition, PLS calibration method was proposed for simultaneous determination of AMD and VAL in pharmaceutical preparation without using any chemical separation step. In this context, two PLS algorithms (PLS-1 and PLS-2) including different pre-processing methodologies were applied to the spectrophotometric data for the analysis of complex mixtures containing the related drugs. The ability of the PLS-1 and PLS-2 were validated by analyzing various synthetic binary mixtures of AMD and VAL. Finally, two PLS algorithms were applied to determine these drugs in tablets and a good agreement was observed.

2 Experimental

2.1 Instruments and Software

The spectrophotometric measurements were carried out using an Aglient 8,453 model UV-VIS spectrophotometer with a diode array detector (DAD) (190–1,100 nm). UV spectra of standard and sample solutions were recorded in 1 cm quartz cells. Microsoft EXCEL and PLS Toolbox_35 in Matlab 7.0 software were used for the statistical treatment of the data and the application of PLS methods.

2.2 Commercial Tablet Preparation

A commercial pharmaceutical preparation (Exforge® Tablets, Novartis Pharma Stein AG, Schaffhauserstrasse, CH-4332 Stein-Switzerland, Batch No: S0051A)

containing 10 mg of amlodipine and 160 mg of valsartan was studied. In this commercial tablet formulation, 10 mg of amlodipine corresponds to 13.87 mg of amlodipine besylate salt. The inactive ingredients in tablets are colloidal silicon dioxide, crospovidone, magnesium stearate and microcrystalline cellulose. The film coating contains hypromellose, iron oxides, polyethylene glycol, talc and titanium dioxide.

2.3 *Standard Solutions*

Stock standard solutions were separately prepared by dissolving 100 mg of amlodipine besylate salt equals to 72.09 mg of AMD and 100 mg of VAL in 100 mL methanol (MeOH). A calibration set of 16 standard mixture solutions consisting of 0.00–17.27 $\mu\text{g/mL}$ for AMD and 0.00–35.00 $\mu\text{g/mL}$ for VAL was made from stock solutions. An independent validation set or test set containing ten synthetic mixture solutions of AMD and VAL in the concentration range of 1.08–17.27 $\mu\text{g/mL}$ was prepared, respectively.

2.4 *Preparation of the Commercial Tablet Samples*

In this sample preparation, ten tablets were accurately weighed by using an electronic balance and tablet content was powdered in a mortar. An amount containing AMD and VAL equivalent to one tablet content was dissolved in MeOH and made up in 100 mL calibrated flask. This solution containing AMD and VAL was mechanically shaken for 25 min. After shaking, part of the flask content was centrifuged at 3,500 rpm for 15 min. Appropriate solutions were prepared by taking suitable aliquots of the clear supernatant and diluting them with MeOH to give final concentration. This sample preparation was repeated eight replicates. Their absorption spectra were recorded and processed by PLS-1 and PLS-2 methods for the determination of AMD and VAL in tablets.

3 Results and Discussions

The absorption spectra of AMD and VAL compounds, and their binary mixture were recorded with the intervals of $\Delta\lambda = 1.0 \text{ nm}$ in the spectral region of 200–300 nm (see Fig. 1). As can be seen from Fig. 1, the resolution of the two-component mixture presents a difficult problem because of the overlapping spectra of the related drugs in the same spectral range. In additional procedure, FWT method was applied to the spectral data vectors of AMD, VAL and two-component mixture in order to increase the peak amplitudes with data reduction in the spectral range of 200–327 nm. This FWT spectrum was presented in Fig. 2. However, using the absorption spectra,

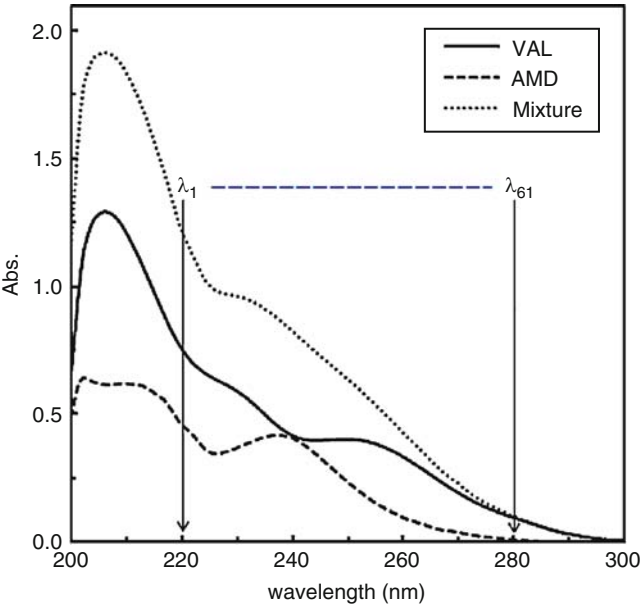


Fig. 1 Absorption spectra of AMD (14 $\mu\text{g/mL}$), VAL (11 $\mu\text{g/mL}$) and their binary mixture in methanol

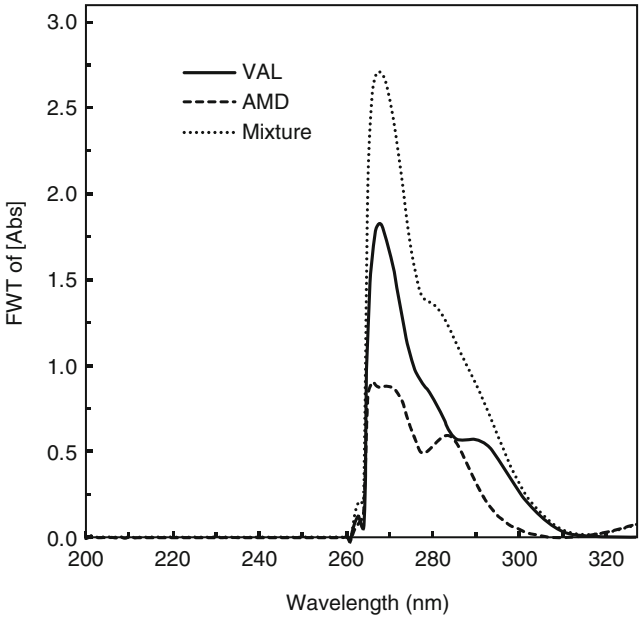


Fig. 2 FWT spectra of AMD (14 $\mu\text{g/mL}$), VAL (11 $\mu\text{g/mL}$) and their binary mixture (alfa = 1.5, type = “ + O”, J = 1)

the PLS-1 and PLS-2 methods allow the quantitative resolution of tablet samples containing AMD and VAL together with inactive ingredients in tablets. The spectral region between 220 and 280 nm, containing 61 wavelength values (at 1.0 nm intervals), was selected to construct PLS calibrations. The above appropriated wavelengths regions were selected by checking different wavelength ranges.

3.1 PLS Calibration Models and Prediction

A calibration set in the concentration ranges of 0.00–17.27 µg/mL for AMD and 0.00–35.00 µg/mL for VAL was prepared. As shown in Table 1, this random calibration set design with 16 standard mixture samples was made, including binary mixtures. The objective of this design was to span the concentration of the drugs in the samples of the calibration set in order to obtain high and low levels of concentration variability.

Absorbances of the prepared calibration set were measured at 61-wavelength set with $\Delta\lambda = 1.0$ nm intervals in the spectral region between 220 and 280 nm (see Fig. 1). As a results the absorbance data matrix were obtained. PLS regressions including two algorithm forms, PLS-1 and PLS-2 calculated by using the mathematical relationship between the calibration set (y-block) and absorbance data matrix (x-block). Both algorithms were applied to mean centered data of the concentration and absorbance data. The simultaneous prediction of the related drugs in samples was performed by applying PLS calibration models.

In the prediction step, a test set consisting of the AMD-VAL mixtures was used to test the applicability of PLS-1 and PLS-2. These test samples were prepared as indicated in given in Table 2. Two PLS models were applied to predict the concentration

Table 1 Concentration data for the calibration set

Standard No.	Set composition (µg/mL)	
	VAL	AMD
1	27.00	0.00
2	3.00	1.44
3	19.00	1.44
4	3.00	0.00
5	35.00	1.44
6	0.00	2.16
7	32.00	8.63
8	0.00	8.63
9	11.00	1.44
10	0.00	17.27
11	32.00	1.08
12	35.00	0.00
13	27.00	1.44
14	19.00	0.00
15	32.00	2.16
16	32.00	17.27

Table 2 Recovery results obtained by application of PLS-1 and PLS-2 to the synthetic mixtures

Mixture (mg/mL)		Predicted concentration µg/mL				Recovery (%)			
		PLS-1		PLS-2		PLS1		PLS2	
VAL	AMD	VAL	AMD	VAL	AMD	VAL	AMD	VAL	AMD
3.00	1.44	2.90	1.47	2.89	1.46	96.6	102.5	96.4	101.7
11.00	1.44	10.73	1.39	10.82	1.39	97.6	96.7	98.4	96.3
19.00	1.44	19.07	1.43	19.14	1.41	100.4	99.4	100.8	98.0
27.00	1.44	27.04	1.48	27.05	1.47	100.1	102.8	100.2	101.9
35.00	1.44	34.28	1.46	33.29	1.45	97.9	101.4	95.1	100.7
32.00	1.08	32.13	1.11	32.12	1.12	100.4	102.9	100.4	103.9
32.00	2.16	31.87	2.10	31.85	2.10	99.6	97.3	99.5	97.2
32.00	4.32	31.49	4.30	31.48	4.31	98.4	99.5	98.4	99.8
32.00	8.63	31.47	8.41	31.46	8.39	98.3	97.4	98.3	97.2
32.00	17.27	31.89	17.50	31.86	17.49	99.6	101.3	99.6	101.3
Mean						98.9	100.1	98.7	99.8
SD						1.31	2.39	1.80	2.52
RSD						1.33	2.39	1.82	2.52

Table 3 Statistical calibration parameters

Parameter	PLS-1		PLS-2	
	VAL	AMD	VAL	AMD
PRESS	1.51715	0.62074	1.5831	0.6131
RMSEC	0.37929	0.15519	0.3958	0.1533
Factor	3	3	3	3

of the drugs in the test set. Table 2 shows these results obtained by application of the regression models to test solutions. As it can be seen, there are very good recovery values for both PLS-1 and PLS-2 algorithms.

The selection of optimal factor for both PLS algorithm was performed by using cross-validation procedure by using 16 absorption spectra of samples. According to the cross-validation process, the different factor numbers were tested and the first three factors having minimum values of PRESS (prediction residual error sum of squares) and RMSEC (residual mean squares error of calibration) were found to be suitable for PLS calibration. By using actual and predicted concentrations, PRESS, RMSEC, correlation coefficient (r), intercept (n) and slope (m) in PLS calibration step using the first four factors were calculated and shown in Table 3.

3.2 Method Applications to Tablet Samples

The proposed PLS-1 and PLS-2 methods were applied for the determination AMD and VAL in commercial pharmaceutical tablets. The results obtained are summarized in Table 4, showing that the predicted concentrations are satisfactory in all cases.

Table 4 Determination results of AMD and VAL in tablets by the proposed chemometric methods

Sample no.	mg/tablet			
	PLS1		PLS2	
	VAL	AMD	VAL	AMD
1	158.80	10.32	158.59	10.27
2	157.98	10.43	157.69	10.40
3	160.44	10.45	160.07	10.35
4	163.87	10.21	163.66	10.21
5	162.34	10.05	162.23	10.04
6	161.73	10.39	161.22	10.10
7	157.47	10.31	157.65	10.30
8	155.53	9.70	155.91	9.68
Mean	159.77	10.23	159.63	10.17
SD	2.81	0.25	2.63	0.23
RSD	1.76	2.44	1.65	2.27
SE	0.89	0.08	0.83	0.07
CL	157.77 $\pm$ 1.74	10.23 $\pm$ 0.15	159.63 $\pm$ 1.63	10.17 $\pm$ 0.14

4 Conclusions

In spite of the overlapping spectra of AMD and VAL in the spectral region of 200–300 nm, the simultaneous determination of the AMD and VAL in their synthetic mixtures and tables was performed by using PLS-1 and PLS-2 calibrations based on the use of spectrophotometric data, without requiring priori separation procedure.

These proposed PLS-1 and PLS-2 calibration methods were found suitable for simple and precise routine quality control analysis of the pharmaceutical tablet samples.

References

1. Chitlange SS, Bagri K, Wankhede SB, Sakarkar DN (2008) Simultaneous spectrophotometric estimation of amlodipine and valsartan in capsule formulation. *Orient J Chem* 24:689–692
2. Chitlange SS, Bagri K, Wankhede SB, Sakarkar DN (2008) Validated spectrophotometric estimation of valsartan and amlodipine in combined dosage form. *J Pharm Res* 7:53–55
3. Celebier M, Kaynak MS, Altinoz S, Şahin S (2008) Validated HPLC method development: the simultaneous analysis of amlodipine and valsartan in samples for liver perfusion studies. *Hacettepe Univ Eczacil Fak Derg* 28:15–30
4. Kramer R (1998) Chemometric techniques in quantitative analysis. Marcel Dekker, New York
5. Beebe KR, Kowalski BR (1987) An introduction to multivariate calibration and analysis. *Anal Chem* 59:1007A–1017A
6. Adams MJ (1995) Chemometrics in analytical spectroscopy. The Royal Society of Chemistry, Graham House, Science Park, Cambridge
7. Thomas EV, Haaland DM (1990) Comparison of multivariate calibration methods for quantitative spectral analysis. *Anal Chem* 62:1091–1099

8. Dinc E, Kanbur M, Baleanu D (2007) Simultaneous determination of chlorotetracycline and benzocaine in bolus by chemometric methods. *Rev Chim* 58:195–198
9. Dinc E, Kanbur M, Baleanu D (2007) Comparative spectral analysis of veterinary powder product by continuous wavelet and derivative transforms. *Spectrochim Acta Part A* 68A: 225–230
10. Dinc E, Baleanu D (2007) Numerical and graphical approaches for the simultaneous quantitative analysis of levamisole and oxytetracycline in veterinary formulation. *Rev Chim* 58:816–821
11. Dinc E, Baleanu D (2007) Continuous wavelet transform and chemometric methods for quantitative resolution of a binary mixture of quinapril and hydrochlorothiazide in tablets. *J Braz Chem Soc* 18:962–968
12. Dinc E, Altynoz S, Baleanu D (2007) Simultaneous determination of quinapril and hydrochlorothiazide in tablets by ratio spectra derivative spectrophotometric and chemometric methods. *Rev Chim* 58:1263–1267
13. Dinc E, Ragno G, Ioele G, Baleanu D (2006) Fractional wavelet analysis for the simultaneous quantitative analysis of lacidipine and its photodegradation product by continuous wavelet transform and multilinear regression calibration. *J AOAC Int* 89:1538–1546
14. Dinc E, Ozdemir A, Aksoy H, Ustunfag O, Baleanu D (2006) Chemometric determination of naproxen sodium and pseudoephedrine hydrochloride in tablets by HPLC. *Chem Pharm Bull* 54:415–421
15. Tosun A, Bahadır O, Dinc E (2007) Determination of anomalin and deltoin in seseli resinosum by LC combined with chemometric methods. *Chromatographia* 66:677–683
16. Dinc E, Ozdemir A, Aksoy H, Baleanu D (2006) Chemometric approach to simultaneous chromatographic determination of paracetamol and chlorzoxazone in tablets and spiked human plasma. *J Liq Chrom Rel Technol* 29:1803–1822
17. Dinc E, Baleanu D, Tas A (2006) Wavelet transform and artificial neural network for the quantitative resolution of ternary mixtures. *Rev Chim* 57: 626–631
18. Dinc E, Baleanu D (2006) A new fractional wavelet approach for the simultaneous determination of ampicillin sodium and sulbactam sodium in a binary mixture. *Spectr Acta Part A* 63A:631–638
19. Dinc E, Aktas AH, Baleanu D, Ustundag O (2006) Simultaneous determination of tartrazine and allura red in commercial preparation by chemometric HPLC method. *J Food Drug Anal* 14:284–291
20. Dinc E, Ustundag O (2005) Application of multivariate calibration techniques to HPLC data for quantitative analysis of a binary mixture of hydrochlorothiazide and losartan in tablets. *Chromatographia* 61:237–244
21. Dinc E, Ozdemir A, Baleanu D (2005) Comparative study of the continuous wavelet transform, derivative, and partial least squares methods applied to the overlapping spectra for the simultaneous quantitative resolution of ascorbic acid and acetylsalicylic acid in effervescent tablets. *J Pharm Biomed Anal* 37:569–575
22. Dinc E, Ozdemir A, Baleanu D (2005) An application of derivative and continuous wavelet transforms to the overlapping ratio spectra for the quantitative multiresolution of a ternary mixture of paracetamol, acetylsalicylic acid, and caffeine in tablets. *Talanta* 65:36–47
23. Dinc E, Baleanu D (2004) Application of the wavelet method for the simultaneous quantitative determination of benazepril and hydrochlorothiazide in their mixtures. *J AOAC Int* 87:834–841

Part V

Fractional Control Systems

Analytical Impulse Response of Third Generation CRONE Control

Rim Jallouli-Khlif, Pierre Melchior, F. Levron, Nabil Derbel,
and Alain Oustaloup

Abstract The present work is dealing with the synthesis of the third generation CRONE control impulse response. The main objective of this research is to extend the application of the preshaping approach to the third generation CRONE control. Preshaping is a feedforward technique used for reducing system oscillation in motion control. Mathematical development is based essentially on the integral residue method. Analytical result puts into proof that a third generation CRONE control impulse response is decomposed into two different sub-system behaviors: oscillatory and aperiodic ones. Simulations are presented to promote this result.

1 Introduction

Much work dealt with input shaping in controlling systems. Shaping command input or preshaping is used for reducing system oscillation in motion control. Input shaping is a feedforward technique that has been successfully applied for controlling flexible structures, and it has been shown to allow flexible structures to be maneuvered with little residual vibration, even in the presence of modelling uncertainties and structural nonlinearities. See [5, 8, 9]. Preshaping has been also applied to explicit fractional derivative systems to improve the second generation CRONE control response time. See [2, 6, 7].

R. Jallouli-Khlif and N. Derbel

Research Unit on Intelligent Control, Design and Optimisation of Complex Systems (ICOS),
University of Sfax, Sfax Engineering School, BP W, 3038 Sfax, Tunisia
e-mail: rim.jalloulikhelif@enis.rnu.tn; n.derbel@enis.rnu.tn

P. Melchior (✉) and A. Oustaloup

IMS (UMR 5218 CNRS, Université Bordeaux 1 - ENSEIRB - ENSCP), Département LAPS,
Bât. A4, 351 cours de la Libération - F33405 Talence cedex, France
e-mail: pierre.melchior@laps.ims-bordeaux.fr; alain.oustaloup@laps.ims-bordeaux.fr

F. Levron

IMB (Institut de Mathématiques de Bordeaux - UMR 5251), Université Bordeaux 1,
351 cours de la Libération - F33405 Talence cedex, France
e-mail: francois.levron@u-bordeaux1.fr

With input shaping control, an input command is convolved with a sequence of impulses designed to produce a resulting command that causes less residual vibration than the original unshaped command. The goal of this method is to determine amplitudes and timing of the impulses to eliminate or reduce residual vibration. Thus, it is necessary to dispose of an analytical time response of the system.

The objective of our work is to extend the input shaping technique to the third generation CRONE control. However, no analytical time response is available. For this reason, as a first step, this paper deals with the synthesis of the analytical impulse response of the third generation CRONE control [2]. Mathematical development, based on integral residue method, is presented.

Section 2 introduces briefly the third generation CRONE control. In Sect. 3 mathematical proofs are detailed to express the third generation CRONE control impulse response. Section 4 describes the third generation CRONE control poles characterization. Final result is presented in Sect. 5. Section 6 shows some simulation results. Finally, in Sect. 7, a conclusion and further works are stated.

2 Third Generation CRONE Control

The third generation CRONE control closed loop transfer function is defined below:

$$T(s) = \frac{\beta(s)}{1 + \beta(s)}, \quad (1)$$

in which, the open loop transfer function $\beta(s)$ is expressed by its theoretical definition given by the Eq. 2, which represents the real part of a complex order integrator:

$$\begin{aligned} \beta(s) &= \operatorname{Re} \left(\left(\frac{\omega_u}{s} \right)^n \right) \\ &= \cosh \left(b \frac{\pi}{2} \right)^{\operatorname{sgn}(b)} \left(\frac{\omega_u}{s} \right)^a \cos \left(b \ln \left(\frac{\omega_u}{s} \right) \right)^{-\operatorname{sgn}(b)} \end{aligned} \quad (2)$$

Parameters a and b are respectively the real part and the imaginary part of the complex order n ($n = a + ib$).

3 Analytical Response Using Residue Method

The analytical impulse response of the third generation CRONE control closed loop is basically calculated through Laplace inverse transform (3):

$$y(t) = L^{-1} [T(s)]. \quad (3)$$

Laplace inverse transform of a transfer function is defined by the following equation:

$$y(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^{ts} T(s) ds. \quad (4)$$

Parameter c is defined as:

$c > \text{Max}(|\text{Re}(\text{poles of } T(s))|)$, let's take $c = 1 + \text{Max}(|\text{Re}(\text{poles of } T(s))|)$.

According to [3,4], the third generation CRONE control has two complex conjugated poles, p_1 and p_2 , in closed loop structure, which are given below:

$$\begin{cases} p_1 = \omega_u \rho e^{j\theta} \\ p_2 = \omega_u \rho e^{-j\theta} \end{cases} \quad (5)$$

so, $c > |\omega_u \rho \cos(\theta)|$.

In order to compute the integral value appearing in Eq. 4, residue method is used. Thus, it is necessary to define an integral curve. s^n operator, n being non integer, is defined only if $s \in \mathbb{C} - \mathbb{R}^-$. γ integral curve is, then, presented in Fig. 1.

Equation 4 is equivalent to the following expression:

$$y(t) = \frac{1}{2\pi j} \lim_{R \rightarrow +\infty} \int_{\gamma_1} e^{ts} T(s) ds. \quad (6)$$

Supposing that the γ integral curve can be decomposed as presented in Fig. 1, $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6$, it can be possible to develop Eq. 6 using different parts of the above γ curve.

$$y(t) = \frac{1}{2\pi j} \lim_{R \rightarrow +\infty} \left[\int_{\gamma} e^{ts} T(s) ds - \int_{\gamma_2 + \gamma_6} e^{ts} T(s) ds - \int_{\gamma_3 + \gamma_5} e^{ts} T(s) ds - \int_{\gamma_4} e^{ts} T(s) ds \right]. \quad (7)$$

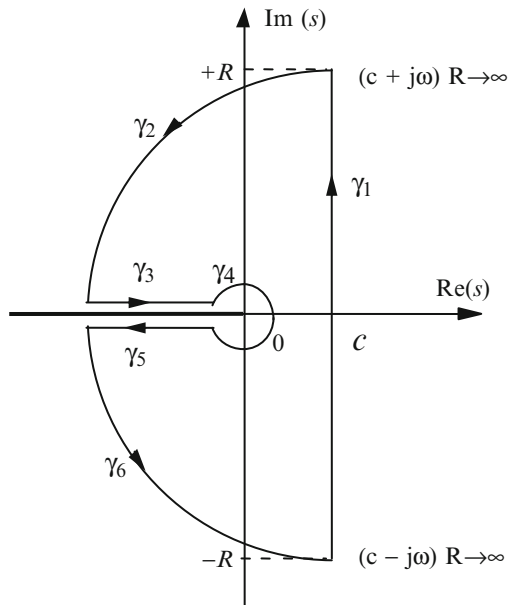


Fig. 1 Integration curve

As $\int_{\gamma} e^{ts} T(s) ds = 2\pi j \sum \text{Residue}$, residues are related to poles internally situated according to γ curve, it comes:

$$y(t) = \sum \text{Residue} - \frac{1}{2\pi j} \lim_{R \rightarrow +\infty} \left[\int_{\gamma_2 + \gamma_6} e^{ts} T(s) ds + \int_{\gamma_3 + \gamma_5} e^{ts} T(s) ds + \int_{\gamma_4} e^{ts} T(s) ds \right]. \quad (8)$$

Calculus lead to the following equation defining the third generation CRONE control closed loop impulse response:

$$y(t) = \sum \text{Residue} - \frac{1}{2\pi j} \lim_{R \rightarrow +\infty} \int_{\gamma_3 + \gamma_5} e^{ts} T(s) ds. \quad (9)$$

Following paragraphs are dealing with the different integral calculus on curve parts defining the total γ integral curve to give a proof for the Eq. 9.

3.1 Integration upon γ_2 and γ_6 Curves

Integration on $\gamma_2 + \gamma_6$ curve can be decomposed as follows:

$$\int_{\gamma_2 + \gamma_6} e^{ts} T(s) ds = \int_{\gamma_2} e^{ts} T(s) ds + \int_{\gamma_6} e^{ts} T(s) ds. \quad (10)$$

Analytical development leads to express Eq. 10 as in Eq. 11. It comes, for all b non null in $\mathbb{R}$:

$$\lim_{R \rightarrow +\infty} \int_{\gamma_2 + \gamma_6} e^{ts} T(s) ds = 0. \quad (11)$$

Proof. On γ_2 and γ_6 curves, operator s and its derivative are defined by $s = c + Re^{j\theta}$ and $ds = jRe^{j\theta} d\theta$, θ is supposed to be the polar angle of the affix point $(c, 0)$. On γ_2 , $\theta \in [\frac{\pi}{2}, \pi]$, and on γ_6 , $\theta \in]-\pi, -\frac{\pi}{2}]$.

Expressing parameter s by its definition, the calculus on γ_2 curve is given by Eq. 12.

$$\int_{\gamma_2} e^{ts} T(s) ds = je^{tc} \int_{\frac{\pi}{2}}^{\pi} Re^{tR \cos(\theta)} e^{j(\theta + tR \sin(\theta))} T(c + Re^{j\theta}) d\theta. \quad (12)$$

If $R \rightarrow +\infty$, Eq. 12 can be written in the following way:

$$\lim_{R \rightarrow +\infty} \int_{\gamma_2} e^{ts} T(s) ds = \lim_{R \rightarrow +\infty} je^{tc} \int_{\frac{\pi}{2}}^{\pi} f(R, \theta) d\theta. \quad (13)$$

The function $f(R, \theta)$ is defined by:

$$f(R, \theta) = Re^{tR \cos(\theta)} e^{j(\theta + tR \sin(\theta))} \frac{\cosh\left(b \frac{\pi}{2}\right)^{\text{sgn}(b)} \left(\frac{\omega_u}{Re^{j\theta}}\right)^a \cos\left(b \ln\left(\frac{\omega_u}{Re^{j\theta}}\right)\right)^{-\text{sgn}(b)}}{1 + \cosh\left(b \frac{\pi}{2}\right) \left(\frac{\omega_u}{Re^{j\theta}}\right)^a \cos\left(b \ln\left(\frac{\omega_u}{Re^{j\theta}}\right)\right)^{-\text{sgn}(b)}}. \quad (14)$$

To manage with Eq. 13, two different cases should be distinguished: $b > 0$ and $b < 0$.

In the first case, $b > 0$, so:

$$|f(R, \theta)| = \frac{Re^{tR \cos(\theta)} |\cosh(b \frac{\pi}{2})| (\frac{\omega_u}{R})^a}{|\cos(b \ln(\frac{\omega_u}{Re^{j\theta}})) + \cosh(b \frac{\pi}{2}) (\frac{\omega_u}{Re^{j\theta}})^a|}. \quad (15)$$

If $R \rightarrow +\infty$, it is possible to write the following equivalence:

$$|\cos(b \ln(\frac{\omega_u}{Re^{j\theta}})) + \cosh(b \frac{\pi}{2}) (\frac{\omega_u}{Re^{j\theta}})^a| \sim |\cos(b \ln(\frac{\omega_u}{Re^{j\theta}}))|.$$

Referring to the appendix (Eq. 69), inequality (16) is deduced:

$$|\sinh(b\pi)| \leq |\cos(b \ln(\frac{\omega_u}{Re^{j\theta}}))|. \quad (16)$$

So, it exists two reals, ε small and R_ε great, verifying that if $R \geq R_\varepsilon$ then:

$$0 < |\sinh(b\pi)| - \varepsilon < |\cos(b \ln(\frac{\omega_u}{Re^{j\theta}})) + \cosh(b \frac{\pi}{2}) (\frac{\omega_u}{Re^{j\theta}})^a|.$$

In this case,

$$|f(R, \theta)| \leq Re^{tR \cos(\theta)} \frac{|\cosh(b \frac{\pi}{2})| (\frac{\omega_u}{R})^a}{|\sinh(b\pi)| - \varepsilon}. \quad (17)$$

The majorant of $|f(R, \theta)|$ in inequation (17) tends towards 0 if R tends towards infinity, so:

$$\lim_{R \rightarrow +\infty} f(R, \theta) = 0.$$

Consequently, for $b > 0$ we have:

$$\lim_{R \rightarrow +\infty} \int_{\gamma_2} e^{ts} T(s) ds = 0. \quad (18)$$

In the case where $b < 0$, we have:

$$|f(R, \theta)| = \frac{Re^{tR \cos(\theta)} (\frac{\omega_u}{R})^a |\cos(b \ln(\frac{\omega_u}{Re^{j\theta}}))|}{|\cosh(b \frac{\pi}{2}) + (\frac{\omega_u}{Re^{j\theta}})^a \cos(b \ln(\frac{\omega_u}{Re^{j\theta}}))|}. \quad (19)$$

According to the appendix (Eq. 69),

$$|f(R, \theta)| \leq Re^{tR \cos(\theta)} \frac{(\frac{\omega_u}{R})^a \cosh(b\pi)}{|\cosh(b \frac{\pi}{2})| - |(\frac{\omega_u}{R})^a \cosh(b\pi)|}. \quad (20)$$

If $R \rightarrow \infty$, the majorant in inequation (20) is equivalent to $e^{tR \cos(\theta)} \frac{\cosh(b\pi) \frac{\omega_u^a}{R^{a-1}}}{\cosh(b \frac{\pi}{2})}$.
Having $\theta \in]-\pi, -\frac{\pi}{2}]$ and $1 < a < 2$ then,

$$\lim_{R \rightarrow +\infty} e^{tR \cos(\theta)} \frac{\cosh(b\pi) \frac{\omega_u^a}{R^{a-1}}}{\cosh(b \frac{\pi}{2})} = 0. \quad (21)$$

Result in Eq. 21 leads to write:

$$\lim_{R \rightarrow +\infty} f(R, \theta) = 0.$$

Consequently, for $b < 0$ we have:

$$\lim_{R \rightarrow +\infty} \int_{\gamma_2} e^{ts} T(s) ds = 0. \quad (22)$$

It can be concluded that, referring to Eqs. 18 and 22, for all b non null in $\mathbb{R}$:

$$\lim_{R \rightarrow +\infty} \int_{\gamma_2} e^{ts} T(s) ds = 0. \quad (23)$$

On γ_6 curve, the following equality can be written:

$$\lim_{R \rightarrow +\infty} \int_{\gamma_6} e^{ts} T(s) ds = \lim_{R \rightarrow -\infty} j e^{tc} \int_{-\pi}^{-\frac{\pi}{2}} f(R, \theta) d\theta \quad (24)$$

$f(R, \theta)$ is already defined in Eq. 14.

Based on the work done for γ_2 and referring to the appendix, the same methodology is applied for the integration upon γ_6 curve. The same result is obtained, so:

$$\lim_{R \rightarrow +\infty} \int_{\gamma_6} e^{ts} T(s) ds = 0. \quad (25)$$

Consequently, for all b non null in $\mathbb{R}$, result in Eq. 11 is proved. $\square$

3.2 Integration upon γ_3 and γ_5 Curves

Integration calculus upon $(\gamma_3 + \gamma_5)$ curve is done according to the following equation:

$$\int_{\gamma_3 + \gamma_5} e^{ts} T(s) ds = \int_{\gamma_3} e^{ts} T(s) ds + \int_{\gamma_5} e^{ts} T(s) ds. \quad (26)$$

The final result is given below:

In case $b > 0$

$$\lim_{R \rightarrow +\infty} \int_{\gamma_3 + \gamma_5} e^{ts} T(s) ds = \alpha \int_0^{+\infty} e^{-tx} \left(\frac{\omega_u}{x} \right)^a \frac{e^{-ja\pi} \cos(\delta + j\gamma) - e^{ja\pi} \cos(\delta - j\gamma)}{D} dx. \quad (27)$$

$\alpha = \cosh(b \frac{\pi}{2})$, parameters δ , γ and D are respectively defined by Eqs. 30, 31 and 38.

In case $b < 0$

$$\lim_{R \rightarrow +\infty} \int_{\gamma_3 + \gamma_5} e^{ts} T(s) ds = \alpha \int_0^{+R} e^{-tx} \left(\frac{\omega_u}{x} \right)^a \frac{e^{-ja\pi} \cos(\delta - j\gamma) - e^{ja\pi} \cos(\delta + j\gamma)}{D1D2} dx \quad (28)$$

$\alpha = \frac{1}{\cosh(b\frac{\pi}{2})}$, parameters δ , γ and $D1D2$ are respectively defined by Eqs. 30, 31 and 41.

Proof. On γ_3 curve, operator s is described as $s = -x = xe^{j\pi}$, on γ_5 , $s = -x = xe^{-j\pi}$, and in both cases $x > 0$. Proceeding, first, by expressing operator s with its definitions, and, supposing, then, equalities from (29) to (35), it is possible to express the integration upon $(\gamma_3 + \gamma_5)$ curve by Eq. 36.

$$\alpha = \cosh\left(b\frac{\pi}{2}\right)^{sgn(b)} \quad (29)$$

$$\delta = b \ln\left(\frac{\omega_u}{x}\right) \quad (30)$$

$$\gamma = b\pi \quad (31)$$

$$N_1 = e^{-ja\pi} (\cos(\delta - j\gamma))^{-sgn(b)} \quad (32)$$

$$N_2 = e^{ja\pi} (\cos(\delta + j\gamma))^{-sgn(b)} \quad (33)$$

$$D_1 = 1 + \alpha \left(\frac{\omega_u}{x} \right)^a e^{-ja\pi} (\cos(\delta - j\gamma))^{-sgn(b)} \quad (34)$$

$$D_2 = 1 + \alpha \left(\frac{\omega_u}{x} \right)^a e^{ja\pi} (\cos(\delta + j\gamma))^{-sgn(b)} \quad (35)$$

$$\int_{\gamma_3 + \gamma_5} e^{ts} T(s) ds = \alpha \int_0^{+R} e^{-tx} \left(\frac{\omega_u}{x} \right)^a \left(\frac{N_1}{D_1} - \frac{N_2}{D_2} \right) dx \quad (36)$$

Two different cases are always distinguished: $b > 0$ and $b < 0$.

If $b > 0$:

$$\frac{N_1}{D_1} - \frac{N_2}{D_2} = \frac{e^{-ja\pi} \cos(\delta + j\gamma) - e^{ja\pi} \cos(\delta - j\gamma)}{D} \quad (37)$$

with:

$$\begin{aligned} D = & \alpha^2 \left(\frac{\omega_u}{x} \right)^{2a} + \cos(\delta - j\gamma) \cos(\delta + j\gamma) \\ & + \alpha \left(\frac{\omega_u}{x} \right)^a [e^{ja\pi} \cos(\delta - j\gamma) + e^{-ja\pi} \cos(\delta + j\gamma)] \end{aligned} \quad (38)$$

So, in case $b > 0$ and according to Eqs. 36, 37 and 38, the equality (27) is expressed.

If $b < 0$:

$$N_1 D_2 = \frac{1}{2} \left(\frac{\omega_u}{x} \right)^a + \frac{\alpha}{2} \left(\frac{\omega_u}{x} \right)^a \cos \left(\frac{\delta}{2} \right) + e^{-ja\pi} \cos(\delta - j\gamma) \quad (39)$$

$$N_2 D_1 = \frac{1}{2} \left(\frac{\omega_u}{x} \right)^a + \frac{\alpha}{2} \left(\frac{\omega_u}{x} \right)^a \cos \left(\frac{\delta}{2} \right) + e^{ja\pi} \cos(\delta + j\gamma) \quad (40)$$

$$\begin{aligned} D_1 D_2 = & 1 + \alpha \left(\frac{\omega_u}{x} \right)^a [e^{-ja\pi} \cos(\delta - j\gamma) + e^{ja\pi} \cos(\delta + j\gamma)] \\ & + \alpha^2 \left(\frac{\omega_u}{x} \right)^{2a} \cos(\delta - j\gamma) \cos(\delta + j\gamma). \end{aligned} \quad (41)$$

Referring to Eqs. 36, 39, 40 and 41 result presented in Eq. 28 is proofed. $\square$

3.3 Integration upon γ_4 Curve

Integration on γ_4 part is given by Eq. 42.

$$\lim_{\varepsilon \rightarrow 0} \int_{\gamma_4} e^{ts} T(s) ds = 0. \quad (42)$$

Calculus are done following the proof below.

Proof. On γ_4 curve, operator s is defined as: $s = \varepsilon e^{j\theta}$ with $\varepsilon \rightarrow 0$; so $ds = j\varepsilon e^{j\theta} d\theta$ and it is, then, possible to express the integral upon γ_4 as follows:

$$\lim_{\varepsilon \rightarrow 0} \int_{\gamma_4} e^{ts} T(s) ds = \lim_{\varepsilon \rightarrow 0} \int_{\pi}^{-\pi} h(\varepsilon, \theta) d\theta \quad (43)$$

with:

$$h(\varepsilon, \theta) = j\varepsilon e^{t\varepsilon e^{j\theta}} \frac{\cosh\left(b \frac{\pi}{2}\right)^{\text{sgn}(b)} \left(\frac{\omega_u}{\varepsilon e^{j\theta}}\right)^a \left(\cos\left(b \ln\left(\frac{\omega_u}{\varepsilon e^{j\theta}}\right)\right)\right)^{-\text{sgn}b} e^{j\theta}}{1 + \cosh\left(b \frac{\pi}{2}\right)^{\text{sgn}(b)} \left(\frac{\omega_u}{\varepsilon e^{j\theta}}\right)^a \left(\cos\left(b \ln\left(\frac{\omega_u}{\varepsilon e^{j\theta}}\right)\right)\right)^{-\text{sgn}b}}. \quad (44)$$

In the first case, $b > 0$ and so:

$$|h(\varepsilon, \theta)| = \frac{|\varepsilon| e^{t\varepsilon \cos(\theta)}}{\left| 1 + \left(\frac{\varepsilon e^{j\theta}}{\omega_u}\right)^a \frac{1}{\cosh\left(b \frac{\pi}{2}\right)} \cos\left(b \ln\left(\frac{\omega_u}{\varepsilon e^{j\theta}}\right)\right) \right|} \quad (45)$$

Referring to the appendix, it is possible to major $|h(\varepsilon, \theta)|$ by the following equation:

$$|h(\varepsilon, \theta)| \leq |\varepsilon| e^{t\varepsilon \cos \theta} \frac{1}{\left| 1 - \left| \frac{\varepsilon}{\omega_u} \right|^a \frac{\cosh\left(b \frac{\pi}{2}\right)}{\cosh\left(b \frac{\pi}{2}\right)} \right|} \quad (46)$$

The majorant of inequation (46) tends to 0 when $\varepsilon \rightarrow 0$, so:

$$\lim_{\varepsilon \rightarrow 0} h(\varepsilon, \theta) = 0. \quad (47)$$

Consequently, if $b > 0$, replacing into Eq. 43, leads to:

$$\lim_{\varepsilon \rightarrow 0} \int_{\gamma_4} e^{ts} T(s) ds = 0 \quad (48)$$

Similarly, in the second case where $b < 0$, it results:

$$|h(\varepsilon, \theta)| = \frac{|\varepsilon| e^{t\varepsilon e^{\cos(\theta)}} \left| \frac{\omega_u}{\varepsilon} \right|^a \cos \left(b \ln \left(\frac{\omega_u}{\varepsilon e^{j\theta}} \right) \right)}{\left| \cosh \left(b \frac{\pi}{2} \right) + \left(\frac{\omega_u}{\varepsilon e^{j\theta}} \right)^a \cos \left(b \ln \left(\frac{\omega_u}{\varepsilon e^{j\theta}} \right) \right) \right|}. \quad (49)$$

By reference to the appendix, it is possible to major $|h(\varepsilon, \theta)|$ by the following inequation:

$$|h(\varepsilon, \theta)| \leq \frac{|\varepsilon| e^{t\varepsilon e^{\cos(\theta)}}}{\left| 1 - \left| \frac{\varepsilon}{\omega_u} \right|^a \frac{\cosh(b \frac{\pi}{2})}{\cosh(b\pi)} \right|}. \quad (50)$$

So, when $b < 0$, $h(\varepsilon, \theta) \rightarrow 0$ if $\varepsilon \rightarrow 0$. In consequence, it concludes: for all b non null in $\mathbb{R}$,

$$\lim_{\varepsilon \rightarrow 0} h(\varepsilon, \theta) = 0. \quad (51)$$

In conclusion, for all b non null in $\mathbb{R}$, the γ_4 integral value is given by Eq. 42. $\square$

4 Poles Characterization

According to the first paragraph, the third generation CRONE control closed loop transfer function is given by:

$$T(s) = \frac{\beta(s)}{1 + \beta(s)}$$

with

$$\beta(s) = \cosh \left(b \frac{\pi}{2} \right)^{\text{sgn}(b)} \left(\frac{\omega_u}{s} \right)^a \cos \left(b \ln \left(\frac{\omega_u}{s} \right) \right)^{-\text{sgn}(b)}.$$

Poles are defined in [3, 4], and they are expressed as follows:

$$\begin{cases} p_1 = \omega_u \rho e^{j\theta} \\ p_2 = \omega_u \rho e^{-j\theta} \end{cases} \quad (52)$$

with:

$$\theta = \pi - \arccos(\xi), \quad (53)$$

where ξ is the system damping ratio.

The module ρ is obtained through Eqs. 54 and 55 [3, 4].

$$F(a, b, \rho, \theta) = 0 \quad (54)$$

and

$$F(a, b, \rho, \theta) = \rho^{-sgn(b)a} e^{-jsgn(b)a\theta} \cosh\left(b\frac{\pi}{2}\right) + \cos(b \ln \rho + jb\theta). \quad (55)$$

Three cases are to take into consideration: $b > 0$, $b < 0$ and $b = 0$.

If $b > 0$, Eqs. 54 and 55 lead, by separating real and imaginary parts, to the following expression:

$$\begin{aligned} \rho^{-a} \cosh\left(b\frac{\pi}{2}\right) \cos(a\theta) + \cosh(b\theta) \cos(b \ln \rho) \\ - j \left[\rho^{-a} \cosh\left(b\frac{\pi}{2}\right) \sin(a\theta) - \sinh(b\theta) \sin(b \ln \rho) \right] = 0. \end{aligned} \quad (56)$$

Resolution of Eq. 56 is fulfilled by putting into zero both real and imaginary parts. That permits to express the poles module by the following expression:

$$\rho = e^{-\frac{1}{b} \arctan[\tan(a\theta) \coth(b\theta)]}. \quad (57)$$

The same resolution method is applied in the case of $b < 0$. The following equation should be solved:

$$\begin{aligned} \rho^a \cosh\left(b\frac{\pi}{2}\right) \cos(a\theta) + \cosh(b\theta) \cos(b \ln \rho) \\ + j \left[\rho^a \cosh\left(b\frac{\pi}{2}\right) \sin(a\theta) + \sinh(b\theta) \sin(b \ln \rho) \right] = 0. \end{aligned} \quad (58)$$

That leads to express the module ρ as follows:

$$\rho = e^{\frac{1}{b} \arctan[\tan(a\theta) \coth(b\theta)]}. \quad (59)$$

The third case deals with b null, in other way it consists of the real case, in which poles are yet complex conjugated and they are defined by:

$$\begin{cases} \rho = 1 \\ \theta = -\frac{\pi}{a} \end{cases}. \quad (60)$$

5 Analytical Impulse Response

By reference to Eq. 8 and to the above calculus, the third generation CRONE control impulse response is already presented in Eq. 9. As a recall, it is expressed as:

$$y(t) = \sum \text{Residue} - \frac{1}{2\pi j} \lim_{R \rightarrow +\infty} \int_{\gamma_3 + \gamma_5} e^{ts} T(s) ds. \quad (61)$$

Depending on the sign of the non null imaginary part (b) of the complex order integrator, the impulse response $y(t)$ can be analytically expressed.

In the first case, with $b > 0$, the above impulse response is obtained by replacing both relations (62) and (63) in Eq. 61.

$$\sum Residue = -\frac{p_1 e^{tp_1}}{a + b \tan\left(b \ln\left(\frac{\omega_u}{p_1}\right)\right)} - \frac{p_2 e^{tp_2}}{a + b \tan\left(b \ln\left(\frac{\omega_u}{p_2}\right)\right)}. \quad (62)$$

and

$$\begin{aligned} \lim_{R \rightarrow +\infty} \int_{\gamma_3 + \gamma_5} e^{ts} T(s) ds &= \alpha \int_0^{+\infty} e^{-tx} \left(\frac{\omega_u}{x}\right)^a [e^{-ja\pi} \cos(\delta + j\gamma) \\ &\quad - e^{ja\pi} \cos(\delta - j\gamma)] / \left[\alpha^2 \left(\frac{\omega_u}{x}\right)^{2a} + \cos(\delta - j\gamma) \cos(\delta + j\gamma) \right. \\ &\quad \left. + \alpha \left(\frac{\omega_u}{x}\right)^a (e^{ja\pi} \cos(\delta - j\gamma) + e^{-ja\pi} \cos(\delta + j\gamma)) \right] dx \end{aligned} \quad (63)$$

Poles p_1 and p_2 are defined by (52), (53) and (57), $\alpha = \cosh\left(b\frac{\pi}{2}\right)$.

In the second case, with $b < 0$, the third generation CRONE control impulse response is obtained by replacing both relations (64) and (65) in Eq. 61.

$$\sum Residue = \frac{p_1 e^{tp_1}}{-a + b \tan\left(b \ln\left(\frac{\omega_u}{p_1}\right)\right)} + \frac{p_2 e^{tp_2}}{-a + b \tan\left(b \ln\left(\frac{\omega_u}{p_2}\right)\right)} \quad (64)$$

and

$$\begin{aligned} \lim_{R \rightarrow +\infty} \int_{\gamma_3 + \gamma_5} e^{ts} T(s) ds &= \alpha \int_0^{+\infty} e^{-tx} \left(\frac{\omega_u}{x}\right)^a [e^{-ja\pi} \cos(\delta - j\gamma) \\ &\quad - e^{ja\pi} \cos(\delta + j\gamma)] / \left[1 + \alpha \left(\frac{\omega_u}{x}\right)^a [e^{-ja\pi} \cos(\delta - j\gamma) \right. \right. \\ &\quad \left. \left. + e^{ja\pi} \cos(\delta + j\gamma)] + \alpha^2 \left(\frac{\omega_u}{x}\right)^{2a} \cos(\delta - j\gamma) \cos(\delta + j\gamma) \right] dx \end{aligned} \quad (65)$$

Poles p_1 and p_2 are defined by (52), (53) and (59), $\alpha = \frac{1}{\cosh(b\frac{\pi}{2})}$.

Equation 61 puts into proof that a third generation closed loop CRONE control impulse response is decomposed into two different sub-system behaviors:

The first term imposes an oscillatory behavior that is due to both conjugated complex poles, the second term generates an aperiodic behavior caused by the calculus of the integral upon the complex plane cut off. It is composed of a continued sum of first order impulse responses.

6 Simulations

Curves in Figs. 2 and 3 present two third generation CRONE control impulse responses in closed loop structure. Simulations are achieved on two systems with parameters given below:

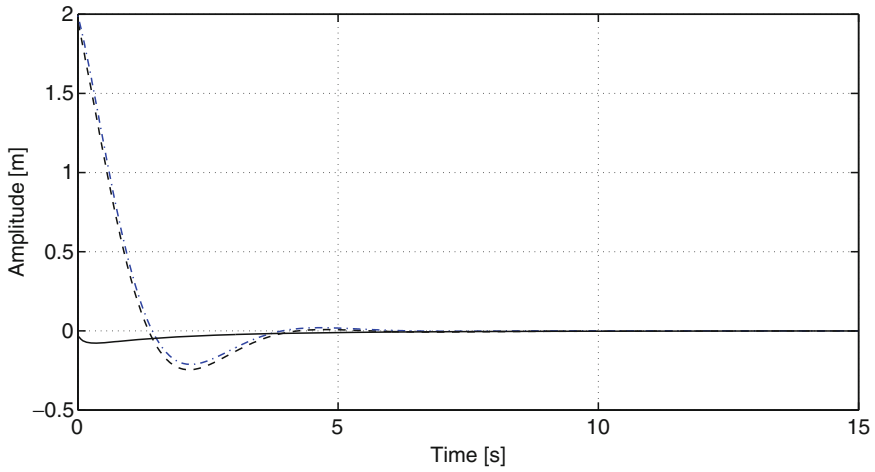


Fig. 2 Impulse response for $a = 1.2$, $b = 1$, $\omega_u = 1$ and $\xi = 0.6$. – aperiodic mode, - . oscillatory mode, - - total response

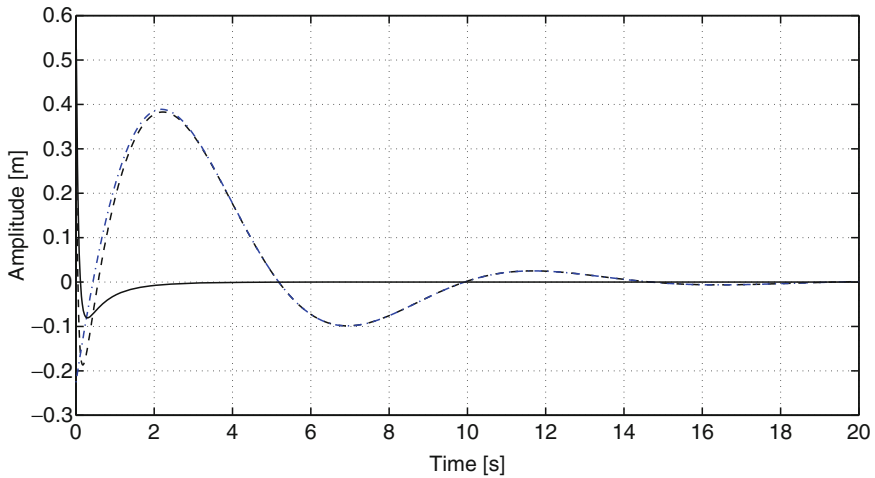


Fig. 3 Impulse response for $a = 1.4$, $b = -1.18$, $\omega_u = 1$ and $\xi = 0.4$. – aperiodic mode, - . oscillatory mode, - - total response

System 1: $a = 1.2$, $b = 1$, $\omega_u = 1$ and $\xi = 0.6$,
 System 2: $a = 1.4$, $b = -1.18$, $\omega_u = 1$ and $\xi = 0.4$.

Both figures show the decomposition of impulse responses into oscillatory and aperiodic dynamics. Referring to both Figs. 2 and 3, it is clear that the global response is closely defined by the oscillatory mode. Aperiodic dynamic rolls rapidly to zero and has an effect only for short times.

7 Conclusion

The purpose of our research is to extend the input shaping technique to the third generation CRONE control. However, no analytical response is available. In this paper, the synthesis of the impulse response of the third generation CRONE control in closed loop structure is developed, in order to permit the achievement of the work. Mathematical development is basically supported by integral residues method. Analytical result puts into proof that a third generation closed loop CRONE control impulse response is decomposed into two different sub-system behaviors: oscillatory and aperiodic ones. Some simulation results are presented.

Thus some prospects are envisaged. The immediate one is to extend basic pre-shaping on the third generation CRONE control. The second one concerns the robustness study of the method and its application on real systems. Then, it is to envisage more complex shaper including more impulses, and to synthesize global impulse response shaper.

Appendix

Considering that:

$$\begin{aligned} \cos \left(b \ln \left(\frac{\omega_u}{R e^{j\theta}} \right) \right) &= \cos \left(b \ln \left(\frac{\omega_u}{R} \right) - j b \theta \right) \\ &= \cos \left(b \ln \frac{\omega_u}{R} \right) \cosh(b\theta) + j \sin \left(b \ln \frac{\omega_u}{R} \right) \sinh(b\theta), \end{aligned} \quad (66)$$

one can write:

$$|\cos \left(b \ln \left(\frac{\omega_u}{R e^{j\theta}} \right) \right)|^2 = \cos^2 \left(b \ln \frac{\omega_u}{R} \right) \cosh^2(b\theta) + \sin^2 \left(b \ln \frac{\omega_u}{R} \right) \sinh^2(b\theta). \quad (67)$$

If $|\theta| < \pi$ then $|b\theta| < |b|\pi$, it is possible to write the following inequations:

$$\begin{aligned} \cosh(|b\theta|) &\leq \cosh(|b|\pi) \\ \sinh(|b\theta|) &\leq \sinh(|b|\pi). \end{aligned}$$

However, $\sinh^2(b\theta) \leq \cosh^2(b\theta)$, so, we can deduce that:

$$\sinh^2(b\pi) \leq |\cos \left(b \ln \left(\frac{\omega_u}{R e^{j\theta}} \right) \right)|^2 \leq \cosh^2(b\pi) \quad (68)$$

and:

$$|\sinh(b\pi)| \leq |\cos \left(b \ln \left(\frac{\omega_u}{R e^{j\theta}} \right) \right)| \leq \cosh(b\pi). \quad (69)$$

References

1. Jallouli-Khlif R, Melchior P, Levron F, Derbel N, Oustaloup A (2008) Impulse response of third generation CRONE control. 3rd IFAC Workshop on Fractional Differentiation and its Applications, Ankara, Turkey, 05–07 November, 2008
2. Melchior P, Poty A, Oustaloup A (2004) Motion control by ZV shaper synthesis extended for fractional systems and its application to CRONE control. *Int J Nonlinear Dynam Chaos Eng Syst* 38:401–416
3. Oustaloup A (1995) *La dérivation non entière: théorie, synthèse et applications*. Editions Hermès, Paris, France. ISBN 2866014561
4. Oustaloup A (1999) *La commande CRONE, du scalaire au multivariable*. Hermès, Paris
5. Pao L Y, Chang T N and Hou E (1997) Input shaper designs for minimizing the expected level of residual vibration in flexible structures. American Control Conference, June 4–6, 1997, Albuquerque Convention Center, The Albuquerque Hyatt Regency and Doubletree Hotels New Mexico, USA
6. Poty A, Melchior P, Levron F, Orsoni B, Oustaloup A (2003) Motion control by preshaping: extension for explicit generalized derivative systems. *Signals, Systems, Decision & Information Technology (SSD '03)*, March 26–28, 2003, Sousse, Tunisia
7. Poty A, Melchior P, Levron F, Orsoni B, Oustaloup A (2003) ZV and ZVD shapers for explicit fractional derivative systems. *The 11th International Conference on Advanced Robotics (ICAR 2003)*, June 30–July 3, 2003, Coimbra, Portugal
8. Singer NC, Seering WP (1990) Preshaping command inputs to reduce system vibration. *J Dyn Syst Meas Control* 112:76–82
9. Singhose W, Singer N, Seering W (1995) Comparison of command shaping methods for reducing residual vibration. 3rd European Control Conference, (ECC 95), Rome, Italy, 5–8 September, 1995

Stability Analysis of Fractional Order Universal Adaptive Stabilization

Yan Li and YangQuan Chen

Abstract In this paper, we study the asymptotic stability of three fractional systems by the method of universal adaptive stabilization. Moreover, when $\alpha \in (2, 3]$ and $\lambda > 0$, Mittag-Leffler function $E_\alpha(-\lambda k^\alpha)$ is shown to be Nussbaum function. Finally, two simulation results are provided to illustrate the concepts.

1 Introduction

Adaptive control has received a lot of attention due to its potential application in systems with high complexity and uncertainty. The primary objective in adaptive control is to design a controller which can achieve the prespecified control objectives for a given class of systems. As a special case of adaptive control, universal adaptive stabilization (UAS) is proposed and studied for systems with state-space representations. By using UAS, it can be guaranteed that the systems are asymptotically stable [1, 2].

Fractional calculus plays an important role in control theory. For example, in [3], the authors study the fractional calculus applications in control systems. In [4], the author introduces the concept of fractional PID controller. In [5], the author proposes a fundamental idea of the fractional control strategy. Motivated by the application of fractional calculus in control community, we propose fractional UAS, which enriches the knowledge of both control theory and fractional calculus. Lastly, the early version of this manuscript is [6].

Y. Li (✉)

Institute of Applied Math, School of Mathematics and System Sciences,
Shandong University, Jinan 250100, P. R. China
e-mail: liyan.sdu@gmail.com

Y. Chen

Center for Self-Organizing & Intelligent Systems (CSOIS), Electrical &
Computer Engineering Dept., Utah State Univ., Logan, UT 84322, USA
e-mail: yqchen@ieee.org.

2 Preliminaries

2.1 Fractional Calculus and Mittag–Leffler Function

Fractional calculus plays an important role in modern science [7]. In this contribution, we choose Riemann–Liouville and Caputo fractional operators as our main mathematical tools. The uniform formula of fractional integral with $\alpha \in (0, 1)$ is defined as

$${}_a\mathcal{D}_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad (1)$$

where $f(t)$ is an arbitrary integrable function, ${}_a\mathcal{D}_t^{-\alpha}$ is the α th fractional integrator on $[a, t]$, and $\Gamma(\cdot)$ denotes the Gamma function. For an arbitrary real number p , the Riemann–Liouville and Caputo fractional operators are defined respectively as

$${}_aD_t^p f(t) = \frac{d^{[p]+1}}{dt^{[p]+1}} \left[{}_aD_t^{-([p]-p+1)} f(t) \right] \quad (2)$$

and

$${}_a^C D_t^p f(t) = {}_aD_t^{-([p]-p+1)} \left[\frac{d^{[p]+1}}{dt^{[p]+1}} f(t) \right], \quad (3)$$

where $[p]$ stands for the integer part of p , D and ${}^C D$ are Riemann–Liouville and Caputo fractional operators, respectively.

Similar to the exponential function frequently used in the solutions of integer-order systems, a function frequently used in the solutions of fractional order systems is the Mittag–Leffler function defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)},$$

where $\alpha > 0$. The Mittag–Leffler function in two parameters has the following form

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)},$$

where $\alpha, \beta > 0$. When $\alpha > 0$ and $\beta = 1$, $E_\alpha(z) = E_{\alpha,1}(z)$.

Moreover, the Laplace transform of Mittag–Leffler function in two parameters is

$$\mathcal{L}\{t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}, \quad (\Re(s) > |\lambda|^{\frac{1}{\alpha}}), \quad (4)$$

where s is the variable in Laplace domain, $\Re(s)$ denotes the real part of s , $\lambda \in \mathbb{R}$ and $\mathcal{L}\{\cdot\}$ stands for the Laplace transform.

2.2 Nussbaum Function

Definition 1. A piecewise right continuous and locally Lipschitz function $N(\cdot)$:

$[k', \infty) \rightarrow \mathbb{R}$ is called a Nussbaum function if $\sup_{k > k_0} \frac{1}{k - k_0} \int_{k_0}^k N(\tau) d\tau = +\infty$ and

$\inf_{k > k_0} \frac{1}{k - k_0} \int_{k_0}^k N(\tau) d\tau = -\infty$, for some $k_0 \in (k', \infty)$, where $k' \in \mathbb{R}$ [1].

3 Stability Analysis of Fractional UAS

In this section, we discuss the asymptotic stability of three fractional scalar systems by using the method of fractional UAS.

3.1 Fractional Dynamics with Integer-Order Control Strategy

Consider the fractional scalar system described by

$$\begin{cases} \dot{x}(t) = ax(t) + bu(t) \\ y(t) = c {}_0D_t^\alpha x(t), \end{cases} \quad x(0) = x_0 \quad (5)$$

and use the control strategy as

$$\begin{cases} u(t) = -N(k(t))x(t) \\ \dot{k}(t) = x^2(t), \end{cases} \quad k(0) \in \mathbb{R}, \quad (6)$$

where $a, b, c, x_0 \in \mathbb{R}$ are unknown, $\alpha \in (0, 1)$ and $N(\cdot)$ is an arbitrary Nussbaum function. Applying (6) to (5) yields the following closed-loop system

$$\begin{cases} \dot{x}(t) = [a - bN(k(t))]x(t), & x(0) = x_0, \\ \dot{k}(t) = x^2(t), & k(0) \in \mathbb{R}. \end{cases} \quad (7)$$

Lemma 1. In (7), $k(t)$ is bounded and $x(t), \dot{x}(t) \in L_2(0, \infty)$.

Proof. It is straightforward to have $\frac{d}{dt} \left[\frac{1}{2} x^2(t) \right] = x(t) \dot{x}(t) = [a - bN(k)]x^2(t) = [a - bN(k)]\dot{k}$, integration over $[0, t]$ and substituting $\tau := k(t)$ yields

$$\frac{1}{2} [x^2(t) - x_0^2] = \int_{k(0)}^{k(t)} [a - bN(\tau)] d\tau \leq [k(t) - k(0)] \left[a - \frac{b}{k(t) - k(0)} \int_{k(0)}^{k(t)} N(\tau) d\tau \right], \quad (8)$$

provided $k(t) > k(0)$. It can be shown that $\frac{b}{k(t)-k(0)} \int_{k(0)}^{k(t)} N(\tau) d\tau$ takes arbitrary large negative and positive values if $k(t) \rightarrow +\infty$. Therefore, $k(t)$ is bounded, since otherwise the right hand side of (8) takes negative values, contradicting the boundedness from below of the left hand side. Thus $x(t) \in L_2(0, \infty)$ and it follows from the first equation of (7) that $\dot{x}(t) \in L_2(0, \infty)$.

Corollary 1. For system (7), $x(0)x(t) \geq 0$.

Proof. Obviously, the first equation of (7) is an integer-order nonautonomous system with $x = 0$ as the equilibrium point. Therefore, if $x(t_0) = 0$, then $x(t) = 0$ for any $t \geq t_0$, i.e. $x(0)x(t) \geq 0$.

Corollary 2. There exist $t_1 \in [0, t]$ and $M_{\max} = \max\{[a - bN(k(t))]|t \geq t_1\} < 0$ such that $0 \leq x(0)x(t) \leq x^2(0)e^{M_{\max}t}$.

Proof. Since $x(t) \in L_2(0, \infty)$, it follows from the second equation of (7) that there exists a constant $k_{\max}$ satisfying that the increasing function $k(t) \leq k_{\max}$. Therefore, there exists a constant t_1 such that for any $t \geq t_1$, $[a - bN(k(t))] < 0$. Let $M_{\max} = \max\{[a - bN(k(t))]|t \geq t_1\}$. Use the first equation of (7), it follows from Corollary 1 that $x(0)\dot{x}(t) = [a - bN(k(t))]x(0)x(t) \leq M_{\max}x(0)x(t)$. There must exist a nonnegative function $m(t)$ satisfying

$$x(0)\dot{x}(t) + m(t) = M_{\max}x(0)x(t). \quad (9)$$

Applying the Laplace transform to (9) gives

$$x(0)x(s) - x^2(0) + m(s) = M_{\max}x(0)x(s), \quad (10)$$

where $x(s)$ and $m(s)$ are the Laplace transform of $x(t)$ and $m(t)$, respectively. Applying the inverse Laplace transform to (10) yields $x(0)x(t) + m(t) * e^{M_{\max}t} = x^2(0)e^{M_{\max}t}$, where $*$ denotes the convolution operation on $[0, t]$. Because both $m(t)$ and $e^{M_{\max}t}$ are nonnegative functions, it follows from Corollary 1 that $0 \leq x(0)x(t) \leq x^2(0)e^{M_{\max}t}$. In addition, $M_{\max} = \max\{[a - bN(k(t))]|t \geq t_1\} < 0$, which implies that $x(0)x(t)$ is exponentially decreasing to zero as $t_1 \leq t \rightarrow \infty$.

Corollary 3. In (6), $\lim_{t \rightarrow +\infty} u(t) = 0$ [6].

In the previous part of this subsection, we have already proved that $k(t)$ is bounded, $x(t), \dot{x}(t) \in L_2(0, \infty)$ and $\lim_{t \rightarrow +\infty} u(t) = 0$. In the following of this subsection, we study the asymptotic stability of system (5) by using the control strategy (6).

Lemma 2. For a continuous function $x(t)$, $\lim_{t \rightarrow +\infty} {}_0D_t^\alpha x(t) = \lim_{t \rightarrow +\infty} {}_0^C D_t^\alpha x(t)$, where $\alpha \in (0, 1)$.

Proof. By using the composition law of fractional operators, it follows that

$$\lim_{t \rightarrow +\infty} {}_0 D_t^\alpha x(t) = \lim_{t \rightarrow +\infty} \left\{ {}_0 D_t^{\alpha-1} \frac{d}{dt} x(t) + \sum_{j=1}^2 \left[{}_0 D_t^{1-j} x(t) \right]_{t=0} \frac{t^{1-\alpha-j}}{\Gamma(2-\alpha-j)} \right\}.$$

Obviously, $\lim_{t \rightarrow +\infty} \sum_{j=1}^2 \left[{}_0 D_t^{1-j} x(t) \right]_{t=0} \frac{t^{1-\alpha-j}}{\Gamma(2-\alpha-j)} = 0$. It then follows that

$$\lim_{t \rightarrow +\infty} {}_0 D_t^\alpha x(t) = \lim_{t \rightarrow +\infty} {}_0 D_t^{\alpha-1} \frac{d}{dt} x(t) = \lim_{t \rightarrow +\infty} {}_0^C D_t^\alpha x(t).$$

Lemma 3. If $\lim_{t \rightarrow +\infty} x_1(t) = 0$ and $x_2(t)$ is bounded on $t \in [0, \infty)$, then

$$\lim_{t \rightarrow \infty} \int_0^{t_1} x_1(t-\tau) x_2(\tau) d\tau = 0 \text{ for an arbitrary constant } t_1 \in \mathbb{R} \text{ [6].}$$

Lemma 4. For system (5) with control strategy (6), $\lim_{t \rightarrow \infty} {}_0^C D_t^\alpha x(t) = \lim_{t \rightarrow \infty} {}_{t_1}^C D_t^\alpha x(t)$, where $\alpha \in (0, 1)$ and $t_1 \in \mathbb{R}$ is an arbitrary constant.

Proof. Using (1) and (3), it follows that ${}_0^C D_t^\alpha x(t) = {}_0 \mathcal{D}_t^{\alpha-1} \dot{x}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha} d\tau$. Because $\dot{x}(t) \in L_2(0, \infty)$, it follows from Lemma 3 that $\lim_{t \rightarrow \infty} \int_0^{t_1} \frac{\dot{x}(\tau)}{(t-\tau)^\alpha} d\tau = 0$, which implies $\lim_{t \rightarrow \infty} {}_0^C D_t^\alpha x(t) = \lim_{t \rightarrow \infty} {}_0 \mathcal{D}_t^{\alpha-1} \dot{x}(t) = \lim_{t \rightarrow \infty} {}_{t_1} \mathcal{D}_t^{\alpha-1} \dot{x}(t) = \lim_{t \rightarrow \infty} {}_{t_1}^C D_t^\alpha x(t)$.

Lemma 5. For any two integrable functions $f(t)$ and $g(t)$, $\int_{t_1}^t f(\tau-t_1)g(t-\tau)d\tau = \int_0^{t-t_1} f(\tau)g(t-t_1-\tau)d\tau$ [6].

Lemma 6. In the closed-loop system (7), $\lim_{t \rightarrow \infty} {}_{t_1} \mathcal{D}_t^{\alpha-1} \dot{x}(t) = \lim_{t \rightarrow \infty} {}_{t_1}^C D_t^\alpha x(t) = 0$, where t_1 has the same definition in Corollary 2.

Proof. From Corollary 1 and (7), there exist two negative constants, $M_{\max} = \max\{[a - bN(k(t))]|t \geq t_1\}$ and $M_{\min} = \min\{[a - bN(k(t))]|t \geq t_1\}$ satisfying

$$M_{\min} x(0)x(t) \leq x(0)\dot{x}(t) = [a - bN(k(t))]x(0)x(t) \leq M_{\max} x(0)x(t) \leq 0. \quad (11)$$

By applying ${}_{t_1} \mathcal{D}_t^{\alpha-1}$ to (11), it follows from Corollary 2 that

$$M_{\min} x^2(0) {}_{t_1} \mathcal{D}_t^{\alpha-1} e^{M_{\max} t} \leq {}_{t_1} \mathcal{D}_t^{\alpha-1} x(0)\dot{x}(t) \leq M_{\max} {}_{t_1} \mathcal{D}_t^{\alpha-1} x(0)x(t) \leq 0. \quad (12)$$

Applying Lemma 5 to (12) yields

$$\begin{aligned} {}_{t_1}\mathcal{D}_t^{\alpha-1}e^{M_{max}t} &= \frac{e^{M_{max}t_1}}{\Gamma(1-\alpha)} {}_{t_1}\mathcal{D}_t^{\alpha-1}e^{M_{max}(t-t_1)} = \frac{e^{M_{max}t_1}}{\Gamma(1-\alpha)} {}_0\mathcal{D}_{t-t_1}^{\alpha-1}e^{M_{max}(t-t_1)} \\ &= \frac{e^{M_{max}t_1}}{\Gamma(1-\alpha)} (t-t_1)^{1-\alpha} E_{1,2-\alpha}[M_{max}(t-t_1)]. \end{aligned}$$

Because $\mathcal{L}\{t^{1-\alpha}E_{1,2-\alpha}(M_{max}t)\} = \frac{s^{\alpha-1}}{s-M_{max}}$ and the pole of $\frac{1}{s-M_{max}}$ is in the left half plane (LHP), by using final value theorem of Laplace transform, it follows that $\lim_{t \rightarrow +\infty} t^{1-\alpha}E_{1,2-\alpha}(M_{max}t) = \lim_{s \rightarrow 0} s \frac{s^{\alpha-1}}{s-M_{max}} = 0$, Which implies

$$\lim_{t \rightarrow \infty} {}_{t_1}\mathcal{D}_t^{\alpha-1}e^{M_{max}t} = 0. \quad (13)$$

Applying (13) to (12) yields $0 \leq \lim_{t \rightarrow \infty} {}_{t_1}\mathcal{D}_t^{\alpha-1}x(0)\dot{x}(t) \leq 0$, which implies $\lim_{t \rightarrow \infty} {}_{t_1}\mathcal{D}_t^{\alpha-1}\dot{x}(t) = \lim_{t \rightarrow \infty} {}^C D_t^\alpha x(t) = 0$.

Theorem 1. *The fractional system (5) is asymptotically stable when using the control strategy (6) [6].*

3.2 Fractional Dynamics with Fractional Control

Consider the fractional dynamics

$$\begin{cases} \dot{x}(t) = a_1 x(t) + a_2 {}^C D_t^\alpha x(t) + bu(t) \\ y(t) = cx(t), \end{cases} \quad x(0) = x_0 \quad (14)$$

with the fractional control strategy

$$\begin{cases} u(t) = -N(k(t))y(t) - (cb)^{-1}a_2 {}^C D_t^\alpha y(t) \\ \dot{k}(t) = y^2(t), \end{cases} \quad k(0) \in \mathbb{R} \quad (15)$$

where $a_1, a_2, b, c, x_0 \in \mathbb{R}$ are unknown, $N(k)$ is an arbitrary Nussbaum function with respect to k , and $\alpha \in (0, 1)$. Applying (15) to (14) yields the closed-loop system

$$\begin{cases} \dot{y}(t) = [a_1 - cbN(k(t))]y(t) \\ \dot{k}(t) = y^2(t), \end{cases} \quad k(0) \in \mathbb{R}. \quad (16)$$

By using Lemma 1, it is obvious that $k(t)$ is bounded and $y(t)$ converges to zero as $t \rightarrow \infty$.

Corollary 4. *For system (16), there exists a t_1 satisfying*

$$0 \leq y(0)y(t) \leq y^2(0)e^{\tilde{M}_{max}t}.$$

Proof. From Corollary 2, there exist t_1 and $\tilde{M}_{max} = \max\{\text{eig}[a - cbN(k(t))]|t \geq t_1\} < 0$ such that $x(t)$ in (16) satisfies $0 \leq y(0)y(t) \leq y^2(0)e^{\tilde{M}_{max}t}$, for all $t \geq t_1$.

Corollary 5. In (15), $u(t) = -N(k(t))y(t) - (cb)^{-1}a_2 {}^C D_t^\alpha y(t)$ tends to zero as $t \rightarrow \infty$.

Proof. From Theorem 1, $x(t)$ and ${}_0 D_t^\alpha x(t)$ in (14) tend to zero as $t \rightarrow \infty$. Therefore, $u(t)$ also tends to zero as $t \rightarrow \infty$.

3.3 Application of Viscoelastic and Electromagnetic Theories in UAS

First, let us introduce the basis of relaxation modulus in viscoelastic theory. The relaxation modulus $K(t)$ has the following property.

Property 1. For $t \geq 0$, $K(t) > 0$, $\dot{K}(t) \leq 0$, $\lim_{t \rightarrow +\infty} K(t) = K_\infty \geq 0$, all the poles of $\mathcal{L}\{K(t) - K_\infty\}$ are in the LHP and $\lim_{s \rightarrow 0} sK(s) = K_\infty$, where $K(s) = \mathcal{L}\{K(t)\}$.

For example [8], the fractional relaxation modulus of viscoelastic material has the following form $K(t) = K_\infty + \rho E_\beta(-\eta t^\beta)$, where $E(\cdot)$ is Mittag-Leffler function, $K_\infty \geq 0$, $\rho > 0$, $\eta > 0$, and $\beta \in (0, 1]$. Therefore, for $t \geq 0$, $K(t) > 0$, $\frac{dK(t)}{dt} = \rho t^{-1} E_{\beta,0}(-\eta t^\beta) \leq 0$, $\lim_{t \rightarrow \infty} K(t) = K_\infty$, all the poles of $\frac{s^{\beta-1}}{s^\beta + \eta}$ are in the LHP [9] and $\lim_{s \rightarrow 0} s\mathcal{L}\{K(t)\} = K_\infty$.

Based on the properties of relaxation modulus, we consider the following dynamics

$$\begin{cases} \dot{x}(t) = ax(t) + bu(t) \\ y(t) = cK(t) * \dot{x}(t), \quad x(0) = x_0 \end{cases} \quad (17)$$

with the following control strategy

$$\begin{cases} u(t) = -N(k(t))x(t) \\ \dot{k}(t) = x^2(t), \quad k(0) \in \mathbb{R} \end{cases} \quad (18)$$

where $a, b, c, x_0 \in \mathbb{R}$ are unknown, $N(k)$ is an arbitrary Nussbaum function, $\alpha \in (0, 1)$, $*$ denotes the usual convolution operation on $t \in [0, t]$, and $K(t)$ is an arbitrary relaxation function satisfying Property 1. Substituting (18) to (17) gives the closed-loop system

$$\begin{cases} \dot{x}(t) = [a - bN(k(t))]x(t), \quad x(0) = x_0, \\ \dot{k}(t) = x^2(t), \quad k(0) \in \mathbb{R}. \end{cases} \quad (19)$$

It then follows from Lemma 1, Corollaries 2 and 3 that $k(t)$ is bounded, $x(t), \dot{x}(t) \in L_2(0, \infty)$, $\lim_{t \rightarrow \infty} u(t) = 0$ and $0 \leq x(0)x(t) \leq x^2(0)e^{\tilde{M}_{max}t}$ for $t \geq t_1$, where

$t_1 \geq 0$ has the same definition in Corollary 2 and $M_{max} = \max\{[a - bN(k(t))]|t \geq t_1\} < 0$.

Theorem 2. For system (17) with control strategy (18), $\lim_{t \rightarrow \infty} y(t) = -cK_{\infty}x(0)$.

Proof. The second equation of (17) can be written as $y(t) = c[K(0)x(t) - K(t)x(0) + \dot{K}(t) * x(t)]$. From Corollary 2, there exists $t_1 \geq 0$ satisfying $0 \leq x(0)x(t) \leq x^2(0)e^{M_{max}t}$, $t \geq t_1$. It follows from Lemma 3 that $\lim_{t \rightarrow \infty} \int_0^{t_1} x(t - \tau) \dot{K}(\tau) d\tau = 0$. Because $\dot{K}(t) \leq 0$, it follows from Corollary 1 that

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_{t_1}^t x^2(0)e^{M_{max}t} \dot{K}(\tau) d\tau &\leq \lim_{t \rightarrow \infty} \dot{K}(t)x(0) * x(t) \\ &= \lim_{t \rightarrow \infty} \int_{t_1}^t x(0)x(t - \tau) \dot{K}(\tau) d\tau \leq 0, \end{aligned}$$

where $M_{max} = \max\{[a - bN(k(t))]|t \geq t_1\} < 0$. Use the final value theorem of the Laplace transform, it follows from Property 1 that

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_{t_1}^t x^2(0)e^{M_{max}t} \dot{K}(\tau) d\tau &= x^2(0) \lim_{s \rightarrow 0} \frac{s[sK(s) - K(0)]}{s - M_{max}} \\ &= x^2(0) \lim_{s \rightarrow 0} \frac{s[K_{\infty} - K(0)]}{s - M_{max}} = 0, \end{aligned}$$

which implies $\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} c[K(0)x(t) - K(t)x(0) + \dot{K}(t) * x(t)] = -cK_{\infty}x(0)$.

Remark 1. In Theorem 2, $x(t)$ and $y(t)$ can be treated by strain and stress in viscoelastic theory. There exists the similar definition of relaxation modulus in electromagnetics. Theorem 2 also holds by treating $x(t)$ and $y(t)$ as current and voltage [10].

4 Nussbaum Function with Mittag–Leffler Form

Nussbaum function plays an crucial role in UAS [1, 11–15]. In this section, we show that the Mittag–Leffler function $E_{\alpha}(-\lambda k^{\alpha})$ for $\lambda > 0$ and $\alpha \in (2, 3]$ is a Nussbaum function which enriches the current results.

Lemma 7. Suppose that $f(k)$ is bounded for all k and $N(k)$ is a Nussbaum function with respect to k . Then $f(k) + N(k)$ is also a Nussbaum function with respect to k [6].

Lemma 8. $e^k \cos(k)$ is a Nussbaum function with respect to k [6].

Theorem 3. $E_\alpha(-\lambda k^\alpha)$ is a Nussbaum function with respect to k , where $\alpha \in (2, 3]$ and $\lambda > 0$.

Proof. From [9, 16], $E_\alpha(-\lambda k^\alpha)$ can be decomposed as $E_\alpha(-\lambda k^\alpha) = f_\alpha(k) + g_\alpha(k)$, where $f_\alpha(k)$ is a completely monotonic function [17] and $g_\alpha(k) = \frac{2}{\alpha} e^{\lambda^{\frac{1}{\alpha}} k \cos(\frac{\pi}{\alpha})} \cdot \cos[\lambda^{\frac{1}{\alpha}} k \sin(\frac{\pi}{\alpha})]$. It follows from $f_\alpha(k)$ is bounded on $k \geq k_0 > 0$ and $g_\alpha(k)$ is a Nussbaum function that $E_\alpha(-\lambda k^\alpha)$ is a Nussbaum function with respect to k , where $\alpha \in (2, 3]$ and $\lambda > 0$.

5 Simulations

Two illustrated simulation results, Figs. 1 and 2, are provided as a proof of concept, where the initial values are defined in Caputo form. This can be realized by

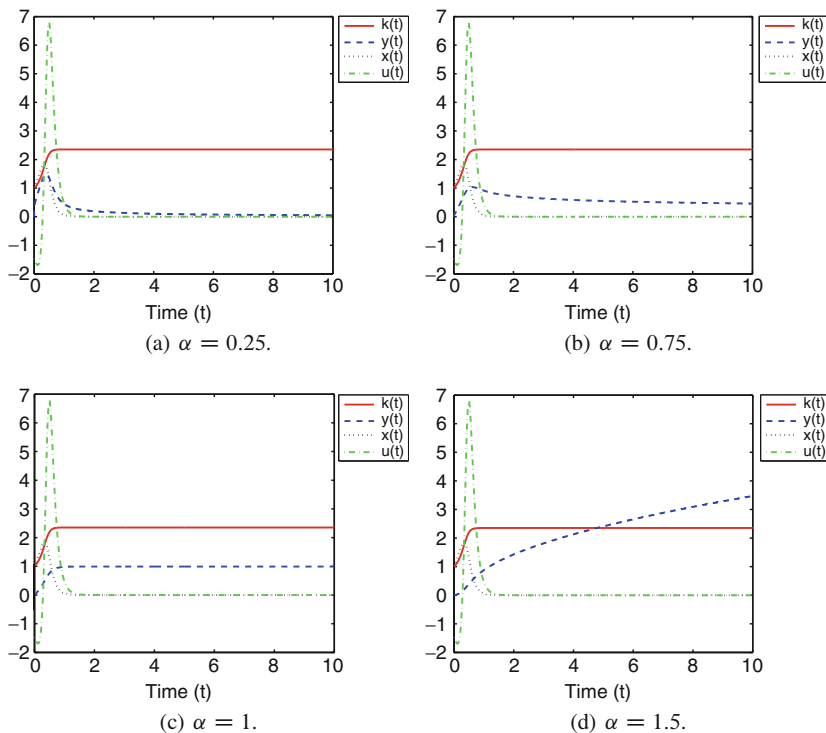


Fig. 1 In (5) and (6), let $N(k) = e^k \cos k$, $a = c = 1$, $b = -1$ and $x(0) = 1$. This figure shows the influence of α to $y(t)$ which reflects the stabilization of system (5) by using (6). It can be seen that $k(t)$ is bounded and $x(t)$ and $u(t)$ tend to zero as $t \rightarrow \infty$

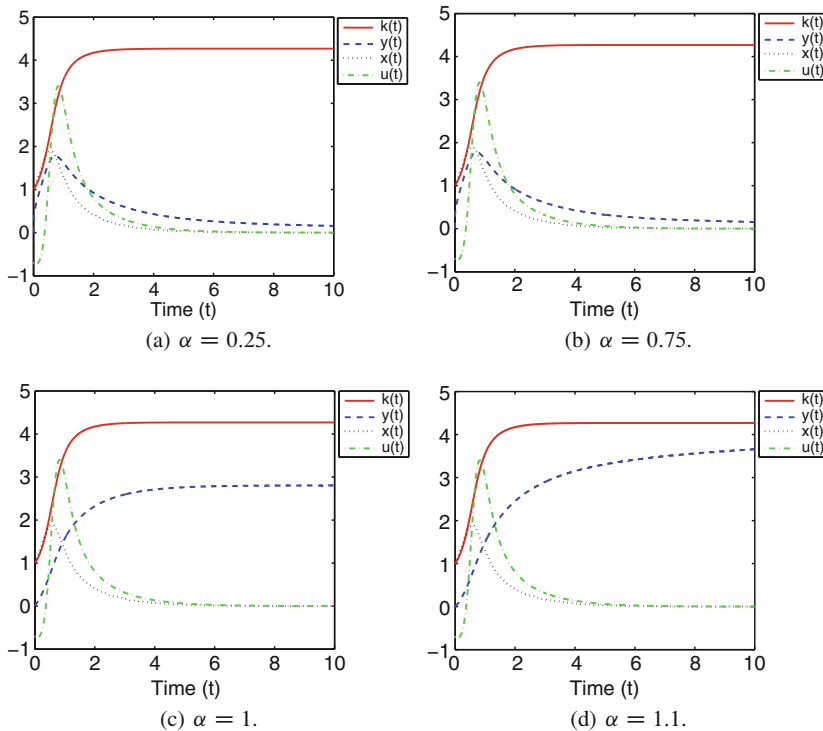


Fig. 2 In (5) and (6), let $N(k) = E_{2.5}(-k^{2.5})$, $a = c = 1$, $b = -1$ and $x(0) = 1$. This figure shows the influence of α to $y(t)$ which reflects the stabilization of system (5) by using (6). It can be seen that $k(t)$ is bounded and $x(t)$ and $u(t)$ tend to zero as $t \rightarrow \infty$

using the following relationship between Caputo and Riemann–Liouville fractional operators:

$${}_0^C D_t^\alpha x(t) = {}_0^R D_t^\alpha x(t) - \frac{t^{-\alpha} x(0)}{\Gamma(1-\alpha)}, \quad (\alpha \in (0, 1)).$$

The fractional block is compiled by the S-Function of Matlab/Simulink.

For system (5) with control strategy (6), let $N(k) = e^k \cos k$, $a = c = 1$, $b = -1$ and $x(0) = 1$. Figure 1 shows the influence of α to $y(t)$ which reflects the stabilization of system (5) by using (6). It can be seen that $k(t)$ is bounded and $x(t)$ and $u(t)$ tend to zero as $t \rightarrow \infty$.

For system (5) with control strategy (6), let $N(k) = E_{2.5}(-k^{2.5})$, $a = c = 1$, $b = -1$ and $x(0) = 1$. Figure 2 shows the influence of α to $y(t)$ which reflects the stabilization of system (5) by using (6). It can be seen that $k(t)$ is bounded and $x(t)$ and $u(t)$ tend to zero as $t \rightarrow \infty$.

6 Conclusion

In this manuscript, we proved the asymptotic stability of three fractional scalar systems by using the method of universal adaptive stabilization. We also proved that the control inputs for these three cases are bounded. We presented that Mittag-Leffler function $E_\alpha(-\lambda k^\alpha)$ is a Nussbaum function for $\alpha \in (2, 3]$ and $\lambda > 0$. Two simulation results are illustrated to support our discussions. Finally, experimental validation of the proposed FO-UAS is presented in [18].

Acknowledgment Yan Li would like to thank Y. Cao for polishing an early version of [6].

References

1. Ilchmann A (1993) Non-identifier-based high-gain adaptive control. Springer-Verlag, New York
2. Shankar S, Bodson M (1989) Adaptive control-stability convergence and robustness. Prentice Hall, Englewood Cliffs, New Jersey
3. Axtell M, Bise ME (1990) Fractional calculus applications in control systems. In: Proceedings of the IEEE 1990 national aerospace electronics conference, New York, pp 563–566
4. Podlubny I (1994) Fractional-order systems and fractional-order controllers. In Proceedings of the Conference Internationale Francophone d'Automatique. UEF-03-94, Inst Exp Phys, Slovak Acad Sci Kosice
5. Oustaloup A (1988) From fractality to non integer derivation through recursivity, a property common to these two concepts: a fundamental idea for a new process control strategy. In: Proceedings of the 12th IMACS world conference, Paris, France 3:203–208, July 1988
6. Yan Li, YangQuan Chen, Yongcan Cao (2008) Fractional order universal adaptive stabilization. In: Proceedings of the 3rd IFAC workshop on fractional differentiation and its applications, Ankara, Turkey, November 2008
7. Igor Podlubny (1999) Fractional differential equations. Academic Press, San Diego
8. Bagley RL, Torvik PJ (1986) On the fractional calculus model of viscoelastic behavior. J Rheol 30:133–155, February 1986
9. Gorenflo R, Mainardi R (1996) Fractional oscillations and Mittag-Leffler functions. [Online Available:] <http://citeseer.ist.psu.edu/gorenflo96fractional.html>
10. Wagner DA, Morman Jr KN (1994) A viscoelastic analogy for solving 2-D electromagnetic problems. Finite Elem Anal Des 15:February 1994
11. Townley S (1992) Generic properties of universal adaptive stabilization schemes. In: Proceedings of the 31st IEEE conference on decision and control 4:3630–3631, September 1992
12. Jianlong Zhang, Petros A Ioannou (2006) Non-identifier based adaptive control scheme with guaranteed stability. In: Proceedings of the 2006 American control conference, pp 5456–5461, June 2006
13. Schuster H, Westermaier C, Schröder D (2006) Non-identifier-based adaptive tracking control for a two-mass system. In: Proceedings of the 2006 American control conference, pp 190–195
14. Hans Schuster, Christian Westermaier, Dierk Schröder (2006) Non-identifier-based adaptive control for a mechatronic system achieving stability and steady state accuracy. In: Proceedings of the 2006 IEEE, pp 1819–1824, October 2006
15. Hao Lei, Wei Lin, Bo Yang (2007) Adaptive robust stabilization of a family of uncertain non-linear systems by output feedback: the non-polynomial case. In: Proceedings of the 2007 American control conference, July 2007

16. Hilfer R and Seybold HJ (2006) Computation of the generalized Mittag-Leffler function and its inverse in the complex plane. *Integr Transf Spec Funct* 17(9):637–652, September 2006
17. Kenneth S Miller, Stefan G Samko (2001) Completely monotonic functions. *Integr Transf Spec Funct* 12(4):389–402
18. Shayok Mukhopadhyay, Yan Li, YangQuan Chen (2008) Experimental studies of a fractional order universal adaptive stabilizer. In: *Proceedings of the 2008 IEEE/ASME international conference on mechatronic and embedded systems and applications*, Beijing, China, October 2008

Position and Velocity Control of a Servo by Using GPC of Arbitrary Real Order

Miguel Romero Hortelano, Inés Tejado Balsera, Blas Manuel Vinagre Jara, and Ángel Pérez de Madrid y Pablo

Abstract This chapter presents the use of conventional and Fractional-Order Generalized Predictive controllers to control a practical system: a servomotor plant. In order to illustrate the procedure for the controllers design and application, several simulations and practical tests on the velocity and position control of the servo are given. From the results, it can be obtained that the use of the fractional predictive controller lets the system get a good performance and offers a set of controlled systems with different dynamics. Therefore, it improves the controlled system performance and gives more flexibility to the control strategy by adding the possibility of tuning the controller fractional orders α and β .

1 Introduction

Model-Based Predictive Control (MPC) has been in use in the process industries during the last 30 years, where it has become an industry standard due to its intrinsic ability to handle input and state constraints for large scale multivariable plants [1, 6, 15].

On the other hand, fractional order integral and differential operators [9, 11] have been introduced in some feedback control methodologies in order to give more flexibility to the control strategy and to improve the controlled system performances [7, 10, 12, 16, 17].

In this work, we shall use a servomotor plant in order to control it using a fractional control strategy, specifically Fractional-Order Generalized Predictive Control (FGPC) [13].

M.R. Hortelano (✉) and A.P. de Madrid y Pablo
Escuela Técnica Superior de Ingeniería Informática, UNED, Spain
e-mail: mromero@scc.uned.es; angel@scc.uned.es

I.T. Balsera and B.M.V. Jara
Industrial Engineering School, University of Extremadura, Spain
e-mail: itejbal@unex.es; bvinagre@unex.es

This chapter is organized as follows. In Sect. 2, GPC and FGPC are introduced. Section 3 describes the experimental platform used here to validate the proposed controllers. In Sect. 4, we shall focus on the controllers design for the servomotor system for the velocity and position control and include the simulation and experimental results. The final section draws the main conclusions.

2 Generalized Predictive Control

2.1 Introduction

GPC stands for Generalized Predictive Control [3, 4], one of the most representative predictive controllers due to its success in industrial and academic applications [2]. All predictive controllers share a common methodology: at each “present” instant t , future process outputs $y(t + k|t)$ are predicted for a certain time window, $k = 1, 2, \dots, N$, using the process model. The optimal control law is obtained by minimizing a given cost function (1) subject to a set of constraints:

$$\begin{aligned}
 & J(\Delta u, t) \\
 &= E \left\{ \sum_{j=N_1}^{N_2} \gamma(j) [r(t + j|t) - y(t + j|t)]^2 + \sum_{j=1}^{N_u} \lambda(j) [\Delta u(t + j - 1|t)]^2 \right\} \\
 & \text{Subject : } H\Delta u \leq h
 \end{aligned} \tag{1}$$

where $E\{\cdot\}$ is the expectation operator, Δ is the increment operator, N_1 and N_2 are the minimum and maximum costing horizons, respectively, N_u represents the control horizon, γ is a future errors weighting sequence, λ is a control weighting sequence, and H and h are a matrix and a vector, respectively. In the minimization process, it is usually assumed that the control signal $u(t)$ remains constant from time instant $t + N_u$ [1, 6, 15].

GPC uses CARIMA (Controlled Auto-Regressive Integrated Moving-Average) models to describe the system dynamics:

$$A(z^{-1})y(t) = B(z^{-1})u(t) + \frac{T(z^{-1})}{\Delta}\xi(t) \tag{2}$$

where $B(z^{-1})$ and $A(z^{-1})$ are the numerator and denominator of the transfer function, respectively, and $\xi(t)$ represents uncorrelated zero-mean white noise. In practice, $T(z^{-1})$ is not considered a model parameter but a controller parameter, i.e. a (pre)filter that is chosen to improve the system robustness rejecting disturbance and noise. If constraints are not defined, this minimization leads to a linear time invariant (LTI) control law that can be computed in advance.

2.2 Fractional-Order Generalized Predictive Control

The task of finding the parameters N_1 , N_2 , N_u , γ , and λ is critical, as they determine the closed loop stability. However, thumb-rules exist that help the user to find initial guesses of their values quickly. It is usually accepted that $N_1 = 1$, $N_2 = 10$, $\gamma = 1$, $\lambda = 10^{-6}$, and N_u equal to the number unstable or badly-damped poles of the system are adequate for a wide range of applications [3].

This recommendation has the disadvantage of keeping constant all terms of the weighting sequences γ and λ . The use of variable terms for these weighting sequences can help to reach closed loop stability as it is shown in [14], where an unstable closed loop system, using default constant setting for γ and λ , is become a stable system choosing weighting sequence λ (3) equal to the binomial coefficients that appear, in a natural way, in the Grünwald–Letnikov formula (4) for fractional-order derivatives [11].

$$\lambda(j) = \left| \binom{\alpha}{N_u - j} \right|, \quad j = 1, 2, \dots, N_u \quad (3)$$

$$D^\alpha f(t)_{t=kh} = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{j=0}^n (-1)^j \binom{\alpha}{j} f(kh - jh) \quad (4)$$

In [13], a new formulation of GPC that uses a fractional-order cost function is proposed. Thus, the so-called Fractional-Order GPC (FGPC) is based on the following cost function:

$$J = \int_{N_1}^{N_2} D^\alpha [r(t) - y(t)]^2 dt + \int_1^{N_u} D^\beta [\Delta u(t - 1)]^2 dt; \quad \alpha, \beta \in \mathbb{R} \quad (5)$$

This cost function can be written in matrix form as follows:

$$J = e\gamma(\alpha, \Delta t)e' + \Delta u\lambda(\beta, \Delta t)\Delta u' \quad (6)$$

where $\gamma(\alpha, \Delta t)$ and $\lambda(\beta, \Delta t)$ are given by the expressions (7) and (8) at the top of the following page, respectively, ()' means the transpose matrix, e is the error signal ($r - y$), Δt is equal to the system sampling time, and ω_k are the so-called coefficients of fractional derivatives (9).

$$\gamma = \Delta t^{1-\alpha} \begin{pmatrix} \omega_m + \omega_{m-1} + \dots + \omega_{N_1-1} & 0 & \dots & 0 & 0 \\ 0 & \omega_{m-1} + \omega_{m-2} + \dots + \omega_{N_1-2} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \omega_1 + \omega_0 & 0 \\ 0 & 0 & \dots & 0 & \omega_0 \end{pmatrix} \quad (7)$$

$$\lambda = \Delta t^{1-\beta} \begin{pmatrix} \omega_{N_u-1} + \omega_{N_u-2} + \dots + \omega_1 & 0 & \dots & 0 & 0 \\ 0 & \omega_{N_u-2} + \omega_{N_u-3} + \dots + \omega_0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \omega_1 + \omega_0 & 0 \\ 0 & 0 & \dots & 0 & \omega_0 \end{pmatrix} \quad (8)$$

where $m = N_2 - N_1$.

$$\omega_k = (-1)^k \binom{\alpha}{k}. \quad (9)$$

3 System Description

To apply the GPC and FGPC strategies to practical systems, an experimental platform has been used in this work, whose connection scheme is shown in Fig. 1, and consists of

- A data acquisition board NI DAQPad-6259 Pinout for USB, by *National Instruments*, which implements a high-performance data acquisition hardware with an easy and friendly software interface (see [8] for key features).
- A computer, where the *Data Acquisition Toolbox* of MATLAB runs for the validation of the proposed controllers.
- A servomotor by *Feedback* with: (a) a mechanical unit, which constitutes the servo, strictly speaking; (b) an analogue unit to connect to the mechanical unit through a 34-way ribbon cable which carries all power supplies and signals and lets a velocity and position control of the servomotor; (c) a power supply for the system (see full description in [5]).

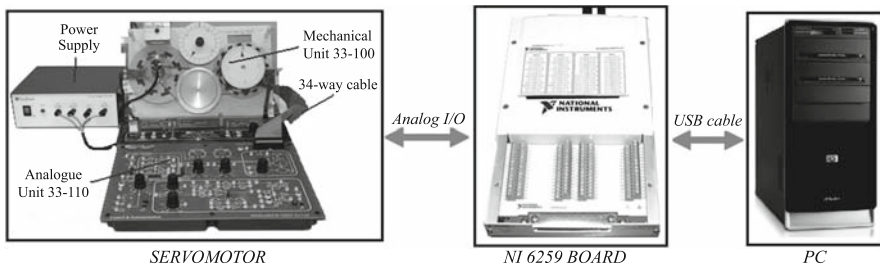


Fig. 1 Connection scheme of the experimental platform

4 Controllers Design and Results

In this section we propose two predictive control strategies to control the velocity and position servomotor: a conventional GPC and a FGPC from the obtained transfer functions for the velocity and position servo. Experimental (actual) and theoretical results are compared in each case to validate the effectiveness of the proposed predictive controllers. These controllers have been tuned (not optimized) for obtaining a stable and fast enough time response.

It is well known that the prefilter T can improve the system robustness against the model-process mismatch and disturbance rejection. Here we use the following prefilter for all GPC and FGPC controllers, considering some guidelines given in [18]: $T(z^{-1}) = (1 - 0.8z^{-1})$.

4.1 Velocity Control

The obtained discrete transformation of the velocity servomotor considering as sampling time $\Delta t = 0.2s$ is:

$$G_V(z^{-1}) = \frac{0.3393z^{-1}}{1 - 0.6255z^{-1}} \quad (10)$$

Taking into account it, we design the GPC controller using the default setting $N_1 = 1$, $N_2 = 10$, $\lambda = 10^{-6}$, $\gamma = 1$, and $N_u = 1$. In order to design the FGPC, we take the same values for the controller parameters N_1 , N_2 , and N_u . However, the values of γ and λ are determined by means of the expressions (7) and (8) from the values of α and β . These values of fractional derivatives must be chosen to get a gain and phase margins similar to the ones obtained previously for GPC. Hence, all gain and phase margins of the control system are represented, considering $\alpha, \beta \in (-5.3, 1)$, in Fig. 2. We have selected $\alpha = \beta = 0.8$ to obtain a gain and phase margins for the system similar to the previous controller.

Figure 3 shows the control signals evolution and the system output responses of the velocity control using the GPC and FGPC controllers, including the theoretical

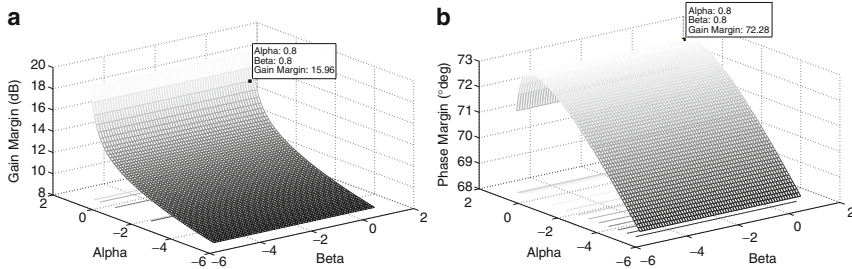


Fig. 2 Gain and phase margins according to α and β for velocity control: (a) gain and (b) phase

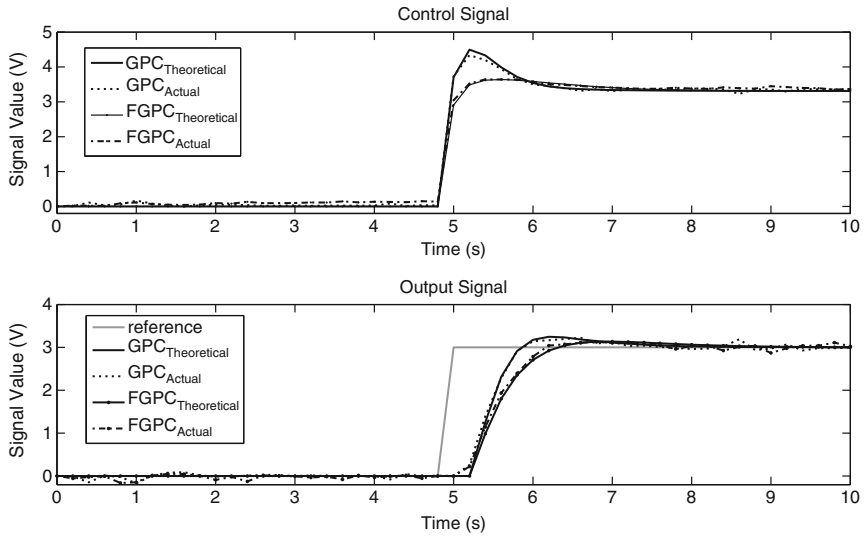


Fig. 3 Velocity control using the proposed controllers

and experimental signals in both cases. As we can see, the system output with both designed predictive controllers is stable but a bit noisy and presents a gain margins equal to 12.14 dB and 15.96 dB, and a phase margin equal to 72.67° and 72.28° , using the GPC and the FGPC, respectively. According to these results, the behaviour of the velocity servo system by the proposed predictive controllers is really similar, although the system response with the GPC is a bit faster. However, it is possible to find a FGPC to control the plant as well as GPC controller can do it, choosing other values of α and β .

4.2 Position Control

The obtained transfer function of position servo for controllers design purpose is the following (in the same conditions as the velocity servo):

$$G_P(z^{-1}) = \frac{0.364z^{-1}}{1 - 1.64z^{-1} + 0.64z^{-2}} = \frac{0.364z^{-1}}{(1 - z^{-1})(1 - 0.64z^{-1})} \quad (11)$$

As in the case of velocity servomotor control, GPC is designed from the default values of parameters N_1 , N_2 , λ , γ , and N_u . To design the FGPC controller, we will use again the default settings for controller parameters. As mentioned above, the values of γ and λ are determined by means of the expressions (7) and (8), considering $\alpha, \beta \in (-5.3, 1)$. Figure 4 exhibits all gain and phase margin for this controlled system. We shall choose $\alpha = \beta = 0.5$ in order to obtain a gain and phase margins for the system similar to the previous controller.

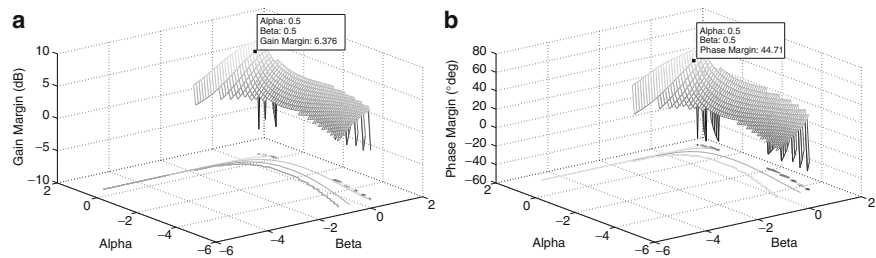


Fig. 4 Gain and phase margins according to α and β for position control: (a) gain and (b) phase

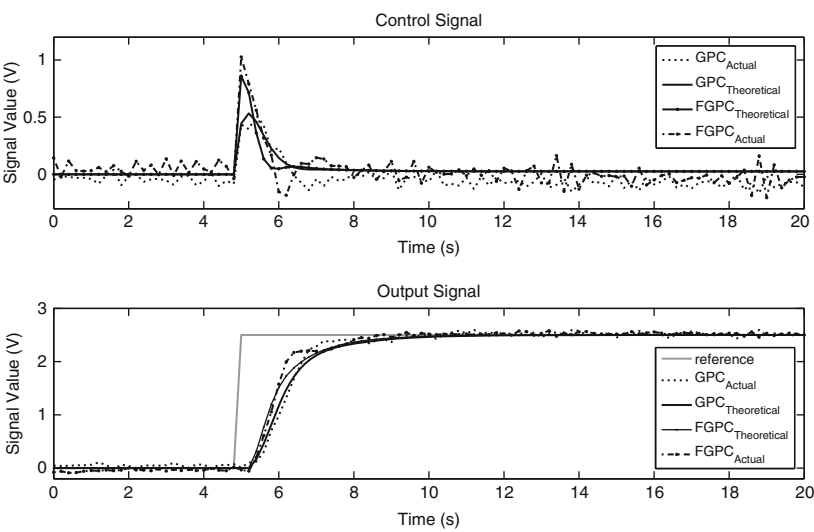


Fig. 5 Position control using the proposed controllers

In Fig. 5 we can observe the performance of servo system applying the proposed controllers. In both cases, the system is stable although there is a bit difference between the theoretical and the actual responses. The system presents a gain margins equal to 8.63 dB and 6.38 dB, and a phase margins equal to 43.84° and 44.71°, using the GPC and FGPC controllers, respectively. Therefore, comparing the position control by the designed controllers, system output using FGPC is a bit faster.

5 Conclusions

This chapter has focused on the design and application of the Fractional-Order Predictive Control (FGPC) to a servomotor system (velocity and position servo controls). Simulation and experimental results have been compared with the ones obtained using conventional GPC strategy.

It has been shown how FGPC can reach a good performance both in system output and robust stability. Moreover, different weighting sequences in the cost function J can be easily defined using just two parameters, the fractional orders α and β . In this way, it is possible to fine-tune the closed loop to achieve performance or robustness objectives with ease.

To sum up, FGPC has proved to be a versatile, valuable and alternative control strategy that deserves a deeper study.

Acknowledgements This work has been supported by the Spanish Ministry Research Grants DPI2004-05903 and DPI2005-07980-C03-03.

References

1. Camacho EF, Bordóns C (2004) Model predictive control. 2nd edn. Springer, London
2. Clarke DW (1988) Application of generalized predictive control to industrial process. *IEEE Contr Syst Mag* 122:49–55
3. Clarke DW, Mohtadi C, Tuffs PS (1987) Generalized predictive control. Part I. The Basic Algorithm. *Automatica* 23(2):137–148
4. Clarke DW, Mohtadi C, Tuffs PS (1987) Generalized predictive control. Part II. Extensions and Interpretations. *Automatica* 23(2):149–160
5. Feedback Instruments (1997) Analogue Servo. Fundamentals trainer 33-002. Feedback Instruments Ltd, UK
6. Maciejowski JM (2002) Predictive control with constraints. Prentice Hall, Harlow, UK
7. Monje CA, Vinagre BM, Feliu V, Chen YQ (2008) Tuning and auto-tuning of fractional order controllers for industry applications. *Contr Eng Pract* 16:798–812
8. National Instruments (2006) DAQ M Series. M Series User Manual. National Instruments, USA
9. Oldham KB, Spanier J (1974) The fractional calculus. Academic, New York
10. Petráš I (1999) The fractional-order controllers: methods for their synthesis and application. *J Elec Eng* 50(9–10): 284–288
11. Podlubny I (1999) Fractional differential equations. In: Mathematics in science and engineering, Vol. 198. Academic, San Diego, USA
12. Podlubny I (1999) Fractional-order systems and $PI^\lambda D^\mu$ controllers. *IEEE Trans. Autom Control* 44(1):208–214
13. Romero M, de Madrid AP, Mañoso C, Hernández R (2008) Fractional-order generalized predictive control. In: Proceedings of the 2008 fractional differentiation and its applications. Ankara, Turkey
14. Romero M, Vinagre BM, de Madrid AP (2008) GPC Control of a Fractional-order plant: improving stability and robustness. In: Proceedings of the 17th IFAC world congress. Seoul, Korea
15. Rossiter JM (2003) Model-based predictive control. A practical approach. CRC, Boca Raton, Florida
16. Vinagre BM, Petráš I, Podlubny I, Chen YQ (2002) Using fractional order adjustment rules and fractional order reference models in model-reference adaptive control. *Nonlinear Dynam* 29(1–4):269–279
17. Vinagre BM, Feliu V (2007) Optimal fractional controllers for rational order systems: a special case of the Wiener-Hopf spectral factorization method. *IEEE Trans Automat Contr* 52(12):2385–2389
18. Yoon TW, Clarke DW (1995) Observer design in receding-horizon predictive control. *Int J Contr* 61(1):171–191

Decentralized CRONE Control of $m \times n$ Multivariable System with Time-Delay

Dominique Nelson-Gruel, Patrick Lanusse, and Alain Oustaloup

Abstract An extension to $m \times n$ multivariable (non-square multivariable) plants with time-delay of the CRONE control system design is presented. To avoid difficult design and implementation, this paper proposes the design of a robust decentralized controller based on the use of the block relative gain for the pairing of the manipulated inputs and controlled outputs. The main advantage of the proposed methodology is the ease of hand-tuning and also of understanding by non-specialist.

1 Introduction

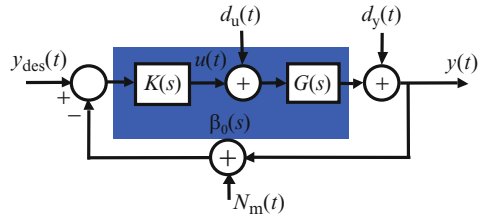
In many industrial plants the step before the use of a “fully” multivariable controller is the design of a decentralized controller. Several ways are used to decouple some manipulated variables to some controlled variables:

1. Control design based on the research of a decoupling matrix $D(s)$ placed between the plant $G(s)$ and the controller $K(s)$, this matrix simplifying the design of the controller.
2. Control design based on neglecting the coupling elements of the plant (multi-SISO or multi-loop control).
3. Design of a simplified MIMO controller by using a simplified model of the system.

One extension of the SISO (single-input single-output) CRONE (Commande Robuste d'Ordre Non-Entier) methodology to non-square multivariable plants already exists [2, 4]. The aim of this non-square MIMO (multivariable) CRONE CSD (control system design) is to robustify the closed loop dynamic performance through a robust either damping factor, or a robust resonant peak of the closed loop

D. Nelson-Gruel (✉), P. Lanusse, and A. Oustaloup
IMS, 351 cours de la libération, 33405 Talence cedex, France
e-mail: dominique.nelson@ims-bordeaux.fr; patrick.lanusse@ims-bordeaux.fr;
alain.oustaloup@ims-bordeaux.fr

Fig. 1 Common CRONE control-system diagram



system. Fractional differentiation is used to define the nominal and optimal open-loop, $\beta_0(s) = G_0(s).K(s)$. The design is based on the unity-feedback configuration, Fig. 1.

The controller $K(s)$ is defined by the product $G_0^\dagger(s).\beta_0(s)$ where $G_0^\dagger(s)$ is the Moore–Penrose pseudo-inverse of nominal plant $G_0(s)$ [5]. This approach is a “fully” multivariable one. Key issues behind the use of this strategy are the availability of a precise enough model and great understanding of control theory.

This article proposed a decentralized CRONE control approach to control non-square multivariable plant based on block relative gain analyzed. It has advantages of easier tuning and implementation. Section 2 briefly presented the mathematical background uses in this article. Section 3 outlines the extension of the CRONE CSD methodology and the design of a decentralized controller for non-square multivariable plants with time-delay.

2 Mathematical Backgrounds

Consider the equation:

$$b = Ay \text{ with } b \in C^m, y \in C^n, A \in C^{m \times n} \quad (1)$$

Moore (1920) and Penrose (1955) show that a solution of this relation exists, $A^\dagger$ named the Moore–Penrose Pseudo-Inverse. When $m = n$, $A^\dagger = A^{-1}$ and for $m \neq n$ Wiberg’s theorem gives the proof of unicity of the solution [7].

Theorem 1 (Wiberg). *Consider the Eq. (1), define $y_0 = A^\dagger b$. Let $y_1 \in \{y \mid \|Ay - b\|_2 \geq \|Ay_0 - b\|_2\}$ and $y \neq y_0$ if $\|Ay_1 - b\|_2 = \|Ay_0 - b\|_2$ then $\|y_1\|_2 > \|y_0\|_2$.*

When $m > n$ (more outputs than inputs when A is a plant transfer function) the solution of (1) is the unique $A^\dagger$ such as $\|b - Ay\|$ have the minimum norm.

When $m < n$ (more inputs than outputs) the solution is not unique. The chosen solution is the one with $\|y\|_2$ minimum.

Graybill in [1] establishes simple expressions for calculating the pseudo inverse of a matrix, A with full rank.

Theorem 2 (Graybill). Let $A \in \mathbb{C}_r^{m \times n}$ a complex $m \times n$ matrix of rank r :

i. if $A \in \mathbb{C}_m^{m \times n}$, then $A^\dagger = A^H (AA^H)^{-1}$ and $AA^\dagger = I_{mxm}$, (2)

ii. if $A \in \mathbb{C}_n^{m \times n}$, then $A^\dagger = (A^H A)^{-1} A^H$ and $A^\dagger A = I_{n \times n}$, (3)

where A^H is the Hermitian transpose of the A .

In [4] we have developed simple expression to calculate the pseudo-inverse of a transfer matrix with m outputs, n inputs and time-delay.

Theorem 3. Let $G_0(s)$ a transfer function matrix with m outputs, n inputs and time-delay:

$$G_0(s) = \begin{bmatrix} g_{011}(s) & & \cdots & g_{01n}(s) \\ & \ddots & & \\ & & g_{0ij}(s) & \vdots \\ \vdots & & & \\ g_{0m1}(s) & \cdots & & g_{0mn}(s) \end{bmatrix}, \quad (4)$$

where:

- $g_{0ij}(s) = h_{ij}(s)e^{-L_{ij}s}$
- $h_{ij}(s)$ is a strictly proper time-delay free transfer function
- L_{ij} is a positive constant

Then:

$$G_0^\dagger(s) = P(s) = \begin{bmatrix} p_{11}(s)e^{\gamma_{11}s} & \cdots & p_{1m}(s)e^{\gamma_{1m}s} \\ \vdots & p_{ij}(s)e^{\gamma_{ij}s} & \vdots \\ p_{n1}(s)e^{\gamma_{n1}s} & \cdots & p_{nm}(s)e^{\gamma_{nm}s} \end{bmatrix}_{1 \leq i \leq n, 1 \leq j \leq m}, \quad (5)$$

where:

- $p_{ij}(s)$ is non-zero polynomial of s with time-delay
- γ_{ij} is a positive or null number

3 CRONE Control Methodology and MIMO Systems

The aim of the CRONE CSD is to find a diagonal open-loop transfer matrix whose m elements are fractional order transfer functions. It is parametered to satisfy the four following objectives:

- Perfect decoupling for the nominal plant
- Accuracy specifications at low frequencies

- Required nominal stability margins of the closed loops (behaviours around the required cut-off frequencies)
- Specifications on the n control efforts at high frequencies

$$\mathbf{f}_{i0}(s) = \begin{bmatrix} \mathbf{f}_{i0_{11}}(s) & 0 & \cdots & 0 \\ 0 & \ddots & & \\ & & \mathbf{f}_{i0_{ii}}(s) & \cdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{f}_{i0_{mm}}(s) \end{bmatrix}. \quad (6)$$

After an optimisation of the diagonal open-loop transfer matrix (6) (see below for explanation on the optimisation), frequency-domain system identification is carried out to obtain the rational form of the fractional controller. Open-loop transfer functions $\beta_{0i}(s)$ are used to satisfy the three other objectives.

3.1 Definition of the Diagonal Open-Loop Transfer Function Elements

Open-loop transfer function behaviors can be described by using the third generation CRONE CSD methodology. This generation of CRONE CSD uses complex non-integer order integration over a chosen frequency range $[\omega_A, \omega_B]$. The complex fractional order, $n_f = a + ib$ allows creating a straight line of any direction in the Nichols chart which is called *the generalized template* (Fig. 2).

The real part of n_f determines its phase location at frequency ω_{cg} , that is $-\text{Re}(n_f)\pi/2$, and the imaginary part determines its direction. The generalized

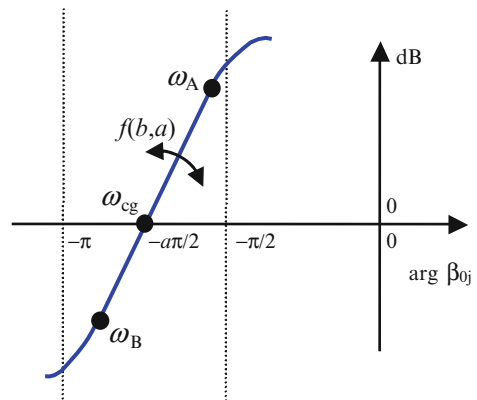


Fig. 2 Generalized template in the Nichols plane

template is described by the limitation in the operational plane C_j of the complex non-integer integrator transfer function:

$$\beta_{0_i}(s) = \left[\left(\frac{\omega_{cg}}{s} \right)^{n_f} \right]_{C_j} \quad (7)$$

with $s = \sigma + j\omega \in C_j$ and $n_f = a + ib \in C_i$.

This transfer function can be described as based on band-limited complex non-integer integration:

$$\beta_{0_i}(s) = C^{\text{sign}(b)} \left(\frac{1 + s/\omega_h}{1 + s/\omega_l} \right)^a \left(\text{Re}_{/i} \left\{ \left(C_g \frac{1 + s/\omega_h}{1 + s/\omega_l} \right)^{ib} \right\} \right)^{-q \text{sign}(b)}, \quad (8)$$

with

$$C = ch \left[b \left(\arctan \left(\frac{\omega_{cg}}{\omega_l} \right) - \arctan \left(\frac{\omega_{cg}}{\omega_h} \right) \right) \right], \quad (9)$$

and

$$C_g = \left(\frac{1 + \left(\frac{\omega_{cg}}{\omega_l} \right)^2}{1 + \left(\frac{\omega_{cg}}{\omega_h} \right)^2} \right)^{1/2}. \quad (10)$$

The corner frequencies are placed around the extreme frequencies ω_A and ω_B such that:

$$\omega_l < \omega_A < \omega_{cg} < \omega_B < \omega_h. \quad (11)$$

For stable and minimum phase plant the generalized template is taken account in the open-loop transfer function as follows:

$$\beta_{0_{ii}}(s) = \beta_{l_i}(s) \beta_{0_i}(s) \beta_{h_i}(s), \quad (12)$$

with

$$\beta_{l_i}(s) = C_{l_i} \left(\frac{\omega_{l_i}}{s} + 1 \right)^{n_{l_i}}, \quad (13)$$

the order n_{l_i} fixes the accuracy of each closed-loop,

$$\beta_{h_i}(s) = \frac{C_{h_i}}{\left(\frac{s}{\omega_{h_i}} + 1 \right)^{n_{h_i}}}, \quad (14)$$

the order n_{h_i} permits the elements of the controller to be proper. This third generation CRONE control open-loop transfer function has been defined using the gain-crossover open-loop frequency; however this definition can be made using the closed loop resonant frequency too. Then ω_r replace ω_{cg} .

3.2 Decoupling and Optimised Controller

Let $G_0(s)$ be the nominal plant transfer matrix such that:

$$\beta_0 = G_0 K = \text{diag} [\beta_{0ij}]_{i=j} = \text{diag} \left[\frac{n_{ij}}{d_{ij}} \right]_{i=j \in N} \quad (15)$$

where:

- $g_{ij}(s)$ is a strictly proper transfer function
- $N = \{1, \dots, m\}$
- $\beta_{0ii} = \beta_{0jj} = \frac{n_{jj}}{d_{jj}}$ the element of the j th column and j th row

As mentioned above the aim of CRONE control for MIMO plants is to find a decoupling controller for the nominal plant. G_0 being not diagonal, the problem is to find a decoupling and stabilizing controller K [6]. This controller exists if and only if the following hypotheses are true:

$$H_1 : [G(s)]^\dagger \text{ exist}, \quad (16)$$

$$H_2 : Z_+[G(s)] \cap P_+[G(s)] = 0, \quad (17)$$

where $Z_+[G(s)]$ and $P_+[G(s)]$ indicate the positive real part zero and pole sets.

The controller $K(s)$ is given by:

$$K(s) = G_0^\dagger(s) \beta_0(s) = G_0^\dagger(s) \text{diag} \left[\frac{n_{jj}(s)}{d_{jj}(s)} \right]_{j \in N}, \quad (18)$$

Where $G_0^\dagger(s)$ is the Moore–Penrose Pseudo-inverse transfer matrix.

Thus, with theorem 3 the controller is written:

$$k_{ij}(s) = p_{ij}(s) e^{\gamma_{ij}s} \beta_{0ij}(s). \quad (19)$$

The relation above, implies that time-delay, RHP zeros and unstable poles of p_{ij} must appear in β_{0ij} to make the controller achievable and stable. The open-loop transfer matrix will now be defined.

3.3 Research of Time-Delay of Each Open-Loop

We define the delay of the non-zero transfer function $a(s)$ by $\tau(a(s))$ which is the smallest time-delay of $a(s)$. It is easy to verify:

- $\tau(a_1 a_2) = \tau(a_1) + \tau(a_2)$,
- $\tau(a^{-1}) = -\tau(a)$,
- $\tau(e^{\eta s}) = \tau(e^{-(\eta s)}) = -\eta$.

$\forall a, a_1$ and a_2 non-zero transfer functions

The following condition must be verified:

$$\tau(k_{ij}) \geq 0 \quad \forall i \in [1, n], j \in [1, m]. \quad (20)$$

For this to be true, with (19):

$$\tau(\beta_{0_{jj}}(s)) \geq \gamma_{ij}, \forall j \in [1, m], \quad (21)$$

where:

$$\gamma_{ij} = \alpha_{q_1} - \tau \left(\left[G_0^H \cdot \lfloor Q^{ji} \rfloor \right]_{i,j} \right) = \alpha_{q_1} - \delta_{0_{ji}}, \quad (22)$$

consequently:

$$\tau(k_{ij}) \geq 0 \quad \forall i \in [1, n], j \in [1, m]. \quad (23)$$

The time-delay of the j th open-loop transfer function must satisfy all the following relations:

$$\begin{aligned} \tau(\beta_{0_{jj}}) &\geq \alpha_{q_1} - \tau \left(\left[G_0^H \cdot \lfloor Q^{ji} \rfloor \right]_{1j} \right) \\ \tau(\beta_{0_{jj}}) &\geq \alpha_{q_1} - \tau \left(\left[G_0^H \cdot \lfloor Q^{ji} \rfloor \right]_{2j} \right) \\ &\vdots \\ \tau(\beta_{0_{jj}}) &\geq \alpha_{q_1} - \tau \left(\left[G_0^H \cdot \lfloor Q^{ji} \rfloor \right]_{nj} \right) \end{aligned} \quad (24)$$

Finally:

$$\tau(\beta_{0_{jj}}) \geq \alpha_{q_1} - \tau_j \quad \forall j \in [1, m], \quad (24a)$$

with

$$\tau_j = \min_{j \in N} \left(\tau \left(\left[G_0^H \cdot \lfloor Q^{ji} \rfloor \right]_{ij} \right) \right). \quad (25)$$

Relation (24a) implies that the j th open-loop transfer function must have a time-delay higher or equal to the difference between the maximum time-delay of $Q(s)$ and, considering the j th column of the transfer matrix $[G_0^H[Q^{ji}]]$ the smallest time-delay.

3.4 Open-Loop Transfer Matrix Optimization

The nominal sensitivity and the nominal complementary sensitivity transfer function matrices are

$$S_0(s) = [I + \beta_0(s)]^{-1} = \text{diag} [S_{0j}(s)]_{1 \leq j \leq m}, \quad (26)$$

$$T_0(s) = [I + \beta_0(s)]^{-1} \beta_0(s) = \text{diag} [T_{0j}(s)]_{1 \leq j \leq m}, \quad (27)$$

with

$$T_{0j}(s) = \frac{\beta_0(s)}{(1 + \beta_0(s))}, \quad (28)$$

$$S_{0j}(s) = \frac{1}{(1 + \beta_0(s))}. \quad (29)$$

For plants other than the nominal, the closed-loop transfer matrices $T(s)$ and $S(s)$ are no longer diagonal. Each diagonal element $T_{jj}(s)$ and $S_{jj}(s)$ could be interpreted as closed loop transfer functions coming from a scalar open-loop transfer function $\beta_{jj}(s)$ called *equivalent open-loop transfer function*:

$$\beta_{jj}(s) = \frac{T_{jj}(s)}{1 - T_{jj}(s)} = \frac{1 - S_{jj}(s)}{S_{jj}(s)}. \quad (30)$$

For each nominal open-loop $\beta_{0i}(s)$, many generalized templates can border the same required magnitude-contour of the Nichols chart or the same resonant peak M_{p0j} . The optimal one minimizes the robustness cost function:

$$J = \sum_{j=1}^m \left(M_{p\max_j} - M_{p\min_j} \right)^2 \quad (31)$$

where:

$$M_{p\max_j} = \max_G \sup_{\omega} (T_{jj}(j\omega)), \quad (32)$$

$$M_{p\min_j} = \min_G \sup_{\omega} (T_{jj}(j\omega)), \quad (33)$$

while respecting the following set of inequalities for $\omega \in \mathbb{R}$ and $i, j \in N$:

$$\begin{aligned} \inf_G |T_{ij}(j\omega)| &\geq T_{ij_l}(\omega), \\ \sup_G |T_{ij}(j\omega)| &\leq T_{ij_u}(\omega), \\ \sup_G |S_{ij}(j\omega)| &\leq S_{ij_u}(\omega), \\ \sup_G |KS_{ij}(j\omega)| &\leq KS_{ij_u}(\omega), \\ \sup_G |SG_{ij}(j\omega)| &\leq SG_{ij_u}(\omega), \end{aligned} \quad (34)$$

where G is the nominal and perturbed plant.

In addition, the controller must be proper and permit the rejection of low amplitude disturbances.

Thus using (19), the controller is proper if

$$\deg(d_j(s)) + \deg(D_{p_{ij}(s)}) \geq \deg(N_{p_{ij}(s)}) + \deg(n_j(s)), \quad (35)$$

$$\forall j \in [1, m], \text{ with } p_{ij}(s) = \frac{N_{p_{ij}(s)}}{D_{p_{ij}(s)}}.$$

As a consequence:

$$\deg(d_j(s)) - \deg(n_j(s)) \geq \max_{j \in N} (\deg(N_{p_{ij}(s)}) - \deg(D_{p_{ij}(s)})), \quad (36)$$

and at high frequencies

$$\deg(d_j(s)) = n_{h_j} \quad (37)$$

gives

$$n_{h_j} \geq \max_{i \in N} (\deg(N_{p_{ij}(s)}) - \deg(D_{p_{ij}(s)})). \quad (38)$$

Moreover, considering η_z a function who gives the number of roots of the transfer function $H(s)$ in z . $\eta_z(H(s))$ is positive if $H(s)$ has more zeros than poles.

At low frequencies, disturbances are rejected if transfer function SG contains no poles at $z = 0$:

$$\eta_0(S(s)) + \eta_0(G_0(s)) \geq 0, \quad (39)$$

and $S(s)$ is equivalent to $\beta_{0i}(s)$:

$$\eta_0(S(s)) \approx \eta_0(\beta_{0_j}(s)) = n_{l_j}. \quad (40)$$

Also:

$$\eta_0(G_0(s)) = \eta_0(N_{g_{ij}(s)}) - \eta_0(D_{g_{ij}(s)}), \quad (41)$$

with:

$$g_{ij}(s) = \frac{N_{g_{ij}(s)}}{D_{g_{ij}(s)}}. \quad (42)$$

Therefore:

$$n_{l_j} \geq \eta_0(N_{g_{ij}(s)}) - \eta_0(D_{g_{ij}(s)}), \forall j \in [1, m], \quad (43)$$

finally:

$$n_{l_j} \geq \max_{i \in N} (\eta_0(N_{g_{ij}(s)}) - \eta_0(D_{g_{ij}(s)})). \quad (44)$$

3.5 Decentralized CRONE Controller

As mentioned in the introduction, the methodology proposed could be difficult to be implemented and tuned in an industrial framework. Specialist and accurate-enough model is needed. To simplify this methodology we use the block relative gain (BGR) method [4] to select block pairings.

Definition 1. Let $G(s)$ a $m \times n$ transfer function matrix. The block relative gain is a way to measure interaction between inputs and outputs of a “block” of the transfer function matrix $G(s)$. It is defined as the ratio of the open-loop block gain and apparent block gain in the same loop when all other control loops are closed:

$$[\Lambda_B(s)]_{ii} = G_{ii}(s) \left[G^\dagger(s) \right]_{ii}, \quad (45)$$

where:

- $G_{ii}(s)$ is the $m_1 \times m_1$ transfer function matrix relating the first m_1 inputs and outputs of $G(s)$,
- $[G^\dagger(s)]_{ii}$ the corresponding block of $G^{-1}(s)$.

Caused by this definition the ideal value of the BRG will be the least square approximation to the identity matrix for a given subsystem.

Thus for a given system $G_0(s)$, we use the BRG method to evaluate the different block interaction. To simplify the methodology proposed we consider all block with a weak BRG equal to zero and define a new nominal transfer function, G'_0 .

Consequently, calculating pseudo-inverse of the new nominal transfer function, $G'^\dagger_0(s)$ and finding RHP zero or poles and time-delay that must appear in β_{0ij} to make the controller achievable and stable, will be easier. Moreover, as some elements $G'^\dagger_0(s)$ will be equal to zero, using (19), the controller will be a decentralized controller:

$$K(s) = G'^\dagger_0 \beta_0(s) \quad (46)$$

However, even if the nominal plant taken into account has been simplified (some elements are null), the optimisation of the nominal open-loop is achieved by taken into account the perturbed full transfer function matrix G .

4 Application

Consider the mixing tank [3], that has three inputs streams and two outputs, the height of liquid in the tank h , and the concentration c of the liquid (Fig. 3).

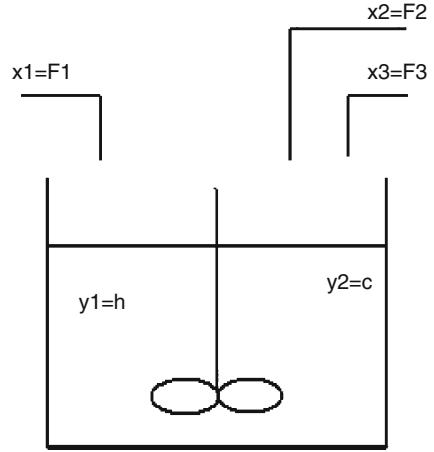
The model of the system is expressed by

$$Y(s) = G(s)X(s), \quad (47)$$

with:

$$G(s) = \begin{bmatrix} \frac{4}{20s+1}e^{-\theta_1 s} & \frac{4}{20s+1}e^{-\theta_2 s} & \frac{4}{20s+1}e^{-\theta_3 s} \\ \frac{3}{10s+1}e^{-\theta_1 s} & \frac{-3}{10s+1}e^{-\theta_2 s} & \frac{5}{10s+1}e^{-\theta_3 s} \end{bmatrix}, \quad (48)$$

$$0 \leq \theta_1 \leq 2, \quad 0 \leq \theta_2 \leq 1, \quad 0 \leq \theta_3 \leq 0.5. \quad (49)$$

Fig. 3 Mixing tank

The nominal plant is chosen with $\theta_1 = 2$, $\theta_2 = 1$ and $\theta_3 = 0.5$. The concept of BRG is applied to this system. From BRG h is paired with x_1 and x_2 and c is paired with x_3 . Thus the nominal plant uses to design the controller is

$$G'_0(s) = \begin{bmatrix} \frac{4}{20s+1}e^{-\theta_1 s} & \frac{4}{20s+1}e^{-\theta_2 s} & 0 \\ 0 & 0 & \frac{5}{10s+1}e^{-\theta_3 s} \end{bmatrix}. \quad (50)$$

4.1 Research of Time-Delay

Using matlab we compute the pseudo-inverse of the nominal transfer matrix:

$$G'^0{}^\dagger(s) = \begin{bmatrix} \frac{20s+1}{8}e^{\theta_1 s} & 0 \\ \frac{20s+1}{8}e^{\theta_2 s} & 0 \\ 0 & \frac{10s+1}{5}e^{\theta_3 s} \end{bmatrix} \quad (51)$$

Then we apply the methodology proposed in Section 3 and obtain:

$$\alpha_{q_1} = 0, \quad (52)$$

$$\tau(\lfloor G_0^H \lfloor Q^{ji}(s) \rfloor \rfloor_{11}) = -\theta_1 = -2, \quad (53)$$

$$\tau(\lfloor G_0^H \lfloor Q^{ji}(s) \rfloor \rfloor_{21}) = -\theta_2 = -1, \quad (54)$$

$$\tau(\lfloor G_0^H \lfloor Q^{ji}(s) \rfloor \rfloor_{32}) = -\theta_3 = -0.5. \quad (55)$$

Consequently:

$$\tau_1 = -2 \text{ and } \tau_2 = -0.5. \quad (56)$$

Finally:

$$\tau(\beta_{01}) \geq \alpha_{q1} - \tau_1 = 2, \quad (57)$$

and

$$\tau(\beta_{02}) \geq \alpha_{q1} - \tau_2 = 0.5. \quad (58)$$

4.2 Specifications

For all parametric states, the following specification must be satisfied:

- Zero steady-state error for both outputs
- Settling time as short as possible
- Robustness according to disturbances and parametric variations

With these specifications some elements of the open-loop transfer matrix can be initialized. So, with the maximum relative degree of $G'^{\dagger}_0$, taking into account (1) and all the elements to include in the open-loop transfer matrix, n_h must be greater or equal than 2 for the two loops. A zero steady-state error for both loops is obtained if n_l equals to one.

4.3 Optimisation

Taking into account all specifications, optimal values for the parameters of the first fractional open-loop transfer function are

- $\omega_r = 0.25$ rad/s,
- $\omega_l = 0.03$ rad/s,
- $\omega_h = 31$ rad/s,

$$\|\beta_{01}(j\omega)\|_{\omega=\omega_r} = 0 \text{ dB},$$

- $a = 0.95$, $b = 0.13$.

And for the second

- $\omega_r = 0.25$ rad/s,
- $\omega_l = 0.025$ rad/s,
- $\omega_h = 1.7$ rad/s,

$$\|\beta_{02}(j\omega)\|_{\omega=\omega_r} = 2 \text{ dB},$$

- $a = 1.10$, $b = -0.44$.

Figure 4 show the optimised equivalent open-loop.

The final and rational controller is found by frequency domain identification.

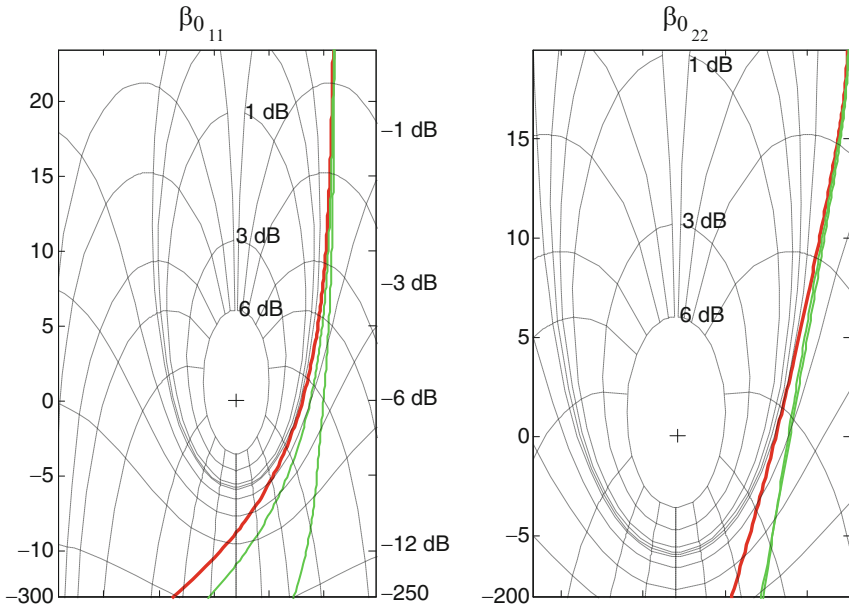


Fig. 4 Nominal open-loop (left) and two equivalent perturbed open-loops (right)

4.4 Results

A simulation is now carried out to assess the controller:

- Reference signals are steps applied respectively at $t = 10$ s for $h_{\text{des}} = 70$ and $t = 50$ s for $c_{\text{des}} = 50$
- Time delay vary in interval defined by the relation (49)
- Disturbance signals are steps applied respectively at $t = 100$ s for $x_1 = 5$ and $t = 150$ s for $x_2 = -5$ and $t = 200$ s for $x_3 = 5$

The simulation results are shown and compared in Fig. 5 for the nominal plant, and Fig. 6 for nominal and perturbed plants. Perturbed parameters are time-delays: for the first perturbed plant $\theta_1 = 0.5$, $\theta_2 = 0.5$ and $\theta_3 = 0.5$ and for the second $\theta_1 = \theta_2 = \theta_3 = 0$. All specifications are satisfied but Figs. 5 and 6 show that, considering IMC [2] and decentralized CRONE approach we have variations greater than 10 cm. This variation is caused by the coupling effect of the plant. Thus, by decreasing the settling time of the second control loop this variation could be less than 10 cm. Figure 5 shows also the perfect decoupling provided by the centralized fully multivariable CRONE controller.

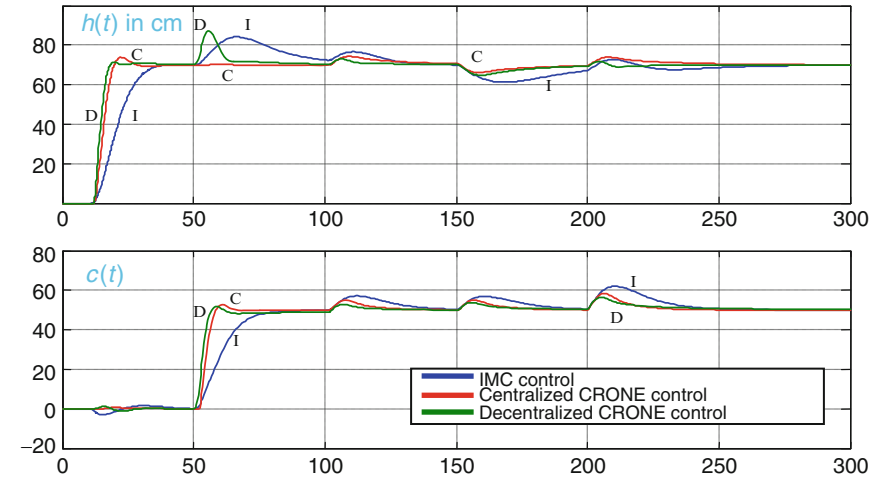


Fig. 5 Control performance of the mixing tank for the nominal plant: (I) IMC controller, (C) CRONE centralized controller and (D) CRONE decentralized controller

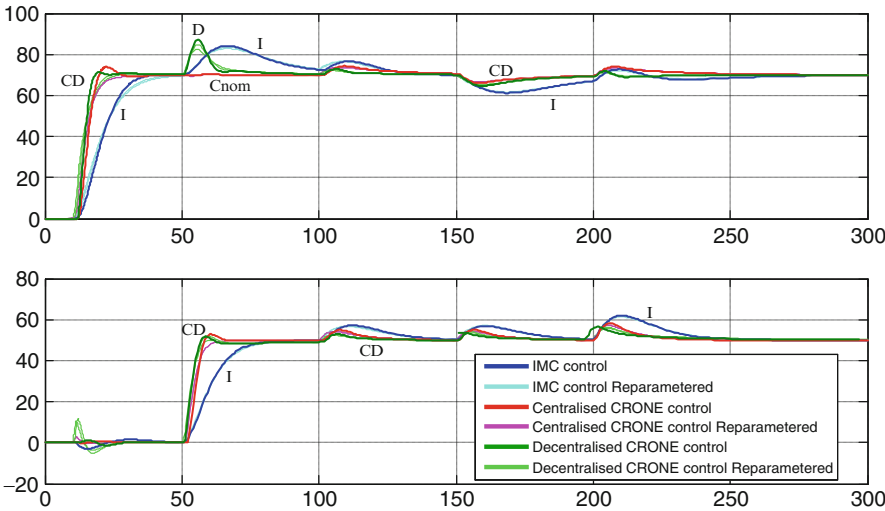


Fig. 6 Control performance of the mixing tank for nominal and perturbed plant: (I) IMC controller, (C and Cnom) CRONE centralized controller and (D) CRONE decentralized controller

5 Conclusion

The multivariable CRONE approach is based on third generation CRONE control system design. Each open-loop transfer function, $\beta_{0i}(s)$ of the matrix $\beta_0(s)$ is thus fractional. For plant with time-delay, RHP zeros or poles, some RHP zeros or poles and time-delay of $G(s)$ or $G^\dagger(s)$ can appear on $K(s)$ or on closed loop transfer

matrices. The controller would not be achievable and stable. So in [3] we have proposed a method for treating this kind of system and to determine a final stable and achievable controller. In an industrial framework, the design, implementation and tuning of this approach could be difficult to achieved. This paper proposed a way to simplify the design and thus the implementation and the tuning of the controller by using BRG (block relative gain). Using BRG method, we simplified the nominal transfer function matrix used to determine RHP zero or poles and time-delay that must appear in $\beta_{0i}(s)$. Nevertheless, the perturbed “fully” multivariable plant is taken into account to optimize the open loop $\beta_0(s)$. The control obtained is then robust and easy to tune.

References

1. Graybill F (1969) Introduction to matrices with applications in statistics. Wadsworth, Belmont, CA
2. Lanusse P, Oustaloup (2000) A Robust control of LTI square MIMO plants using two CRONE control design approaches. 3rd IFAC Symposium on Robust Control Desing (ROCOND 2000). Prague. Czech Republic
3. Loh EJ, Chiu MS (1997) Robust decentralized control of non-square systems. Chem Eng commun 158:157–180
4. Nelson-Gruel D, Lanusse P (2008) Commande robuste systèmes mimo non-carrés retardés. IEEE-CIFA'08. Bucarest
5. Reeves D, Arkun Y 1989 Interaction measures for nonsquare decentralized control structures. *AIChE* 35(4):603–613
6. Vardulkis A (1987) Internal stabilization and decoupling in linear multivariable systems by unity output feedback compensation. IEEE Trans Automat Contr 32:August 1987
7. Wiberg DM (1971) Schaum's Outline of theory and problems of state space and linear systems. Mc Graw-Hill, New York

Fractional Order Adaptive Control for Cogging Effect Compensation

Ying Luo, YangQuan Chen, and Hyo-Sung Ahn

Abstract Fractional calculus is a generalization of the integration and differentiation to the fractional (non-integer) order. In this chapter, for the first time, we devised a fractional-order adaptive compensation (FO-AC) method for cogging effect minimization for permanent magnetic synchronous motors (PMSM) position and velocity servo system. Cogging effect is a major disadvantage of PMSM. In our FO-AC scheme, a new fractional order adaptive compensator for cogging effect is designed to guarantee the boundedness of all signals. Stability properties have been proven for the systems with the traditional integer order adaptive compensation method and the proposed fractional order adaptive compensation method, respectively. Simulation results are presented to illustrate the advantage of the proposed FO-AC method for cogging effect compensation over the conventional integer order scheme.

1 Introduction

Extending classical integer order calculus to non-integer order case leads to the so-called fractional calculus. It has a firm and long-standing theoretical foundation and

Y. Luo (✉)

Center for Self-Organizing and Intelligent Systems (CSOIS), Dept. of Electrical and Computer Engineering, Utah State University, 4160 Old Main Hill, Logan, UT 84322-4160, USA. On leave from the Department of Automation Science and Technology in South China University of Technology, Guangzhou, P. R. China
e-mail: ying.luo@ieee.org

Y. Chen

Center for Self-Organizing and Intelligent Systems (CSOIS), Dept. of Electrical and Computer Engineering, Utah State University, 4160 Old Main Hill, Logan, UT 84322-4160, USA
e-mail: yqchen@ieee.org

H.-S. Ahn

Department of Mechatronics, Gwangju Institute of Science and Technology (GIST), Gwangju, Korea
e-mail: hyosung@gist.ac.kr

the earliest systematic studies of fractional calculus were in the nineteenth century by Liouville, Riemann and Holmgren [10]. In the last several decades, as the rapid development of computer technology and the better understanding of the potential of fractional calculus, the realization of fractional order control system became much easier than before, and the fractional order calculus will be more and more useful in various science and engineering areas. In motion controls, some example applications can be found in [9, 14]. The application of the theory of fractional calculus in adaptive control is just in the beginning, but with more and more research efforts on this subject [2, 3, 12].

In this chapter, which is based on the IFAC FDA'08 conference paper [7], we present a fractional-order adaptive compensation (FO-AC) method for cogging effect minimization of permanent magnetic synchronous motors (PMSM) position and velocity servo system. Cogging effect is a major disadvantage of PMSM, and degrades the servo control performance of application, especially in a low-speed range [4, 7, 13]. In our FO-AC scheme, a new fractional order adaptive compensator for cogging effect is designed to guarantee the boundedness of all signals. Stability properties have been proven for the system with the traditional integer order adaptive compensation method and the proposed fractional order adaptive compensation method, respectively. Simulation results are presented to illustrate the advantage of the proposed FO-AC method for cogging effect compensation over the conventional integer order scheme.

The major contributions of this paper include (1) a new fractional order adaptive compensation scheme for cogging effect minimization; (2) stability proofs of the system with the traditional integer order adaptive compensation method and the proposed fractional order adaptive compensation scheme in frequency domain; (3) simulation verification of FO-AC for multi-harmonic cogging effect on the PMSM position servo system simulation model; (4) demonstration of the advantage of the FO-AC for cogging effect by performing the simulation comparison with the traditional integer order scheme.

The rest of this paper is organized as follows. In Sect. 2, a new fractional order adaptive compensator for cogging effect is designed based on frequency domain stability analysis. Simulation test is presented in Sect. 3, and the performances using the fractional order AC are compared with the traditional integer order AC for cogging effect in the simulation. Conclusion is given in Sect. 4.

2 Fractional Order Adaptive Compensation of Cogging Effect

2.1 Motivations and Problem Formulation

In this section, a fractional order adaptive compensator for cogging is designed. The cogging force can be written as: $-a(\theta)$, where $a(\theta)$ is the function of θ , the angular displacement. In this chapter, to present our ideas clearly, without loss of generality,

The motion control system is modeled as follows

$$\dot{\theta}(t) = v(t), \quad (1)$$

$$\dot{v}(t) = u - \frac{a(\theta)}{J} - T_{l'} - B_1 v, \quad (2)$$

$$u = \frac{1}{J} T_m, T_{l'} = \frac{1}{J} T_l, B_1 = \frac{B}{J},$$

where θ is the angular position; v is the velocity; u is the control input and $a(\theta)$ is the unknown position-dependent cogging disturbance which is repeating in every pole-pitch; J is the moment of inertia; T_m is the ideal case of the electromagnetic torque generated, and T_l is the load torque applied; B is the friction coefficient.

Remark 1. Equations 1 and 2 can be used to describe the mechanical dynamic system of a PMSM in the synchronous rotating reference frame [7].

In this chapter, we also consider the cogging force as the general multi-harmonic form as considered in [7]

$$a(\theta) = \sum_{i=1}^{\infty} A_i \sin(\omega_i \theta + \varphi_i). \quad (3)$$

where A_i is the unknown amplitude of the i -th harmonics, ω_i is the known state-periodic cogging force frequency, and φ_i is the phase angle. From physical consideration, $a(\theta)$ should be bounded as

$$|a(\theta)| \leq b_0. \quad (4)$$

First, before presenting our main results, the following definitions are necessary which are adapted from [1] for self-containing purpose.

Definition 1. The total passed trajectory is given as:

$$s(t) = \int_0^t \left| \frac{d\theta}{d\tau} \right| d\tau = \int_0^t |v(\tau)| d\tau,$$

where θ is the angle position, and v is the velocity. Physically, $s(t)$ is the total passed trajectory, hence it has the following property:

$$s(t_1) \geq s(t_2), \text{ if } t_1 \geq t_2.$$

With notation $s(t)$, the position corresponding to $s(t)$ is denoted as $\theta(s(t))$ and the cogging force corresponding to $s(t)$ is denoted as $a(s(t))$. In our definition, since $s(t)$ is the summation of absolute position increasing along the time axis, just like t and $s(t)$ is a monotonous growing signal. So we have

$$a(\theta(s)) = a(s(t)) = a(t). \quad (5)$$

Then we define

$$e_a(s(t)) = a(s(t)) - \hat{a}(s(t)),$$

where $\hat{a}(s(t))$ is the estimated cogging force, $\hat{a}(s(t)) = \hat{a}(t)$ (note: t is the current time corresponding to the current total passed trajectory $s(t)$). Here, let us change $e_a(s(t)) = a(s(t)) - \hat{a}(s(t))$ into time-domain such as:

$$e_a(s(t)) = a(t) - \hat{a}(t) = e_a(t). \quad (6)$$

In the same way, the following relationships are true:

$$v_d(s(t)) = v_d(t), \quad v(s(t)) = v(t),$$

and the following notations are also defined

$$e_\theta(t) = \theta_d(t) - \theta(t), \quad e_v(t) = v_d(t) - v(t). \quad (7)$$

The control objective is to track or servo the given desired position $\theta_d(t)$ and the corresponding desired velocity $v_d(t)$ with tracking errors as small as possible. In practice, it is reasonable to assume that $\theta_d(t)$, $v_d(t)$ and $\dot{v}_d(t)$ are all bounded signals.

The feedback controller is designed as:

$$u(t) = \dot{v}_d(t) + T_{l'} + \frac{\hat{a}(t)}{J} + \alpha m(t) + \gamma e_v(t), \quad (8)$$

with

$$m(t) := \gamma e_\theta(t) + e_v(t), \quad (9)$$

where α and γ are positive gains; $\hat{a}(t)$ is an estimated cogging force from an adaptation mechanism to be specified later; $\dot{v}_d(t)$ is the desired acceleration; and $e_\theta(s(t)) = e_\theta(t)$; and $m(s(t)) = m(t)$.

Our adaptation law is designed as follows:

$$\dot{\hat{a}}(t) = z - \mu v, \quad (10)$$

μ is a positive design parameter; and the following tuning mechanism is designed for z :

$${}_0D_t^\nu z(t) = \mu[\dot{v}_d(t) + \alpha m(t) + \gamma e_v(t)] + \frac{e_v(t)}{J}, \quad (11)$$

where

$$\nu \in (0, 1], z(t)|_{t=0} = 0.$$

In this study we adopt the following Caputo definition for fractional derivative, which allows utilization of initial values of classical integer-order derivatives with known physical interpretations [11]

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha - n)} \int_0^t \frac{f^{(n)}(\tau) d\tau}{(t - \tau)^{\alpha+1-n}}, \quad (12)$$

where n is an integer satisfying $n - 1 < \alpha \leq n$.

Remark 2. If $\nu \in (0, 1)$, our designed control law (10) is the new fractional order adaptive compensation scheme; if $\nu = 1$, our designed control law (10) is the integer order adaptive compensation scheme [5].

Now, based on the above discussions, the following stability analysis of the proposed integer order and fractional order adaptive compensation schemes are performed in the frequency domain. Consider two cases: (1) when $\nu = 1$ for integer order adaptive compensation method for cogging effect and (2) when $0 < \nu < 1$ for fractional order adaptive compensation method for cogging effect.

2.2 Stability Analysis of Integer Order Case

First, let us consider the case (1) Integer order adaptive compensation scheme when $\nu = 1$.

Theorem 1. *If using the integer order adaptive compensation, the equilibrium points of e_θ and e_v are bounded as $t \rightarrow \infty$.*

Proof. From (2), (8) and our adaptation law (10), we can get

$$\begin{aligned} \dot{v}(t) &= (\dot{v}_d(t) + T_{l'} + \frac{1}{J}\hat{a}(t) + \alpha\gamma e_\theta(t) + (\alpha + \gamma)e_v(t)) \\ &\quad - \frac{1}{J}a(t) - T_{l'} - B_1(v_d(t) - e_v(t)) \\ &= \dot{v}_d(t) + \frac{1}{J}(z - \mu v(t)) + \alpha\gamma e_\theta(t) + (\alpha + \gamma)e_v(t) \\ &\quad - \frac{1}{J}a(t) - B_1(v_d(t) - e_v(t)) \\ &= \dot{v}_d(t) + \frac{1}{J}z + \alpha\gamma e_\theta(t) + (\alpha + \gamma)e_v(t) \\ &\quad - \frac{1}{J}a(t) - (\frac{1}{J}\mu + B_1)(v_d(t) - e_v(t)) \end{aligned}$$

$$\begin{aligned}
&= \dot{v}_d(t) + \alpha\gamma e_\theta(t) + (\alpha + \gamma + \frac{1}{J}\mu + B_1)e_v(t) \\
&\quad - \frac{1}{J}a(t) - (\frac{1}{J}\mu + B_1)v_d(t) + \frac{1}{J}z(t),
\end{aligned} \tag{13}$$

and from (7), we have

$$\begin{aligned}
\dot{e}_v &= \dot{v}_d(t) - \dot{v}(t) \\
&= -\alpha\gamma e_\theta(t) - (\alpha + \gamma + \frac{1}{J}\mu + B_1)e_v(t) \\
&\quad + \frac{1}{J}a(t) + (\frac{1}{J}\mu + B_1)v_d(t) - \frac{1}{J}z(t).
\end{aligned} \tag{14}$$

Then from (1), we further have

$$\begin{aligned}
\dot{e}_\theta &= \dot{\theta}_d(t) - \dot{\theta}(t) \\
&= v_d(t) - v(t) \\
&= e_v(t).
\end{aligned} \tag{15}$$

Substituting (15) into (14) yields

$$\begin{aligned}
\ddot{e}_\theta(t) &= -\alpha\gamma e_\theta(t) - (\alpha + \gamma + \frac{1}{J}\mu + B_1)\dot{e}_\theta(t) \\
&\quad + \frac{1}{J}a(t) + (\frac{1}{J}\mu + B_1)v_d(t) - \frac{1}{J}z(t).
\end{aligned} \tag{16}$$

Now, from (11), using integer order derivative, namely $\nu = 1$, yields

$$\dot{z}(t) = \mu[\dot{v}_d(t) + \alpha m(t) + \gamma e_v(t)] + \frac{e_v(t)}{J}. \tag{17}$$

Then, using formula for the Laplace transform of (16) and (17) leads to

$$\begin{aligned}
&s^2 E_\theta(s) + (\alpha + \gamma + \frac{1}{J}\mu + B_1)s E_\theta(s) + \alpha\gamma E_\theta(s) \\
&= \frac{1}{J}A(s) + (\frac{1}{J}\mu + B_1)V_d(s) + v_d(0) - \frac{1}{J}Z(s),
\end{aligned} \tag{18}$$

$$Z(s) = \frac{\mu}{s}(sV_d(s) - v_d(0)) + \frac{\mu}{s}\alpha\gamma E_\theta(s) + (\mu(\alpha + \gamma) + \frac{1}{J})E_\theta(s). \tag{19}$$

Substituting (19) into (18) we can get

$$\begin{aligned}
 E_\theta(s) &= \frac{1}{s^2 + a_0s + b_0 + \frac{1}{s}c_0} F_0(s) \\
 &= \frac{s}{s^3 + a_0s^2 + b_0s + c_0} F_0(s),
 \end{aligned} \tag{20}$$

where

$$a_0 = \alpha + \gamma + \frac{1}{J}\mu + B_1, \quad b_0 = \alpha\gamma + \frac{1}{J}\mu(\alpha + \gamma) + \frac{1}{J^2}, \quad c_0 = \frac{1}{J}\mu\alpha\gamma,$$

$$F_0(s) = \frac{1}{J}A(s) + B_1V_d(s) + (1 + \frac{1}{J_s}\mu)v_d(0).$$

As α, γ and μ are all positive design parameters, so a_0, b_0 and c_0 are all positive values, and we have

$$\begin{aligned}
 a_0b_0 - c_0 &= (\alpha + \frac{1}{J}\mu + B_1)[\alpha\gamma + \frac{1}{J}\mu\gamma^2 + \frac{1}{J^2}] \\
 &= \gamma[\alpha\gamma + \frac{1}{J}\mu(\alpha + \gamma) + \frac{1}{J^2}] \\
 &> 0.
 \end{aligned} \tag{21}$$

From Routh table technique, we can conclude that the system (20) is stable. Since $a(t)$ and $v_d(t)$ are bounded, from the inverse Laplace transform

$$f_0(t) = L^{-1}[F_0(s)],$$

we can conclude the input signal $f_0(t)$ in system (20) is bounded, so output signal $e_\theta(t)$ in system (20) is also bounded.

Furthermore, from (15), we have

$$\begin{aligned}
 E_v(s) &= sE_\theta(s) \\
 &= \frac{s^2}{s^3 + a_0s^2 + b_0s + c_0} F_0(s).
 \end{aligned} \tag{22}$$

In the same way, we can conclude that the system (22) is stable and the error signal $e_\theta(t)$ is also bounded.

So we can conclude that the equilibrium points of e_θ and e_v , are bounded as $t \rightarrow \infty$.

2.3 Stability Analysis of the Fractional Order Case

Now, let us consider the case (2) Fractional order adaptive compensation scheme as $0 < \nu < 1$. Our major result is summarized in the following theorem. First of all, the following lemma is needed for the proof of Theorem 2.

Lemma 1. *An ordinary input/output relation (with only integer derivatives) can be written in a polynomial representation*

$$\begin{aligned} P(\sigma)\xi &= Q(\sigma)u, \\ y &= R(\sigma)\xi. \end{aligned} \quad (23)$$

where $u \in \mathfrak{R}^{\bar{m}}$ is the control, $\xi \in \mathfrak{R}^{\bar{n}}$ is the partial state, and the $y \in \mathfrak{R}^{\bar{p}}$ is the output; P , Q and R are polynomial matrices in the variable σ of dimensions $\bar{n} \times \bar{n}$, $\bar{n} \times \bar{m}$ and $\bar{p} \times \bar{n}$ respectively; σ can be seen either as the symbol for the usual derivative d^v or s^v , the complex function in Laplace variable s , when all initial conditions are zero.

If the triplet (P, Q, R) of polynomial matrices is minimal, we have the following equivalence: system (23) is bounded-input bounded-output iff $\det(P(\sigma)) \neq 0 \forall \sigma$, $|\arg(\sigma)| < v\frac{\pi}{2}$.

For a proof of Lemma 1, see [8].

Theorem 2. *If choosing proper parameters α , γ and μ to ensure*

$$|\arg(w_i)| > v\frac{\pi}{2},$$

where w_i are the solutions of Eq. 24,

$$w^{2pq+p^2} + aw^{pq+p^2} + bw^{pq} + dw^{p^2} + c = 0, \quad (24)$$

where $v = p/q$, p and q are positive integers,

$$a = \alpha + \gamma + \frac{1}{J}\mu + B_1, \quad b = \frac{1}{J}\mu(\alpha + \gamma) + \frac{1}{J^2},$$

$$c = \frac{1}{J}\mu\alpha\gamma, \quad d = \alpha\gamma,$$

then the equilibrium points of $e_\theta(t)$ and $e_v(t)$ are bounded, as $t \rightarrow \infty$.

Proof. From (13), (14) and (15), we also can get,

$$\begin{aligned} \ddot{e}_\theta(t) &= -\alpha\gamma e_\theta(t) - (\alpha + \gamma + \frac{1}{J}\mu + B_1)\dot{e}_\theta(t) \\ &\quad + \frac{1}{J}a(t) + (\frac{1}{J}\mu + B_1)v_d(t) - \frac{1}{J}z(t). \end{aligned} \quad (25)$$

Since

$${}_0D_t^v z(t) = \mu[v_d(t) + \alpha m(t) + \gamma e_v(t)] + \frac{e_v(t)}{J}, \quad (26)$$

from Eqs. 2.113 and 2.115 in [11]

$${}_a D_t^{-p} ({}_a D_t^p f(t)) = f(t) - [{}_a D_t^{p-1} f(t)]_{t=a} \frac{(t-a)^{p-1}}{\Gamma(p)}, \quad (27)$$

where $0 < p < 1$.

$${}_a D_t^{-p} ({}_a D_t^q f(t)) = {}_a D_t^{q-p} f(t) - \sum_{j=1}^k [{}_a D_t^{q-j} f(t)]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(1+p-j)}, \quad (28)$$

where $0 < p$ and $0 < k-1 < q < k$.

As $z(t)|_{t=0} = 0$, $e_\theta(t)|_{t=0} = 0$ and $e_v(t)|_{t=0} = v_d(0)$, we can get

$$\begin{aligned} z(t) &= {}_0 D_t^{-\nu} ({}_0 D_t^\nu z(t)) \\ &= {}_0 D_t^{-\nu} \left\{ \mu [\dot{v}_d(t) + \alpha m(t) + \gamma e_v(t)] + \frac{e_v(t)}{J} \right\} \\ &= {}_0 D_t^{-\nu} \left\{ \mu [\dot{v}_d(t) + \alpha \gamma e_\theta(t) + (\alpha + \gamma) e_v(t)] + \frac{e_v(t)}{J} \right\} \\ &= \mu {}_0 D_t^{-\nu} \dot{v}_d(t) + \mu \alpha \gamma {}_0 D_t^{-\nu} e_\theta(t) + \left(\mu (\alpha + \gamma) + \frac{1}{J} \right) {}_0 D_t^{-\nu} e_v(t) \\ &= \mu {}_0 D_t^{-\nu} \dot{v}_d(t) + \mu \alpha \gamma {}_0 D_t^{-\nu} e_\theta(t) + \left(\mu (\alpha + \gamma) + \frac{1}{J} \right) {}_0 D_t^{1-\nu} e_\theta(t), \\ &= \mu ({}_0 D_t^{1-\nu} v_d(t) - v_d(0)) + \mu \alpha \gamma {}_0 D_t^{-\nu} e_\theta(t) \\ &\quad + \left(\mu (\alpha + \gamma) + \frac{1}{J} \right) {}_0 D_t^{1-\nu} e_\theta(t). \end{aligned} \quad (29)$$

By substituting (29) into (25)

$$\ddot{e}_\theta(t) + a \dot{e}_\theta(t) + b {}_0 D_t^{1-\nu} e_\theta(t) + c {}_0 D_t^{-\nu} e_\theta(t) + d e_\theta(t) = f(t), \quad (30)$$

where

$$\begin{aligned} a &= \alpha + \gamma + \frac{1}{J} \mu + B_1, \quad b = \frac{1}{J} \mu (\alpha + \gamma) + \frac{1}{J^2}, \\ c &= \frac{1}{J} \mu \alpha \gamma, \quad d = \alpha \gamma, \\ f(t) &= \frac{1}{J} a(t) + \left(\frac{1}{J} \mu + B_1 \right) v_d(t) - \frac{1}{J} \mu ({}_0 D_t^{1-\nu} v_d(t) - v_d(0)). \end{aligned}$$

From Eqs. 2.242 and 2.248 in [11]

$$L\{{}_0 D_t^{-p} f(t); s\} = s^{-p} F(s), \quad (31)$$

where $0 < p$.

$$L\{ {}_0D_t^p f(t); s \} = s^p F(s) - \sum_{k=1}^{n-1} s^k [{}_0D_t^{p-k-1} f(t)]_{t=0}, \quad (32)$$

where $0 \leq n-1 \leq p < n$.

Using formula for the Laplace transform of (30) leads to

$$\begin{aligned} & (s^2 E_\theta(s) - v_d(0)) + a(s + bs^{1-\nu} + cs^{-\nu} + d) E_\theta(s) \\ &= F(s) \\ &= \frac{1}{J} A(s) + \left(\frac{1}{J} \mu + B_1 \right) V_d(s) - \frac{\mu s^{1-\nu}}{J} v_d(s) + \frac{\mu}{J s} v_d(0), \end{aligned} \quad (33)$$

so we can obtain

$$\begin{aligned} & E_\theta(s) \\ &= \frac{1}{s^2 + as^1 + bs^{1-\nu} + cs^{-\nu} + d} (F(s) + v_d(0)) \\ &= \frac{s^\nu}{s^{2+\nu} + as^{1+\nu} + bs + c + ds^\nu} G(s), \end{aligned} \quad (34)$$

where

$$\begin{aligned} G(s) &= F(s) + v_d(0) \\ &= \frac{1}{J} A(s) + \left(\frac{1}{J} \mu + B_1 \right) V_d(s) - \frac{\mu s^{1-\nu}}{J} v_d(s) + \left(\frac{\mu}{J s} + 1 \right) v_d(0). \end{aligned} \quad (35)$$

Denote that

$$\nu = \frac{p}{q}, \quad s^\nu = s^{\frac{p}{q}},$$

where p and q are positive integers. So, we have

$$E_\theta(s) = \frac{s^{\frac{p}{q}}}{s^{2+\frac{p}{q}} + as^{1+\frac{p}{q}} + bs + c + ds^{\frac{p}{q}}} G(s). \quad (36)$$

Denoting

$$w = s^{\frac{1}{pq}},$$

then

$$E_\theta(w) = \frac{w^{p^2}}{w^{2pq+p^2} + aw^{pq+p^2} + bw^{pq} + dw^{p^2} + c} G(w). \quad (37)$$

From Lemma 1, $\bar{m} = \bar{n} = 1$, since we have

$$|arg(w)| > \nu \frac{\pi}{2},$$

then system (37) is bound-input bound-output. Since $a(t)$ and $v_d(t)$ are bounded, from the inverse Laplace transform

$$g(t) = L^{-1}[G(s)].$$

we can conclude that the input signal $g(t)$ in system (36) is bounded, so the output error signal $e_\theta(t)$ in system (36) is also bounded.

Furthermore, from (15), we have

$$\begin{aligned} E_v(w) &= w^{pq} E_\theta(w) \\ &= \frac{w^{p^2+pq}}{w^{2pq+p^2} + a_w w^{pq+p^2} + b_w w^{pq} + d_w w^{p^2} + c} G(w). \end{aligned} \quad (38)$$

Similarly, we can conclude that the system (39) is stable and the error signal $e_\theta(t)$ is also bounded. In summary, we can conclude that the equilibrium points of e_θ and e_v are bounded as $t \rightarrow \infty$.

3 Simulation Illustrations Using PMSM Position Servo System Model

In this section, we present two simulation tests to demonstrate the effectiveness of the proposed fractional order AC for cogging effect on the PMSM position servo control system model. Figure 1 shows the simulation block diagram. Details of this authentic simulation model can be found in [7].

- Case-1: Integer order adaptive compensation for multi-harmonics cogging effect
- Case-2: Fractional order adaptive compensation for the same cogging effect as in Case-1

For our simulation tests, the control gains in (8) were selected as: $\alpha = 50$, $\gamma = 20$ and $\mu = 3$. The motor parameters are given in Table 1 and the $T_l = 1[Nm]$. And the actual cogging force is modeled as the state-period sinusoidal signal of multiple harmonics below:

$$F_{cogging} = 2 \cos(6\theta) + \cos(12\theta) + 0.5 \cos(18\theta).$$

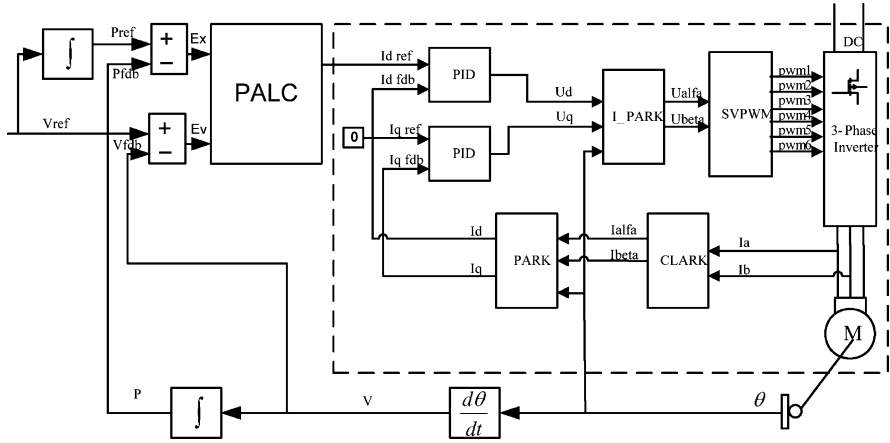


Fig. 1 Block diagram of the cogging adaptive compensation in the PMSM position servo system model

Table 1 PMSM specifications

Rated power	1.64 Kw	Rated speed	2,000 rpm
Rated torque	8 Nm	Stator resistance	2.125 Ω
Stator inductance	11.6 mH	Magnet flux	0.387
Number of poles	6	Moment of inertia	0.00289 kgm ²
Friction coefficient	0.0003 Nms		

3.1 Case-1: Integer Order Adaptive Compensation for Multi-Harmonics Cogging Effect

In this case simulation test, choosing $\nu = 1$ in adaptive law (10), namely, integer order periodic adaptive learning compensation scheme is used for cogging effect minimization. As cogging effect degrades the performance of PMSM seriously in low speed range, and normally, the motor speed below the 3% of the rated speed always can be treated as in the low speed range, in our simulation system model, the rated speed of the PMSM is 2,000 rpm, which is given in Table 1, so we choose the reference speed in this simulation as $5 \text{ rad/s} = 47.77 \text{ rpm} \leq 60 \text{ rpm}$. So, for this case simulation test, the following reference trajectory and velocity signals are used:

$$s_d(t) = 5t \text{ (rad)}, \quad (39)$$

$$v_d(t) = 5 \text{ (rad/s)}. \quad (40)$$

Figure 3(a,b) show the position/speed tracking errors with compensation using integer order adaptive compensation. The integer order adaptive compensation method works efficiently comparing with the tracking errors without compensation in Fig. 2(a, b).

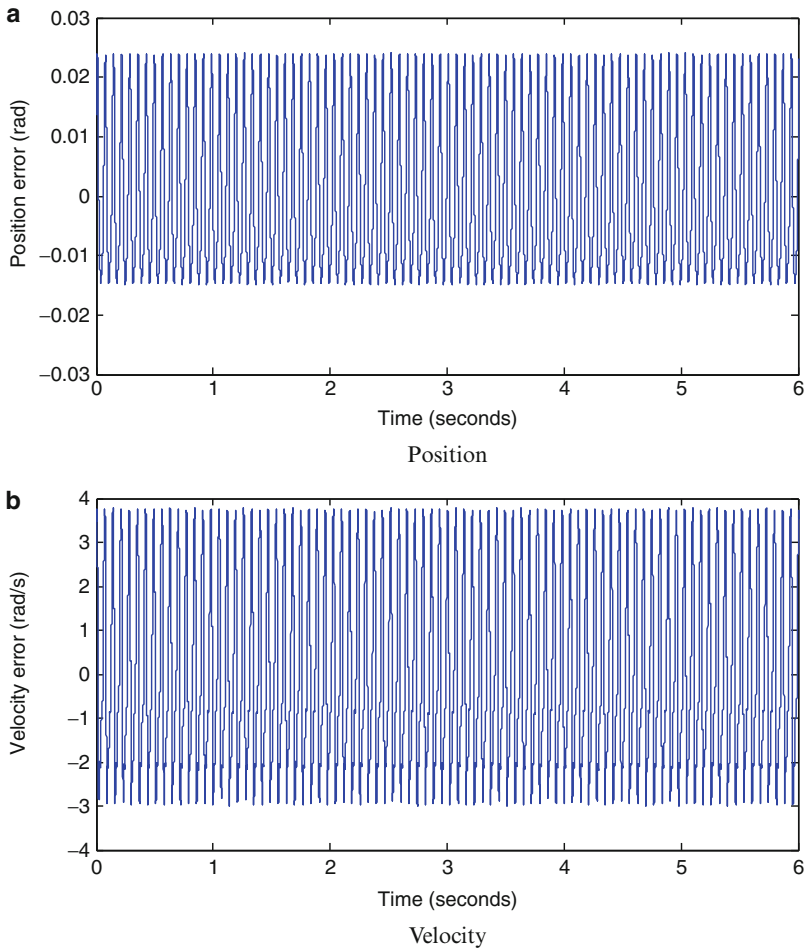


Fig. 2 Tracking errors without compensation

3.2 Case-2: Fractional Order Adaptive Compensation for Multi-Harmonics Cogging Effect

In this case, for a fair comparison, the reference trajectory (40) and velocity (40) signals in Case-1 are used; and we use fractional order adaptive compensation for cogging effect. Here, in fractional order adaptation law (10), we choose $\nu = 0.5$, and $\alpha = 50$, $\gamma = 20$ and $\mu = 3$, substituting the parameter values into (24), we can get five solutions

$$\begin{aligned} w_1 &= 25.68808 + 55.53037i; w_2 = 25.68808 - 55.53037i; \\ w_3 &= 0.01285 + 2.32261i; w_4 = 0.01285 - 2.32261i; \\ w_5 &= -51.40188, \end{aligned}$$

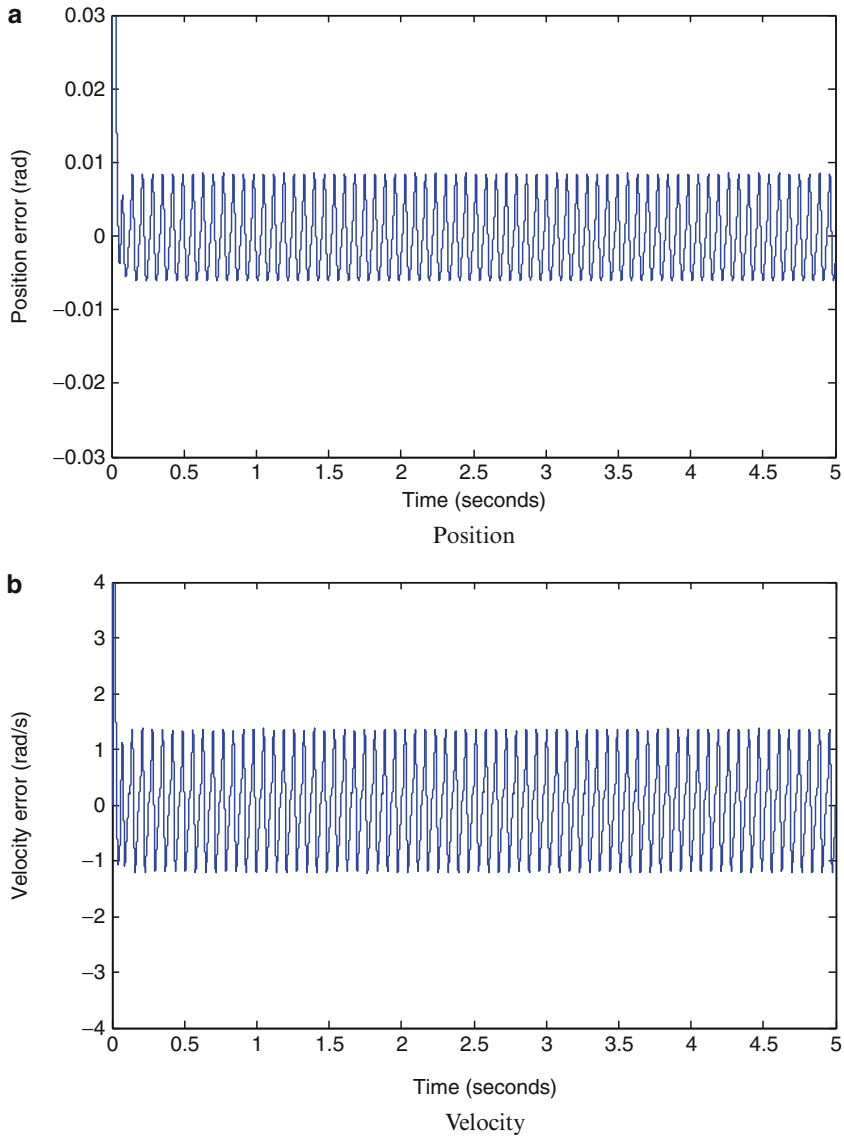


Fig. 3 Tracking errors with integer order adaptive compensation ($\nu = 1$)

so we can get

$$\begin{aligned}
 |arg(w_1)| &= |arg(w_2)| = \frac{67.18^\circ}{180^\circ} \pi = 0.3621\pi \\
 &> \nu \frac{\pi}{2} = 0.5 \frac{\pi}{2} = 0.25\pi;
 \end{aligned}$$

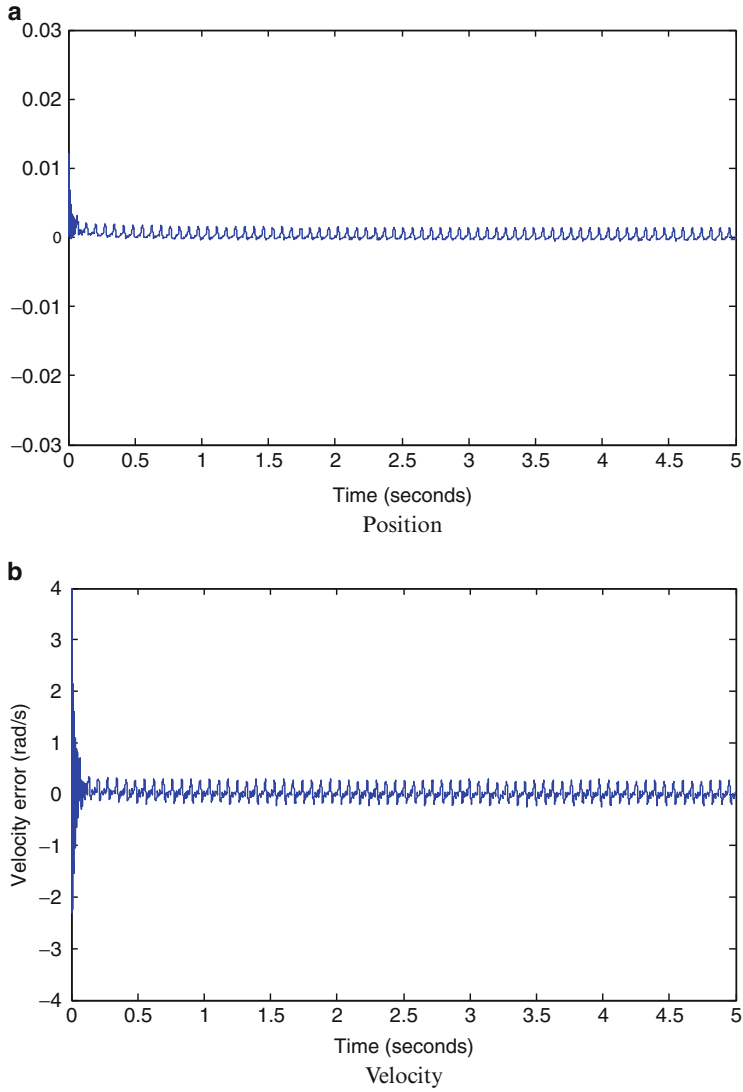


Fig. 4 Tracking errors with fractional order adaptive compensation ($\nu = 0.5$)

$$\begin{aligned}
 |arg(w_3)| &= |arg(w_4)| = \frac{89.68^\circ}{180^\circ} \pi = 0.4982\pi \\
 &> \nu \frac{\pi}{2} = 0.5 \frac{\pi}{2} = 0.25\pi; \\
 |arg(w_5)| &= \pi > \nu \frac{\pi}{2} = 0.5 \frac{\pi}{2} = 0.25\pi.
 \end{aligned}$$

So the system should be bounded-input bounded-output stable according to our stability analysis presented in the last section. In the approximate realization for fractional order derivative, the frequency range of interest is designed [14] as [0.001, 1000]. Figure 4a, b show the position/speed tracking errors using fractional order adaptive compensation. Comparing with Fig. 3a, b, we can clearly see that the performance of using fractional order adaptive compensation method is much better than that of using integer order adaptive compensation for multi-harmonics cogging effect.

Remark 3. In the course of simulation, we observed that changing the parameter μ does not improve the performance of the position and velocity tracking as significantly as changing the fractional order ν using the PMSM position servo simulation model.

4 Chapter Conclusion

In this chapter, for the first time, a fractional order adaptive compensation method is proposed to compensate the cogging effect in PMSM position and velocity servo system. In our FO-AC scheme, a new fractional order adaptive compensator for cogging effect is designed to guarantee the boundedness of all signals. Stability properties have been proven for the system with the traditional integer order adaptive compensator and the proposed fractional order adaptive compensator respectively. Simulation results are presented to illustrate the advantage of the proposed fractional order AC for cogging effect. This new fractional order scheme performs better than the traditional integer order method, the state position/speed tracking errors using the FO-AC is much smaller than that using the IO-AC. Furthermore, although the suggested FO-AC method is developed for the cogging force compensation, our method also can be used to compensate other state-dependent periodic disturbances. Extensive lab experimental validation has been done and will be reported elsewhere.

Acknowledgements Ying Luo would like to thank Dr. Youguo Pi for his valuable guidance of building the PMSM servo system, Dr. Huifang Dou for her expertise in cogging compensation of PMLM (permanent magnet linear motor) and Mr. Yan Li for his valuable suggestions on the fractional order adaptive compensation stability proof and to the China Scholarship Council (CSC) for the financial support. The authors acknowledge the benefits from the weekly Fractional Calculus Reading Group meeting at CSOIS (<http://mechatronics.ece.usu.edu/foc/yan.li/>).

References

1. Hyo-Sung Ahn, YangQuan Chen, and Huifang Dou. State-periodic adaptive compensation of cogging and coulomb friction in permanent-magnet linear motors. *IEEE Transactions on Magnetics*, 41(1):90 – 98, 2005.
2. S. Ladaci and A. Charef. On fractional adaptive control. *Nonlinear Dynamics*, 43(4): 365 – 378, 2006.

3. S. Ladaci and A. Charef. An adaptive fractional PID controller. In *Proceedings TMCE 2006 International Symposium Series on Tools and Methods of Competitive Engineering*, pages 1533 – 1540, Ljubljana, Slovenia, April 18 - 22 2006.
4. Ying Luo, YangQuan Chen, Hyo-Sung Ahn, and YouGuo Pi. Design of dual-high-order dynamic periodic adaptive learning controller for long-term cogging effect compensation. In *IEEE International Conference on Control, Automation, Robotics and Vision (ICARCV08) (accepted)*, Hanoi, Vietnam, 17-20 December 2008.
5. Ying Luo, YangQuan Chen, Hyo-Sung Ahn, and YouGuo Pi. A high order periodic adaptive learning compensator for cogging effect in PMSM position servo system. In *IEEE International Conference on Systems, Man, and Cybernetics (SMC08)(accepted)*, Singapore, 12-15 October 2008.
6. Ying Luo, YangQuan Chen, Hyo-Sung Ahn, and YouGuo Pi. A high order periodic adaptive learning compensator for cogging effect in PMSM position servo system. *IEEE Trans. on Magnetics (submitted)*, 2008.
7. Ying Luo, YangQuan Chen, and YouGuo Pi. Authentic simulation studies of periodic adaptive learning compensation of cogging effect in PMSM position servo system. In *Proceedings of the 20-th Chinese Conference on Decision and Control (CCDC08)*, Yantai, Shandong, China, July 2008.
8. D. Matignon. Stability results for fractional differential equations with applications to control processing. In *Computational Engineering in Systems Applications*, pages 963–968, Lille, France, July 1996.
9. C. A. Monje, B. M. Vinagre, V. Feliu, and Y. Q. Chen. Tuning and auto-tuning of fractional order controllers for industry applications. *IFAC: Control Engineering Practice*, 16(7):798–812, 2008.
10. K. B. Oldham and J. Spanier. *The Fractional Calculus*. Academic Press, New York and London, 1974.
11. I. Podlubny. *Fractional Differential Equations*. Academic Press, 1999.
12. B. M. Vinagre, I. Petras, I. Podlubny, and Y. Q. Chen. Using fractional order adjustment rules and fractional order reference models in model reference adaptive control. *Nonlinear Dynamics*, 29:269 – 279, 2002.
13. J.-X. Xu, S. K. Pands, Y.-J. Pan, and T. H. Lee. A modular control scheme for PMSM speed control with pulsating torque minimization. *IEEE Trans. on Ind. Electron.*, 51:526 – 536, 2004.
14. D. Y. Xue, C. N. Zhao, and Y. Q. Chen. Fractional order PID control of a DC-motor with elastic shaft: a case study. In *Proc. of American Control Conference*, pages 3182–3187, MN, USA, June 2006. AACC.

Generalized Predictive Control of Arbitrary Real Order

Miguel Romero Hortelano, Ángel Pérez de Madrid y Pablo, Carolina Mañoso Hierro, and Roberto Hernández Berlinches

Abstract This paper proposes Fractional–Order Generalized Predictive Control (FGPC), a model-predictive control methodology that makes use of a cost function of arbitrary real order. FGPC uses two scalar parameters that represent fractional-order differentiation. These parameters can be tuned to achieve closed-loop specifications in a way much easier and faster than using the classical GPC weighting sequences.

1 Introduction

Fractional Calculus can be defined as integration and differentiation of non-integer order. Fractional differentiation is the generalization of the derivative operator D^n using real or complex values for the ordinary integer value n [6, 7].

Two definitions used for the general fractional integro-differential calculus are the ones given by Grünwald–Letnikov – GL – (1) and Riemann–Liouville – RL – (2). For a wide class of functions, which appear in real physical and engineering applications, the two definitions GL and RL are equivalent. For this reason, RL is usually used for algebraic manipulations and GL (together with the short memory principle) for numerical integration and simulation [7].

$$D^\alpha f(t)_{t=kh} = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{i=0}^{\infty} (-1)^i \binom{\alpha}{i} f(kh - ih). \quad (1)$$

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau. \quad (2)$$

M.R. Hortelano (✉), A.P. de Madrid y Pablo, C.M. Hierro, and R.H. Berlinches
Escuela Técnica Superior de Ingeniería Informática Dept. Sistemas
de Comunicacion Control, UNED, Juan del Rosal 16. 28040 – Madrid (Spain)
e-mail: mromero@scc.uned.es; angel@scc.uned.es; carolina@scc.uned.es; roberto@scc.uned.es

Model predictive control (MPC), an industry standard, is an advanced process control method in which a dynamical model of the plant is used to predict and optimize the future behavior of the process by means of a cost function minimization [1, 5, 10]. Generalized Predictive Control (GPC) is one of the most representative predictive controllers due to its success in industrial and academic applications [2–4]. Recently, MPC has been proposed to control fractional order plants [8, 9].

In this paper we propose Fractional-Order Generalized Predictive Control (FGPC), a new type of predictive controller based on the minimization of a fractional-order cost function.

This paper is organized as follows: In the next section GPC is introduced. Section 3 gives some mathematical tools that are needed in the rest of the paper. Section 4 introduces FGPC. Next, in Sect. 5 an application example is proposed. Finally, Sect. 6 draws the main conclusions of this work.

2 Generalized Predictive Control

The GPC control law is obtained by minimizing the cost function in (3), where $E\{\cdot\}$ is the expectation operator, Δ is the increment operator, N_1 and N_2 are the minimum and maximum costing horizons, respectively, N_u represents the control horizon, γ is a future errors weighting sequence, and λ is a control weighting sequence. The notation $x(j|t)$ stands for the predicted value of $x(j)$ made at time instant t .

$$J(\Delta u, t) = E \left\{ \sum_{j=N_1}^{N_2} \gamma(j) [r(t+j|t) - y(t+j|t)]^2 + \sum_{j=1}^{N_u} \lambda(j) [\Delta u(t+j-1|t)]^2 \right\} \quad (3)$$

GPC uses CARIMA (Controlled auto-regressive integrated moving-average) models to describe the system dynamics:

$$A(z^{-1})y(t) = B(z^{-1})u(t) + \frac{T(z^{-1})}{\Delta} \xi(t), \quad (4)$$

where $B(z^{-1})$ and $A(z^{-1})$ are the numerator and denominator of the transfer function, respectively, $\xi(t)$ represents uncorrelated zero-mean white noise and $T(z^{-1})$ is a (pre)filter to improve the system robustness.

The task of finding the parameters N_1 , N_2 , N_u , γ , and λ is critical, as they determine the closed loop stability. Thumb-rules given by Clarke et al. [2] are adequate for a wide range of applications: $N_1 = 1$, $N_2 = 10$, $\lambda = 10^{-6}$, $\gamma = 1$, and N_u equal to the number of unstable or badly-damped poles of the system. Sometimes λ is chosen an exponentially increasing or decreasing sequence: a sequence $\lambda(k) < \lambda(k+1)$ gives rise to smooth control and a sequence $\lambda(k) > \lambda(k+1)$ produces tighter controls [1].

3 Some Mathematical Results

In this section we give a discrete method to evaluate

$$\int_a^b D^\alpha f(t) dt, \quad (5)$$

where, for the sake of simplicity in the notation and without loss of generality, a and b are chosen to be multiples of the sampling time Δt . This calculation will be needed in the following section to evaluate the fractional-order cost function of FGPC.

The fractional derivative D^α in (5) can be evaluated using the Grünwald–Letnikov (1) definition:

$$\int_a^b D^\alpha f(t) dt = \int_a^b \left[\Delta t^{-\alpha} \sum_{i=0}^{\infty} (-1)^i \binom{\alpha}{i} f(t - i \Delta t) \right] dt. \quad (6)$$

This expression can be expanded as

$$\begin{aligned} \int_a^b D^\alpha f(t) dt &= \Delta t^{-\alpha} \left[\int_a^b (-1)^0 \binom{\alpha}{0} f(t) dt + \right. \\ &\left. + \int_a^b (-1)^1 \binom{\alpha}{1} f(t - \Delta t) dt + \int_a^b (-1)^2 \binom{\alpha}{2} f(t - 2\Delta t) dt + \dots \right]. \end{aligned} \quad (7)$$

Using the standard notation $\omega_x \equiv (-1)^x \binom{\alpha}{x}$, we write

$$\begin{aligned} \int_a^b D^\alpha f(t) dt &= \Delta t^{-\alpha} \left[\int_a^b \omega_0 f(t) dt + \int_a^b \omega_1 f(t - \Delta t) dt + \right. \\ &\left. + \int_a^b \omega_2 f(t - 2\Delta t) dt + \int_a^b \omega_3 f(t - 3\Delta t) dt + \dots \right]. \end{aligned} \quad (8)$$

The latter expression is an infinite summation of definite integrals $\omega_p \int_a^b f(t - p\Delta t) dt$ that can be evaluated as

$$\omega_p \int_a^b f(t - p\Delta t) dt = \omega_p [F_p(b) - F_p(a)], \quad (9)$$

where F_p is a primitive function of $f(t - p\Delta t)$.

Expression (9) can be expanded using the Grünwald–Letnikov definition

$$\omega_p \int_a^b f(t - p\Delta t) dt = \omega_p \left[\left(\Delta t \sum_{i=0}^{\infty} (-1)^i \binom{-1}{i} f(b - (i + p)\Delta t) \right) - \left(\Delta t \sum_{i=0}^{\infty} (-1)^i \binom{-1}{i} f(a - (i + p)\Delta t) \right) \right], \quad (10)$$

and (10) is now evaluated for each different value of p . The general term is

$$\omega_p \int_a^b f(t - p\Delta t) dt = \omega_p \Delta t [f(b - p\Delta t) - f(a - p\Delta t) + f(b - (2 + p)\Delta t) - f(a - (2 + p)\Delta t) + f(b - (3 + p)\Delta t) - f(a - (3 + p)\Delta t) + \dots]. \quad (11)$$

Now (8) can be summed up:

$$\int_a^b D^\alpha f(t) dt = \Delta t^{1-\alpha} \cdot \bar{k} \cdot \bar{f}', \quad (12)$$

where

$$\bar{f} = (..., f(-\Delta t), f(0), f(\Delta t), ..., f(a - \Delta t), f(a), ..., f(b - \Delta t), f(b)), \quad (13)$$

$$\begin{aligned} \bar{k} = & (\dots, \omega_{B+1} + \omega_B + \dots + \omega_{A+2}, \omega_B + \omega_{B-1} + \dots + \omega_{A+1}, \\ & \omega_{B-1} + \omega_{B-2} + \dots + \omega_A, \dots, \omega_{B-n+1} + \omega_{B-n} + \dots + \omega_{A-n+2}, \\ & \omega_{B-n} + \omega_{B-n-1} + \dots + \omega_{A-n+1}, \dots, \omega_1 + \omega_0, \omega_0), \end{aligned} \quad (14)$$

$$A = a/\Delta t, B = b/\Delta t, n = B - A. \quad (15)$$

The integral (12) has infinite memory as $\bar{f}$ and $\bar{k}$ have an infinite number of terms. However, in practice, due to the short memory principle [7] only a finite number of terms is needed.

4 Fractional-Order Generalized Predictive Control

The cost function of fractional-order generalized predictive control (FGPC) is defined as follows (compared with (3), the expectation operator E and the notation $|t$ have not been explicitly written for the sake of simplicity):

$$J(\Delta u, t) = \int_{N_1}^{N_2} D^\alpha [r(t) - y(t)]^2 dt + \int_1^{N_u} D^\beta [\Delta u(t - 1)]^2 dt. \quad (16)$$

According to (12) we can discretize (16):

$$J(\Delta u, t) = (\Delta t^{1-\alpha} \cdot \bar{e} \cdot \bar{\gamma} \cdot \bar{e}') + (\Delta t^{1-\beta} \cdot \overline{\Delta u} \cdot \bar{\lambda} \cdot \overline{\Delta u}'), \quad (17)$$

with $\bar{\gamma}$ and $\bar{\lambda}$ infinite-dimensional square weighting matrices. Thus, the latter expression has infinite memory and J depends both on an infinite number of past terms of e and Δu and a finite number of predicted future terms of e and Δu .

To take into account just the future values of e and Δu in the intervals of interest, $[N_1, N_2]$ and $[1, N_u]$, respectively, we shall truncate (17) as

$$J(\Delta u, t) \simeq e\gamma(\alpha, \Delta t)e' + \Delta u\lambda(\beta, \Delta t)\Delta u', \quad (18)$$

where $\gamma(\alpha, \Delta t)$ and $\lambda(\beta, \Delta t)$ are given by (19) and (20), respectively, and $m = N_2 - N_1$. By doing so we obtain a classical GPC formulation with weighting sequences γ and λ that are given by the sampling time Δt and the fractional orders of derivation α and β . The optimal control law is obtained by minimizing (18) using conventional GPC techniques [1–5, 10].

$$\gamma = \Delta t^{1-\alpha} \begin{pmatrix} \omega_m + \omega_{m-1} + \dots + \omega_{N_1-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \omega_{m-1} + \omega_{m-2} + \dots + \omega_{N_1-2} & 0 & \dots & 0 & 0 \\ 0 & 0 & \omega_{m-2} + \omega_{m-3} + \dots + \omega_{N_1-3} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \omega_1 + \omega_0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \omega_0 \end{pmatrix} \quad (19)$$

$$\lambda = \Delta t^{1-\beta} \begin{pmatrix} \omega_{N_u-1} + \omega_{N_u-2} + \dots + \omega_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \omega_{N_u-2} + \omega_{N_u-3} + \dots + \omega_0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \omega_{N_u-3} + \omega_{N_u-4} + \dots + \omega_0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \omega_1 + \omega_0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \omega_0 \end{pmatrix} \quad (20)$$

5 FGPC: An Application Example

In the following we shall consider the control of the plant (21) with GPC and FGPC.

$$G(z^{-1}) = \frac{1 - 2z^{-1}}{1 - 0.9z^{-1}}z^{-1}. \quad (21)$$

This plant is a non-minimum phase process discretized with $\Delta t = 1$ s. In order to control this kind of plant it is necessary to ensure that both N_u and N_2 are sufficiently large [10].

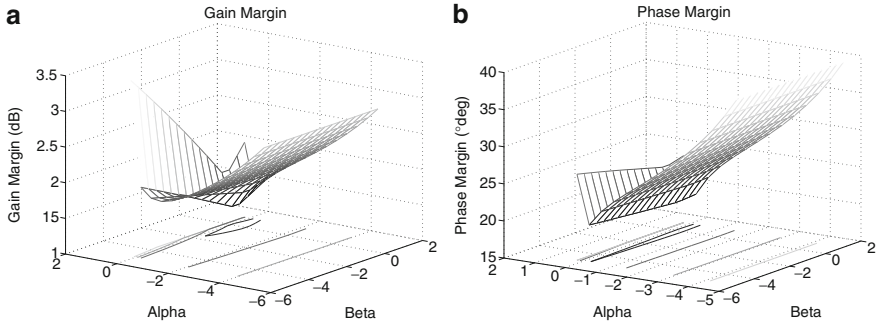


Fig. 1 Gain and phase margins of the FGPC system

In the following GPC and FGPC controllers will be calculated using (21) as the process model and settings $N_1 = 1$, $N_u = 2$, $N_2 = 10$, and $T(z^{-1}) = 1$. No model-process mismatch will be assumed.

GPC is tuned with constant sequences $\lambda = 10^{-6}$ and $\gamma = 1$ (default settings). The performance of this control system is illustrated in Fig. 2. It is stable and presents a gain margin of 1.50 dB and a phase margin of 21.60° .

For FGPC, the weighting sequences γ and λ are determined from the corresponding fractional-differentiation orders, α and β , using (19) and (20). To illustrate FGPC, Fig. 1 depicts all the FGPC gain and phase margins in the interval $\alpha, \beta \in (-5, 1)$. Each pair (α, β) defines a different FGPC controller. For instance, $(\alpha, \beta) = (-0.2, 0.2)$ gives rise to a “fast” system with a gain margin of 1.15 dB and a phase margin of 18.85° , whereas the pair $(\alpha, \beta) = (-3.4, 0.6)$ produces a “slow” system with a gain margin of 1.30 dB and a phase margin of 31.67° . For these controllers the weighting sequences $\gamma(\alpha, \Delta t)$ and $\lambda(\beta, \Delta t)$ are as follows:

$$\begin{cases} \gamma(-0.2, 1) = \text{diag} (0.7123 \ 1.675 \ 1.6342 \ 1.5888 \ 1.5375 \\ \quad \quad \quad 1.4784 \ 1.408 \ 1.32 \ 1.2 \ 1) \\ \lambda(0.2, 1) = \text{diag} (-0.2 \ 1) \end{cases} \quad (22)$$

$$\begin{cases} \gamma(-3.4, 1) = \text{diag} (358.96 \ 261.26 \ 183.34 \ 123.40 \ 78.769 \\ \quad \quad \quad 46.886 \ 25.344 \ 11.88 \ 4.4 \ 1) \\ \lambda(0.6, 1) = \text{diag} (-0.6 \ 1) \end{cases} \quad (23)$$

The fractional-differentiation orders α and β can be considered as high-level parameters to define weighting sequences $\gamma(\alpha, \Delta t)$ and $\lambda(\beta, \Delta t)$. In this way, α and β can be used to achieve closed-loop specifications (performance, robustness, etc.) in a way much easier than trying to find non-constant γ and λ sequences by trial and error or by optimization techniques.

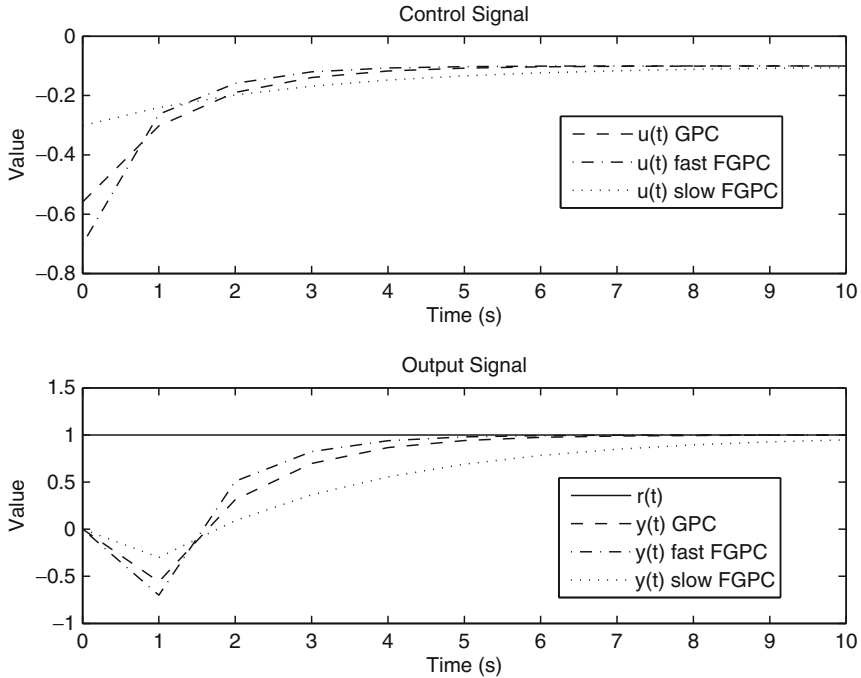


Fig. 2 GPC and FGPC performance

6 Conclusions

This paper has introduced fractional-order GPC, a predictive controller that makes use of a fractional-order cost function. This controller has two new tuning parameters, α and β , that represent fractional-differentiation orders of the predicted future errors and control efforts, respectively. They are used instead of the GPC weighting sequences γ and λ .

It has been shown that α and β define two conventional weighting sequences $\gamma(\alpha, \Delta t)$ and $\lambda(\beta, \Delta t)$. In this way, FGPC is equivalent to a GPC controller with non-constant weights given by α and β . However, α and β can be easily used to numerically optimize the closed-loop performance and fulfill given specifications.

Instead of two (long) weighting sequences γ and λ , just two scalar parameters α and β are needed.

References

1. Camacho EF, Bordóns C (2004) Model predictive control, 2nd ed. Springer, London
2. Clarke DW, Mohtadi C, Tuffs PS (1987a) Generalized predictive control. Part I. The basic algorithm. *Automatica* 23(2):137–148

3. Clarke DW, Mohtadi C, Tuffs PS (1987b) Generalized predictive control. Part II. Extensions and interpretations. *Automatica* 23(2):149–160
4. Clarke DW (1988) Application of generalized predictive control to industrial process. *IEEE Cont Syst Mag* 122:49–55
5. Maciejowski JM (2002) Predictive control with constraints. Prentice Hall, Harlow, UK
6. Oldham KB, Spanier J (1974) The fractional calculus. Academic, New York
7. Podlubny I (1999) Fractional differential equations. Mathematics in science and engineering. Academic, San Diego, California
8. Romero M, de Madrid AP, Mañoso C, Hernández R (2007) Application of generalized predictive control to a fractional Order plant. Proceedings of IDETC07. Las Vegas, USA
9. Romero M, Vinagre BM, de Madrid AP (2008) GPC control of a fractional-order plant: improving stability and robustness. Proceedings of 17th IFAC world congress. Seoul, Korea
10. Rossiter JM (2003) Model-based predictive control. A practical approach. CRC, Boc Raton

Frequency Response Based CACSD for Fractional Order Systems

Robin De Keyser, Clara Ionescu, and Corneliu Lazar

Abstract Computer Aided Design methods are nowadays very attractive, especially for non-experts in control engineering, due to their highly interactive graphics. Matlab offers currently a controller design tool based on the root locus technique. This contribution presents a Computer Aided Control System Design (CACSD) software, based on the use of Nichols frequency charts: FRtool. Apart from its easy-to-use and spec-oriented control system design, FRtool tackles generalized order processes and controllers. The CACSD tool is illustrated on two non-integer order processes.

1 Introduction

In the past, theoretical insight on closed-loop behaviour has been extensively used as an analysis tool – however, analysis implies that a controller is already available, irrespective of its design method. Nowadays, thanks to the computational and graphical power of modern computers, many of these theories can be implemented as interactive graphical design tools. In this way, control engineering moves away from being an abstract and mathematical-oriented discipline and it evolves gradually towards a mature engineering discipline.

Good examples of teaching automatic control in an interactive manner are Ictools and CCSdemo [7, 12] and SysQuake [10]. Their net advantage of allowing direct object manipulation is an attempt of demystifying abstract mathematical concepts. This new way of interactive control education provides practical insights into control

R. De Keyser (✉) and C. Ionescu
Ghent University, Department of Electrical energy, Systems and Automation,
Technologiepark 913, B9052 Gent, Belgium
e-mail: rdk@autoctrl.UGent.be; clara@autoctrl.UGent.be

C. Lazar
Gh Asachi Technical University of Iasi, Department of Automation, Mangeron Blvd 53A,
Iasi, Romania
e-mail: clazar@ac.tuiasi.ro

systems fundamentals. Such combinations of interactive environment and animation bring visualization to a new level and aid learning and active participation by control engineering students.

Recently, a Root Locus toolbox (rltool) has been introduced in Matlab, as well as a Frequency Response toolbox (FRtool) [3]. Although some other CACSD tools based on frequency response are available [1, 4, 5], the one presented in [3] has several advantages. It is highly interactive, graphical, easy-to-use and posing an elegant simplicity (especially for non-experts, as limited control engineering insight is sufficient). It can also tackle the problem of time-delay systems by operating with frequency diagrams (Nichols charts) – and in this way, the dead-time can be treated without any approximation just as a simple phase shift. The contribution in the current paper improves our frequency response CACSD package in generalizing the transfer functions of the system to both integer and non-integer order Laplace coefficients.

The paper is structured as follows: the FRtool graphical interface is briefly depicted in the next section, followed in the third section by a description of the underlying frequency response concepts for general order systems. In the fourth section, two examples are given on a biomedical application and a final section concludes this contribution.

2 Graphical Interface

Probably the most important feature of the FRtool is the user-friendly graphical interface. It has the possibility to display design specifications as graphical restrictions on the Nichols plot – including a real-time update while dragging controller's poles and zeros. It can also import/export process and controller from or to the Matlab workspace and has options to print Nichols, Nyquist or Bode curves and closed-loop responses.

Some of the traditional design specifications are gain margin and phase margin [8]. However, these specifications have not necessarily a clear physical meaning to a potential user (unless this user is e.g. a control engineer) – they are based on mathematical insight and system theory. Therefore, more practical specifications – which can be easily interpreted by any user – are settling-time and overshoot of the closed-loop time response, and of course robustness of the design. The design specifications can be introduced using the options denoted in Fig. 1 in the bolded dashed green line; e.g. the overshoot (%OS) and the robustness (Ro) specifications are visible in the chart. Closed-loop performance can be evaluated with the options in the dashed-dotted blue line. The upper right window in Fig. 1 is used to design the compensator $C(s)$ by dragging compensator's poles and zeros with the mouse. Additional to the process $P(s)$, pre-filter $F(s)$ or feedback $H(s)$ transfer functions can be added within the control scheme depicted in Fig. 2.

In Fig. 1, the PHC-curve corresponds to the loop Nichols curve, with a default controller equal to 1 (and in this example also $H = 1$) [3]. The user has then

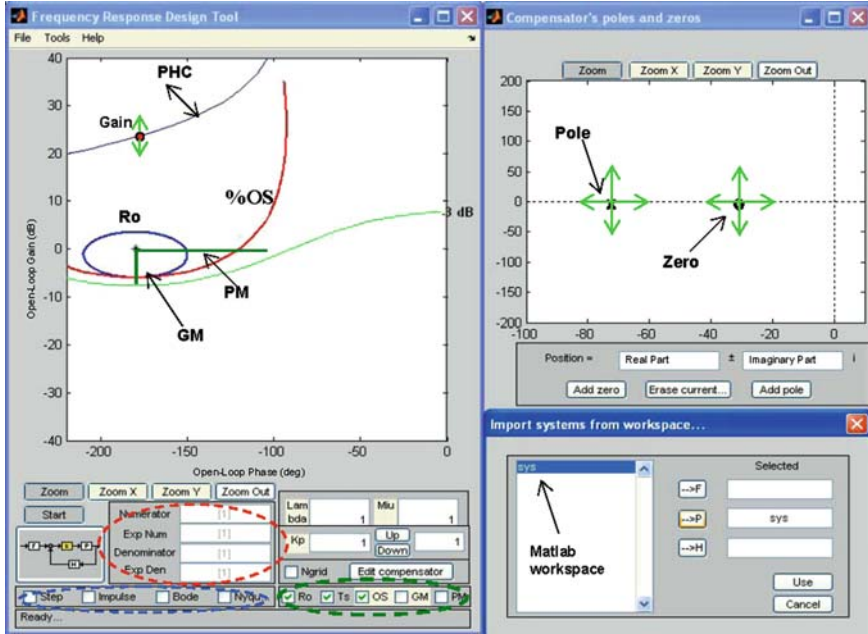


Fig. 1 Graphical interface of FRtool. After the system has been imported from Matlab workspace (see window in the lower right part) or explicitly in the area denoted by the red dashed line, it appears as a curve in the Nichols chart corresponding to the loop frequency response (PHC)

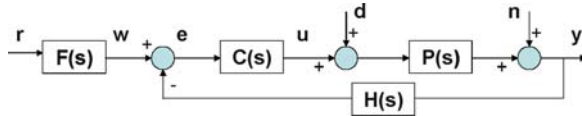


Fig. 2 General scheme of a control loop: r – reference, w – setpoint, e – error, u – the manipulated variable, d and n – disturbances, y – the controlled variable

to play with the controller's poles and zeros to fulfill the specifications by pure visual inspection in the Nichols chart. The frequency-domain insight on the desired specifications and the influence of playing with the controller's zeros–poles has been published previously [3].

3 Generalized Order Transfer Function

For the typical control scheme depicted in Fig. 2, consider a general order transfer function:

$$P(s) = \frac{a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \dots}{b_1 s^{\beta_1} + b_2 s^{\beta_2} + \dots} e^{-\tau_d s} \quad (1)$$

with τ_d the time delay. This transfer function can be equivalently represented as: $P(j\omega) = Re(j\omega) + j Im(j\omega)$, with the real and imaginary parts given as:

$$Re(j\omega) = \frac{Re_1 Re_2 + Im_1 Im_2}{Re_2^2 + Im_2^2}, \quad Im(j\omega) = \frac{Im_1 Re_2 - Re_1 Im_2}{Re_2^2 + Im_2^2} \quad (2)$$

which contain the real and imaginary parts of the numerator (Re_1 and Im_1), respectively the denominator (Re_2 and Im_2) of (1). For $j = \cos \frac{\pi}{2} + j \sin \frac{\pi}{2}$, we have $s = [\cos \frac{\pi}{2} + j \sin \frac{\pi}{2}] \omega$. Using complex number notation, all fractional differentiators are represented as in:

$$s^k = \omega^k [\cos \frac{k\pi}{2} + j \sin \frac{k\pi}{2}] \quad (3)$$

resulting in:

$$\begin{aligned} Re_1 &= [a_1 \omega^{\alpha_1} \ a_2 \omega^{\alpha_2} \ \dots] \begin{bmatrix} \cos(\alpha_1 \frac{\pi}{2} - \tau_d \omega) \\ \cos(\alpha_2 \frac{\pi}{2} - \tau_d \omega) \\ \dots \end{bmatrix}, \\ Im_1 &= [a_1 \omega^{\alpha_1} \ a_2 \omega^{\alpha_2} \ \dots] \begin{bmatrix} \sin(\alpha_1 \frac{\pi}{2} - \tau_d \omega) \\ \sin(\alpha_2 \frac{\pi}{2} - \tau_d \omega) \\ \dots \end{bmatrix} \\ Re_2 &= [b_1 \omega^{\beta_1} \ b_2 \omega^{\beta_2} \ \dots] \begin{bmatrix} \cos(\beta_1 \frac{\pi}{2}) \\ \cos(\beta_2 \frac{\pi}{2}) \\ \dots \end{bmatrix}, \\ Im_2 &= [b_1 \omega^{\beta_1} \ b_2 \omega^{\beta_2} \ \dots] \begin{bmatrix} \sin(\beta_1 \frac{\pi}{2}) \\ \sin(\beta_2 \frac{\pi}{2}) \\ \dots \end{bmatrix} \end{aligned} \quad (4)$$

At the present moment, we are able to import systems of any order, with the limitation that in case of non-integer order system, only frequency domain charts can be generated (Nichols, Bode, Nyquist). Step and impulse responses can be plotted only if the equivalent time-domain approximation of the non-integer order systems is used [9].

4 Illustrative Examples

In this section, two illustrative examples will be given, to help the reader understand the advantages of using CACSD tools for tuning controllers instead of analytic methods, such as nonlinear minimization.

4.1 Ventilatory Support in COPD Patients

Consider the following biomedical application of artificial ventilation systems in COPD (Chronic Obstructive Pulmonary Disease) diagnosed patients [2]. These patients need a ventilatory support to ensure the optimal aeration of their lungs with minimal effort. Such a system consists of the patient, the ventilator and the controller, as schematically depicted in Fig. 3.

The transfer function which denotes the impedance of the respiratory system of the COPD patient is given by:

$$P(s) = Ls^\alpha + \frac{1}{Cs^\beta} \quad (5)$$

with $L = 0.25kPas^2/l$ the inertance; $C = 0.034l/kPa$ the compliance; $\alpha = 0.41$ and $\beta = 0.65$. To this transfer function, the effects of the ventilator have to be added [11]. In a simplified form, the mechano-electrical dynamics of the motor for the air-pump and the tubing system can be considered as:

$$V(s) = \frac{1}{(0.01s + 1)(0.02s + 1)(s + 1)} \quad (6)$$

Further on, (5) and (6) are imported in the FRtool environment as a frequency response over the range $(10^{-3}; 10^2)\text{Hz}$. No dynamics are taken into account for $H(j\omega)$, thus the sensor measurements are considered as being very fast ($=1$). For the illustrative example, the specifications required are the following: robustness $Ro > 0.5$ (on a scale from $0 \rightarrow 1$); overshoot $\%OS < 50\%$; and for settling time $Ts < 1s$. Once the specs are set, one can proceed with tuning of the controller. The specifications can be achieved by adding poles and zeros in the compensator's transfer function and adjusting the gain K . For the purpose of this example, a PI controller has been designed. The controller transfer function which satisfied the specs as best as possible in this case was given by:

$$C(s) = 0.7614 \frac{s + 0.0076316}{s} \quad (7)$$

and the corresponding Nichols diagram in FRtool is given in Fig. 4. In order that the settling time specification is fulfilled the little red circle has to be above the 3 dB line. In this case, it can be observed that the specification for the settling time ($Ts <$

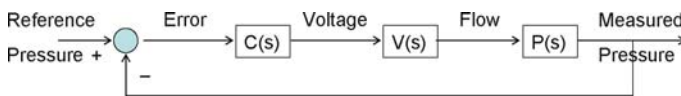


Fig. 3 Schematic representation of the closed loop for artificial ventilation

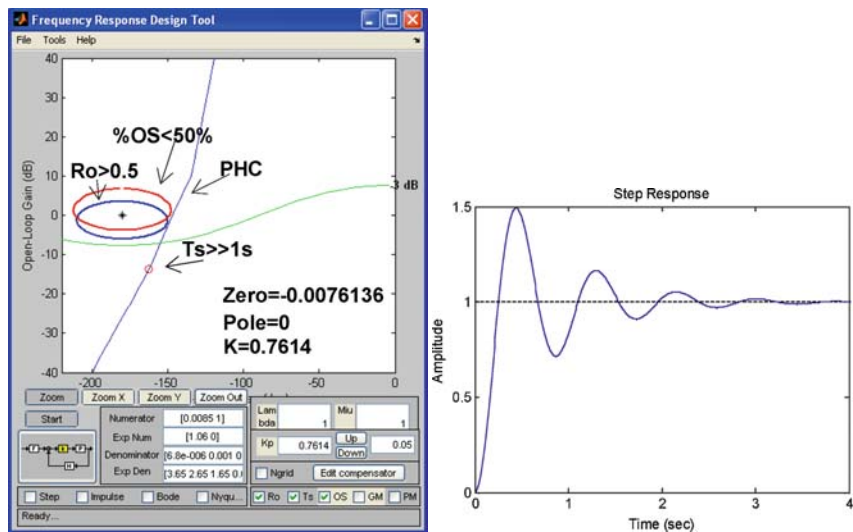


Fig. 4 *Left*: schematic representation of the final design for artificial ventilation in COPD; *right*: closed loop response for a unit step input

Table 1 COPD: transfer function coefficients of the approximated integer-order total transmittance

Order	Numerator COPD	Denominator COPD
6	9.12	1
5	1.9e4	4127
4	1.21e7	3.67e6
3	2.22e9	8.85e8
2	1.93e11	5.8e10
1	5.57e12	1.03e12
0	5.4e13	3.94e12

1s) will not be fulfilled, thus one can expect higher settling time than specified. Further on, the controller is tested in a simulation for the closed loop response. In order to simulate this in time domain, an approximation for the fractional order system is performed and the corresponding transfer function coefficients are given in Table 1 [9].

The response to a unit step input is depicted in Fig. 4. It can be seen that the controller does not reach the reference in the desired settling time ($< 1s$). In this case, the conclusion is that a PI controller cannot fulfill the specifications (only two degrees of freedom to change the dynamics: the location of the zero and the gain). Better results are obtained with a model-free adaptive controller, as presented recently in [6].

4.2 Ventilation in Anesthetised Animals

The second application concerns ventilation for anesthetised cats/dogs, during surgery. The model is then given in the form:

$$P(s) = R + Ls + \frac{1}{Cs^\beta} \quad (8)$$

with $R = 6.48kPas/l$, $L = 0.091kPas^2/l$, $C = 0.022l/kPa$ and $\beta = 0.823$. The same ventilator transfer function has been applied as in previous section. In this application, the intention is to design a PID controller, such that the following specifications are satisfied: $Ro > 0.7$; $\%OS < 20\%$ and $T_s < 1s$. The result in FRtool CAD package is depicted in Fig. 5, and the corresponding controller transfer function is given by:

$$C(s) = 0.19203 \frac{(s + 14.5517)^2}{s} \quad (9)$$

The controller performance in closed loop has to be evaluated and we need the time domain approximation of the transfer function from (8), which is given in Table 2. Recall that we need this approximation to do time-domain analysis.

In order to test the robustness of the closed loop, consider that the resistance changed its value to $R = 3.63kPas/l$ and the fractional order to $\beta = 0.95$ (increased viscosity). The step response for the nominal and changed transfer function (without varying controller parameters) is given in Fig. 5.

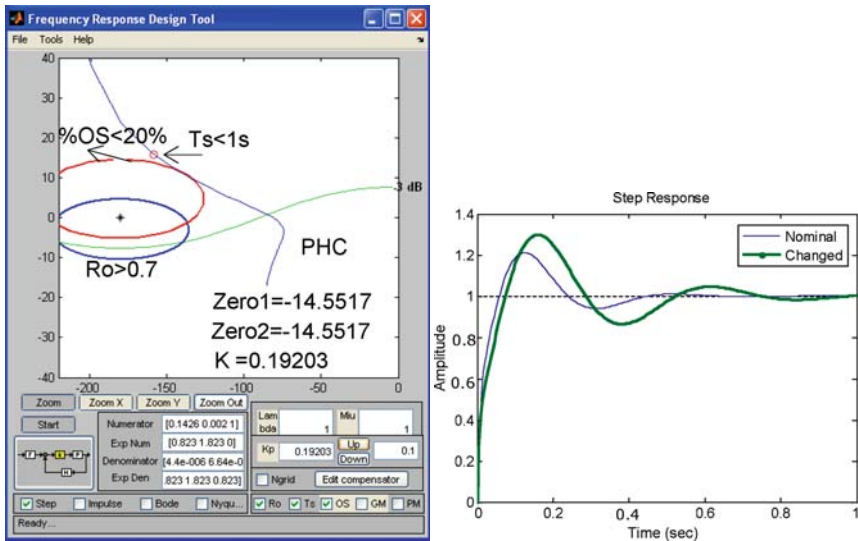


Fig. 5 Left: schematic representation of the final design for anesthesia; right: the unit step response for the closed-loop in the nominal case (continuous line) and in the changed case (bolded line), without re-tuning the controller parameters

Table 2 Transfer function coefficients of the approximated integer-order total transmittance

Order	Numerator	Denominator
6	578.6	1
5	7.69e5	7532
4	3.24e8	8.29e6
3	5.11e10	2.81e9
2	5.12e12	2.21e11
1	2.25e14	5.35e12
0	3.33e15	1.89e13

It can be observed that in the nominal case, we have an overshoot of 21.2% and settling time about 0.7 s; while in the changed case we have 29.8% overshoot and settling time of about 1 s. Thanks to the fact that a specification for high robustness has been given ($Ro = 0.7$, on a scale from 0 to 1) when the controller has been tuned, the system is still stable and has reasonably good performance, although significant changes in gain and time constants occurred.

5 Conclusions

This contribution presents an upgrade of our in-house Computer Aided Control System Design (CACSD) package for frequency response controller design for non-integer order transfer functions. The CACSD package is based on the frequency response (Nichols chart) representation of systems and entitled FRtool – Frequency Response toolbox. The toolbox can perform controller design with given practical specifications such as overshoot, settling time, robustness (and also 'classical' gain and phase margins as an option). Another attractive feature of FRtool is that it deals with time-delay systems in an elegant manner (phase shift). The use of FRtool for CACSD has been also illustrated on two biomedical examples.

Acknowledgements The authors would like to acknowledge the technical support of Miss Mirabela Tun.

References

1. Balakrishnan V, Boyd S (1994) Trade-offs in frequency-weighted H8-control. In: Proceedings of IEEE/IFAC joint symposium on computer aided control systems design. Arizona, pp 469–474
2. Barnes PJ (2000) Chronic obstructive pulmonary disease. *N Engl J Med* 343(4):269–280
3. De Keyser R, Ionescu C (2006) FRtool: a frequency response tool for CACSD in MatLab. In: IEEE conference on computer aided control systems design (CACSD-CCA-ISIC). Munchen, Germany, pp 2276–2280
4. Dormido S (2004) Control learning: present and future. *Annu Rev Control* 28:115–136

5. Gasparyan O (2006) Computer-aided analysis and design of linear and nonlinear multivariable control systems: a classical approach. In: IEEE conference on computer aided control systems design (CACSD-CCA-ISIC). Munchen, Germany, pp 1958–1963
6. Ionescu C, De Keyser R (2008) Model-free adaptive control in frequency domain: application to mechanical ventilation, chapter 13. In: *Frontiers in Adaptive Control*, pp 253–270. Available via i-Techonline. <http://intechweb.org/>. Accessed 15 Dec 2008
7. Johansson M, Gäfvert M, Åström KJ (1998) Interactive tools for education in automatic control. *IEEE Control Syst Mag* 18:33–40
8. Nise N (1995) *Control systems engineering*, 2nd edn. Addison-Wesley, chapter 10
9. Oustaloup A, Levron F, Mathieu B, Nanot F (2000) Frequency-based complex noninteger differentiator: characterization and synthesis. *IEEE Trans on Circ Syst – part I* 47:25–39
10. Piguet Y, Holmberg U, Longchamp R (1999) Instantaneous performance visualization for graphical control design methods. In: 14th IFAC world congress. Beijing, China
11. Polak A, Mroczka J (2006) Nonlinear model for mechanical ventilation of human lungs. *Comput Biol Med* 36:41–58
12. Wittenmark B, Hägglund H, Johansson M (1998) Dynamic pictures and interactive learning. *IEEE Cont Syst Mag* 18:26–32

Resonance and Stability Conditions for Fractional Transfer Functions of the Second Kind

Rachid Malti, Xavier Moreau, and Firas Khemane

Abstract Fractional transfer functions of the second kind are studied in this chapter. First, stability conditions are established in terms of pseudo-damping factor and commensurable order. Then, resonance conditions are determined. An open-loop analysis of the equivalent closed-loop transfer function of the second kind confirms the analysis.

1 Introduction

Commensurable fractional systems can be represented in a transfer function form as:

$$H(s) = \frac{T(s^\nu)}{R(s^\nu)} = \frac{\sum_{j=0}^{m_B} b_j s^{\nu j}}{1 + \sum_{i=1}^{m_A} a_i s^{\nu i}}, \quad (1)$$

where $(a_i, b_j) \in \mathbb{R}^2$, the commensurable order $\nu \in \mathbb{R}^{+*}$ is a strictly positive reel number, m_B and m_A are respectively numerator and denominator degrees, with $m_A > m_B$ for strictly causal systems.

The commensurable transfer function (1) can always be decomposed in a modal form:

$$H(s) = \sum_{k=1}^N \sum_{q=1}^{\nu_k} \frac{A_{k,q}}{(s^\nu + p_k)^q}, \quad (2)$$

R. Malti (✉), X. Moreau, and F. Khemane
Bordeaux University – IMS, 351, cours de la Libération, 33405 Talence Cedex, France
e-mail: rachid.malti@ims-bordeaux.fr; xavier.moreau@ims-bordeaux.fr;
firas.khemane@ims-bordeaux.fr

where $(-p_k)$, with $k = 1, \dots, N$, represent the s^ν -poles of integer multiplicity ν_k . The representation (2) is constituted of elementary transfer functions of the first kind studied in [1]:

$$\tilde{F}(s) = \frac{\tilde{K}}{s^\nu + b}. \quad (3)$$

with $\tilde{K} \in \mathbb{C}$, $b \in \mathbb{C}$, and $\nu \in \mathbb{R}^{+*}$.

When two complex conjugate s^ν -poles are present, the following elementary transfer function of the second kind, written in a canonical form, is generally used as it contains only real-valued parameters:

$$F(s) = \frac{K}{1 + 2\zeta \left(\frac{s}{\omega_n}\right)^\nu + \left(\frac{s}{\omega_n}\right)^{2\nu}}. \quad (4)$$

with $K \in \mathbb{R}$, $\zeta \in \mathbb{R}$, $\omega_n \in \mathbb{R}^{+*}$, and $\nu \in \mathbb{R}^{+*}$.

As in rational systems, K and ω_n represent respectively steady-state gain and natural frequency of $F(s)$. However, the parameter ζ does not have the same meaning as in rational systems: it is a damping factor only if $\nu = 1$. It will be referred to as a *pseudo-damping factor* in this chapter.

Sabatier et al. [4] have studied the time-domain performances (maximum overshoot and settling time) of the elementary transfer functions of the first and the second kinds in terms of the differentiation order ν and, for the transfer function of the second kind, the pseudo-damping factor ζ (denoted $\cos(\theta)$ in [4]). This study was then used for modal placement control of fractional systems.

Plotting the asymptotic frequency response of a transfer function such as (2) requires usually to decompose (2) into elementary transfer functions of the first and the second kind and the contribution of each elementary function is plotted as it appears depending on the transitional frequencies. The elementary transfer function of the first kind has already been studied in [1, 4]. On the other hand, the properties of rational second order systems, (4) with $\nu = 1$, are well known:

- The system is stable if the damping factor $\zeta > 0$.
- The system is resonant if $0 < \zeta < \frac{\sqrt{2}}{2}$.
- The system has two complex conjugate poles if $0 < \zeta < 1$.
- The system has a real double pole if $\zeta = 1$.
- The system is overdamped if $\zeta > 1$.

The main concern of this chapter is to study the elementary properties of the fractional transfer functions of the second kind written in the canonical form (4). First, stability conditions are established in terms of the pseudo-damping factor ζ and the commensurable differentiation order ν . Then, resonance conditions are established.

The authors came to this study when they wanted to simulate a resonant and stable transfer function of the second kind as in (4). They noticed then that the above-mentioned properties of second order rational systems do not always apply to fractional systems.

2 Stability of Fractional Transfer Functions of the Second Kind

The objective of this section is to determine stability conditions of the fractional transfer function of the second kind (4) in terms of the pseudo-damping factor ζ and the commensurable order ν . Matignon [2] stability theorem is used.

Theorem 1 ([2]). *A commensurable transfer function with a commensurable order ν , as in (1), with T and R two coprime polynomials, is stable if and only if (iff) $0 < \nu < 2$ and $\forall p \in \mathcal{C}$ such as $R(p) = 0$, $|\arg(p)| > \nu \frac{\pi}{2}$. $\square$*

To apply Theorem 1 on (4), it is required to compute both s^ν -poles:

$$s_{1,2}^\nu = \omega_n^\nu \left(-\zeta \pm \sqrt{\zeta^2 - 1} \right), \quad (5)$$

which can either be real if $|\zeta| \geq 1$, or complex conjugate if $|\zeta| < 1$. Hence, two cases are distinguished.

2.1 Case $|\zeta| \geq 1$

Two real s^ν -poles are present. According to Theorem 1, the transfer function (4) is stable if both s^ν -poles are negative:

$$s_{1,2}^\nu < 0 \Rightarrow \omega_n^\nu \left(-\zeta \pm \sqrt{\zeta^2 - 1} \right) < 0 \quad (6)$$

$$\Rightarrow -\zeta \pm \sqrt{\zeta^2 - 1} < 0. \quad (7)$$

Inequality (7) holds only for positive ζ . Consequently:

$$1 \leq \zeta < +\infty. \quad (8)$$

2.2 Case $|\zeta| < 1$

Two complex conjugate s^ν -poles are present:

$$s_{1,2}^\nu = \omega_n^\nu \left(-\zeta \pm j \sqrt{1 - \zeta^2} \right) \quad (9)$$

$$= \omega_n^\nu e^{\pm j\theta}, \quad (10)$$

where θ , restricted without loss of generality to $]0, \pi[$, is given by:

$$\theta = \begin{cases} \arctan \left(\frac{\sqrt{1-\zeta^2}}{-\zeta} \right) & \text{if } -1 < \zeta \leq 0 \\ \arctan \left(\frac{\sqrt{1-\zeta^2}}{-\zeta} \right) + \pi & \text{if } 0 \leq \zeta < 1. \end{cases} \quad (11)$$

According to Theorem 1, the system is stable iff:

$$0 < \nu \frac{\pi}{2} < \theta < \pi. \quad (12)$$

Both conditions expressed in (11) are treated below.

2.2.1 Subcase $-1 < \zeta \leq 0$

In this case,

$$0 < \nu \leq 1 \quad \text{and} \quad (13)$$

$$\frac{\sqrt{1-\zeta^2}}{-\zeta} > 0. \quad (14)$$

Consequently, θ is in the first quadrant: $\theta \in (0, \frac{\pi}{2}]$. Furthermore, from (12):

$$0 < \nu \frac{\pi}{2} < \theta \leq \frac{\pi}{2}. \quad (15)$$

Substituting (11) in (15) yields

$$\tan\left(\nu \frac{\pi}{2}\right) < \frac{\sqrt{1-\zeta^2}}{-\zeta} \leq \tan\left(\frac{\pi}{2}\right) \quad (16)$$

$$\tan^2\left(\nu \frac{\pi}{2}\right) < \frac{1-\zeta^2}{\zeta^2} \leq \infty \quad (17)$$

$$\zeta^2 < \cos^2\left(\nu \frac{\pi}{2}\right). \quad (18)$$

Since ζ is negative, it must satisfy the following inequalities:

$$-1 < -\cos\left(\nu \frac{\pi}{2}\right) < \zeta \leq 0. \quad (19)$$

2.2.2 Subcase $0 \leq \zeta < 1$

In this case,

$$1 \leq \nu < 2 \quad \text{and} \quad (20)$$

$$\frac{\sqrt{1-\zeta^2}}{-\zeta} < 0. \quad (21)$$

Hence, θ is in the second quadrant: $\theta \in \left[\frac{\pi}{2}, \pi\right)$. Furthermore, from (12):

$$\frac{\pi}{2} \leq \nu \frac{\pi}{2} < \theta < \pi. \quad (22)$$

Substituting (11) in (22) yields:

$$\tan\left(\nu \frac{\pi}{2} - \pi\right) < \frac{\sqrt{1-\xi^2}}{-\xi} < \tan(0) \quad (23)$$

$$\tan^2\left(\nu \frac{\pi}{2}\right) > \frac{1-\xi^2}{\xi^2} > 0 \quad (24)$$

$$\xi^2 > \cos^2\left(\nu \frac{\pi}{2}\right). \quad (25)$$

Since ξ is positive, it must satisfy the following inequalities:

$$0 < -\cos\left(\nu \frac{\pi}{2}\right) < \xi < 1. \quad (26)$$

2.3 Summary

Combining inequalities (8), (19), and (26) yields the following corollary of Theorem 1.

Corollary – Stability of fractional transfer functions of the second kind (4). *The transfer function (4) is stable if and only if:*

$$-\cos\left(\nu \frac{\pi}{2}\right) < \xi < \infty \quad \text{and} \quad 0 < \nu < 2. \quad (27)$$

□

This corollary is in accordance with the rational case, as it is well known that a second order rational transfer function, as in (4) with $\nu = 1$, is stable if:

$$0 < \xi < \infty. \quad (28)$$

3 Resonance of Fractional Transfer Functions of the Second Kind

The frequency response of (4) is given by:

$$F(j\omega) = \frac{K}{1 + 2\xi \left(\frac{j\omega}{\omega_n}\right)^\nu + \left(\frac{j\omega}{\omega_n}\right)^{2\nu}}. \quad (29)$$

Define $\Omega = \frac{\omega}{\omega_n}$ as the normalized frequency. Since the resonance does not depend on the gain K , it will be set to one in the following. Hence, define:

$$\mathcal{F}(j\Omega) = \frac{1}{1 + 2\zeta (j\Omega)^\nu + (j\Omega)^{2\nu}}. \quad (30)$$

The gain of $\mathcal{F}(j\Omega)$ ¹ is given by

$$|\mathcal{F}(j\Omega)| = \frac{1}{\left| 1 + 2\zeta e^{j\nu\frac{\pi}{2}} \Omega^\nu + e^{j\nu\pi} \Omega^{2\nu} \right|}, \quad (31)$$

$$|\mathcal{F}(j\Omega)| = \frac{1}{\left| (1 + 2\zeta \cos(\nu\frac{\pi}{2}) \Omega^\nu + \cos(\nu\pi) \Omega^{2\nu}) + j (2\zeta \sin(\nu\frac{\pi}{2}) \Omega^\nu + \sin(\nu\pi) \Omega^{2\nu}) \right|}} \quad (32)$$

The gain in dB is now given by

$$|\mathcal{F}(j\Omega)|_{\text{dB}} = -10 \log \left[\left(1 + 2\zeta \cos\left(\nu\frac{\pi}{2}\right) \Omega^\nu + \cos(\nu\pi) \Omega^{2\nu} \right)^2 + \left(2\zeta \sin\left(\nu\frac{\pi}{2}\right) \Omega^\nu + \sin(\nu\pi) \Omega^{2\nu} \right)^2 \right] \quad (33)$$

$$|\mathcal{F}(j\Omega)|_{\text{dB}} = -10 \log \left[\Omega^{4\nu} + 4\zeta \cos\left(\nu\frac{\pi}{2}\right) \Omega^{3\nu} + 2(2\zeta^2 + \cos(\nu\pi)) \Omega^{2\nu} + 4\zeta \cos\left(\nu\frac{\pi}{2}\right) \Omega^\nu + 1 \right] \quad (34)$$

In case $F(s)$ is resonant, $|\mathcal{F}(j\Omega)|_{\text{dB}}$ has at least a maximum at a positive normalized frequency. Hence, $F(s)$ is resonant if the equation $\frac{d|\mathcal{F}(j\Omega)|_{\text{dB}}}{d\Omega} = 0$ has at least one real and strictly positive solution corresponding to that maximum. Based on (34), all real and strictly positive solutions of the following equation need to be computed:

$$\frac{d|\mathcal{F}(j\Omega)|_{\text{dB}}}{d\Omega} = 0 \Rightarrow \Omega^{4\nu-1} + 3\zeta \cos\left(\nu\frac{\pi}{2}\right) \Omega^{3\nu-1} + (2\zeta^2 + \cos(\nu\pi)) \Omega^{2\nu-1} + \zeta \cos\left(\nu\frac{\pi}{2}\right) \Omega^{\nu-1} = 0. \quad (35)$$

¹ The multi-valued function s^ν becomes holomorphic in the complement of its branch cut line as soon as a branch cut line, i.e. $\mathbb{R}_-$ is specified. The following restrictions on arguments of s are imposed: $-\pi < \arg(s) < \pi$. Hence, the only possible argument of j is $\frac{\pi}{2}$.

Since $\Omega = 0$ is not an acceptable solution, the common factor $\Omega^{\nu-1}$ in (35) can be simplified, so as to obtain:

$$\frac{d|\mathcal{F}(j\Omega)|_{\text{dB}}}{d\Omega} = 0 \Rightarrow \Omega^{3\nu} + 3\zeta \cos\left(\nu \frac{\pi}{2}\right) \Omega^{2\nu} + (2\zeta^2 + \cos(\nu\pi)) \Omega^\nu + \zeta \cos\left(\nu \frac{\pi}{2}\right) = 0. \quad (36)$$

One can check easily that for rational systems, $\nu = 1$, (36) reduces to

$$\Omega^3 + (2\zeta^2 - 1) \Omega = 0, \quad (37)$$

which strictly positive solution is given, as expected, by

$$\Omega_r = \sqrt{1 - 2\zeta^2}, \quad (38)$$

provided that the following known condition is satisfied:

$$1 - 2\zeta^2 > 0 \Rightarrow \zeta < \frac{\sqrt{2}}{2}. \quad (39)$$

The third order equation in Ω^ν (36), can have positive real-valued, negative real-valued or complex-valued solutions. The number of resonant frequencies of the studied system, zero one or two, depends on the number of strictly positive real-valued solutions corresponding to maxima of $|\mathcal{F}(j\Omega)|$. Care must be taken, because some of the strictly positive solutions correspond to minima of $|\mathcal{F}(j\Omega)|$, especially when a double resonance is present, then the gain presents a minimum between the two maxima (see Example 2, p. 442).

Solving (36) analytically is not easy. A numerical solution is obtained for various combinations of ν and ζ and plotted in Fig. 1 (yellow and green regions represent combinations of ν and ζ which produce resonant systems). Hence, the main result of this chapter is plotted in Fig. 1 and summarized below.

- If $0 < \nu \leq 0.5$ and $-\cos \nu \frac{\pi}{2} < \zeta < 0 \Rightarrow$ the stable transfer function $F(s)$ in (4) is always resonant.
- If $0 < \nu \leq 0.5$ and $0 \leq \zeta \Rightarrow$ the stable transfer function $F(s)$ is never resonant (for $0 < \nu \leq 0.5$ and $\zeta = 0$ it can straightforwardly be proven that (36) has no strictly positive solution).
- If $0.5 < \nu \leq 1$ and $-\cos \nu \frac{\pi}{2} < \zeta < \infty \Rightarrow$ the stable transfer function $F(s)$ is resonant if an additional condition is satisfied $\zeta < \zeta_0$, where ζ_0 is computed numerically and plotted in Fig. 1 as the upper contour of the resonance region in the interval $\zeta \in]0.5, 1]$. In the particular case of rational systems, when $\nu = 1$, it is well known that $\zeta_0 = \frac{\sqrt{2}}{2}$.
- If $1 < \nu < 2$ and $-\cos \nu \frac{\pi}{2} < \zeta < \infty \Rightarrow$ the stable transfer function $F(s)$ is always resonant.

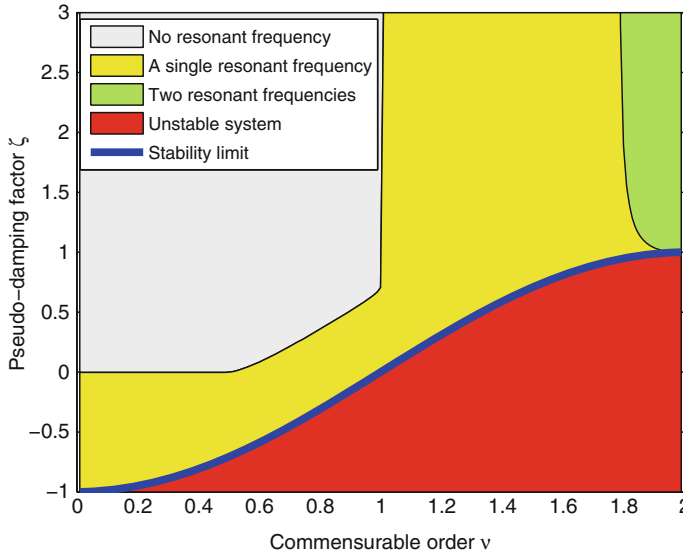


Fig. 1 Stability and resonance regions of the fractional system (4) in the ζ versus ν plane

- If $\nu_0 < \nu < 2$ and $\zeta_0 < \zeta < \infty$ where ν_0 and ζ_0 are computed numerically and plotted as the lower-left contours of the upper-right green region of Fig. 1 $\Rightarrow$ the stable transfer function $F(s)$ has two resonant frequencies.

Based on the relation (5), stability and resonance regions are plotted for different values of ν in the complex s^ν -plane in Figs. 2, 3, and 4. As shown in Fig. 2, when $0 < \nu \leq 0.5$, the stable transfer function (4) is resonant if its complex conjugate s^ν -poles are in the right hand complex s^ν -plane except the instability region. It is shown in Fig. 3, when $0.5 < \nu \leq 1$, that the resonance region is a sector which can be determined for different values of ν . Finally, as soon as $\nu > 1$, the resonance region is the whole stability region, as shown in Fig. 4.

4 Open-Loop – Closed-Loop Analysis

The transfer function of the second kind (4) can be viewed as two nested closed-loop transfer functions of Fig. 5 with two fractional integrators of order ν and gains,

$$K_1 = \frac{\omega_n^\nu}{2\zeta}, \quad (40)$$

$$K_2 = 2\zeta\omega_n^\nu. \quad (41)$$

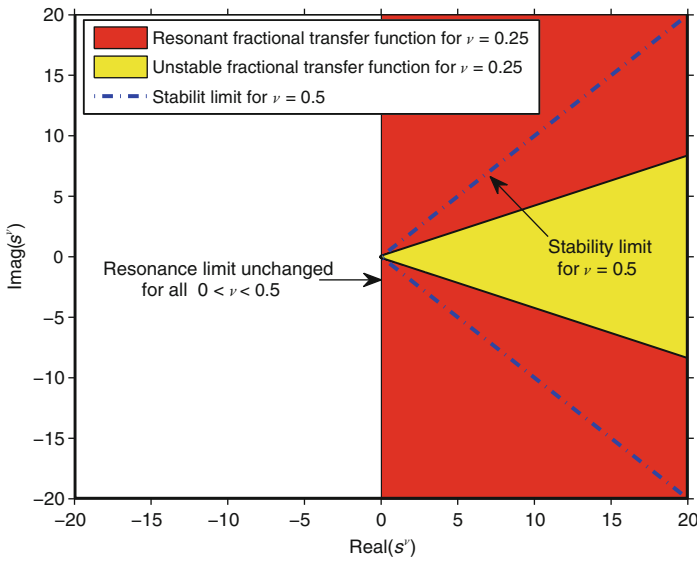


Fig. 2 Stability and resonance regions of the fractional system (4) with $\nu = 0.25$ in the complex s^ν -plane

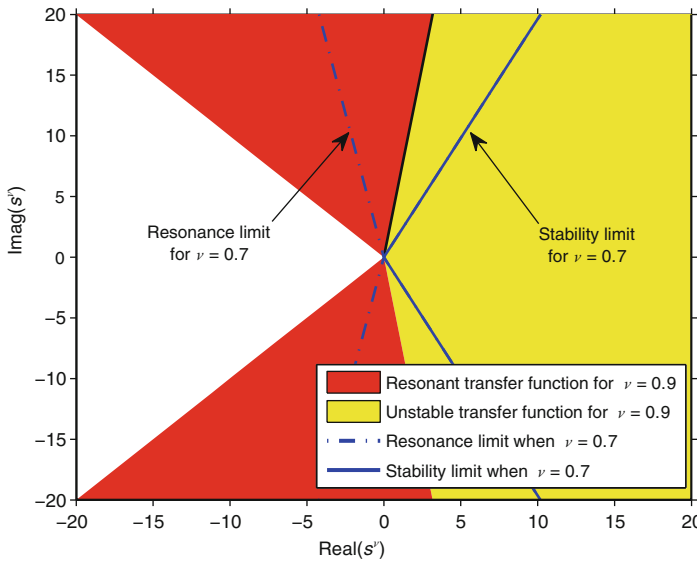


Fig. 3 Stability and resonance regions of the fractional system (4) with $\nu = 0.90$ in the complex s^ν -plane

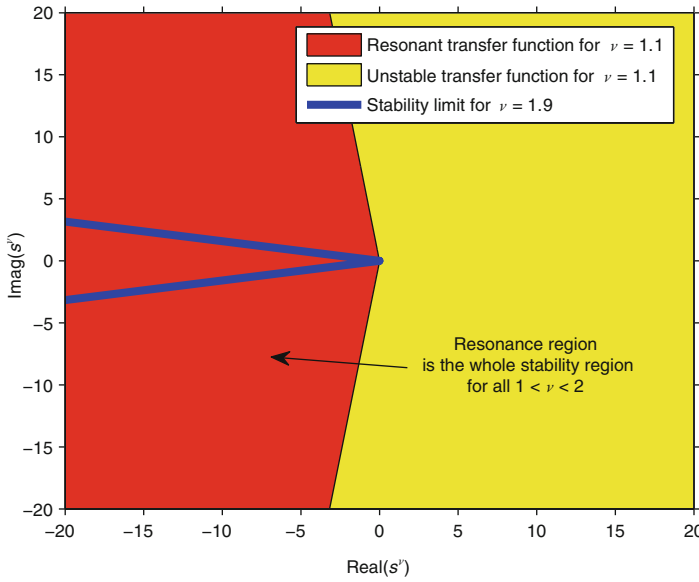


Fig. 4 Stability and resonance regions of the fractional system (4) with $\nu = 1.1$ in the complex s^ν -plane

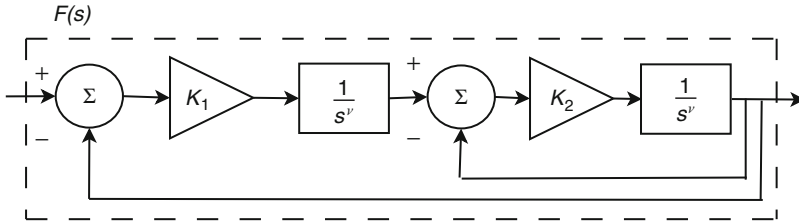


Fig. 5 Closed-loop transfer function equivalence

It can also be viewed by considering only the outer loop, as the closed-loop system of Fig. 6.

$$F(s) = \frac{\beta(s)}{1 + \beta(s)}, \quad (42)$$

where the open-loop transfer function $\beta(s)$ is given by:

$$\beta(s) = \frac{K_1 K_2}{s^{2\nu} (1 + \frac{K_2}{s^\nu})}, \quad (43)$$

K_1 and K_2 being defined in (40) and (41). The open-loop transfer function $\beta(s)$ is studied for different values of ν and ζ and its frequency response plotted for $\zeta = -0.7$, $\zeta = +0.7$, and $\zeta = 2$ in the Nichols charts of Figs. 7, 8, and 9.

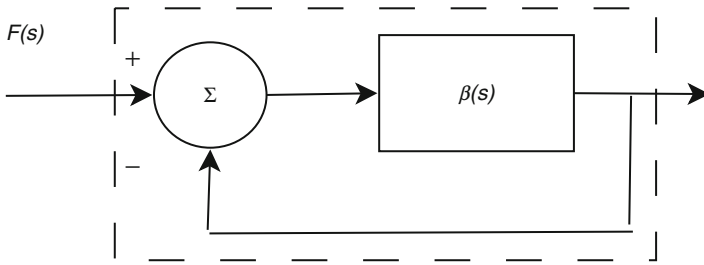


Fig. 6 Closed-loop transfer function equivalent to $F(s)$ in (4)

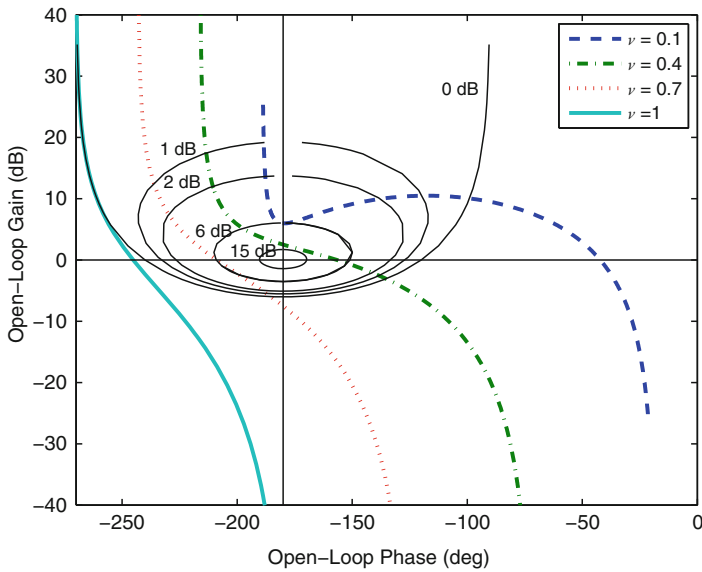


Fig. 7 Nichols charts for $\zeta = -0.7$ and different values of ν

For negative ζ , see for instance Fig. 7 with $\zeta = -0.7$, the steady state gain of the open-loop transfer function, $\beta(s)$, is negative. Hence, in low frequencies, the Nichols chart of $\beta(s)$ is inside the Nichols magnitude contours. When stability condition (27), expressed in terms of ν , is satisfied, here $\nu > \arccos(0.7)\frac{2}{\pi} = 0.51$, $\beta(s)$ passes on the right of the critical point; otherwise it passes on its left. Thus, for negative ζ , the system can either be stable and resonant or unstable.

For $0 < \zeta \leq 1$, see for instance Fig. 8 with $\zeta = +0.7$, the steady state gain of $\beta(s)$ is positive. When $\nu < \nu_0$, in this example $\nu_0 \approx 1$, the Nichols chart remains outside the 0 dB Nichols magnitude contour. For $\nu_0 < \nu < 2$, the Nichols chart crosses the 0 dB Nichols magnitude contour which makes the closed-loop system resonant. The system is stable if condition (27) is satisfied, here $\nu > \arccos(-0.7)\frac{2}{\pi} = 1.50$.

For $\zeta > 1$, see for instance Fig. 9 with $\zeta = 2$, the steady state gain of $\beta(s)$ is positive. When $\nu \leq 1$, the Nichols chart remains outside the 0 dB Nichols magnitude contour. For $1 < \nu < 2$, the Nichols chart crosses the 0 dB Nichols magnitude

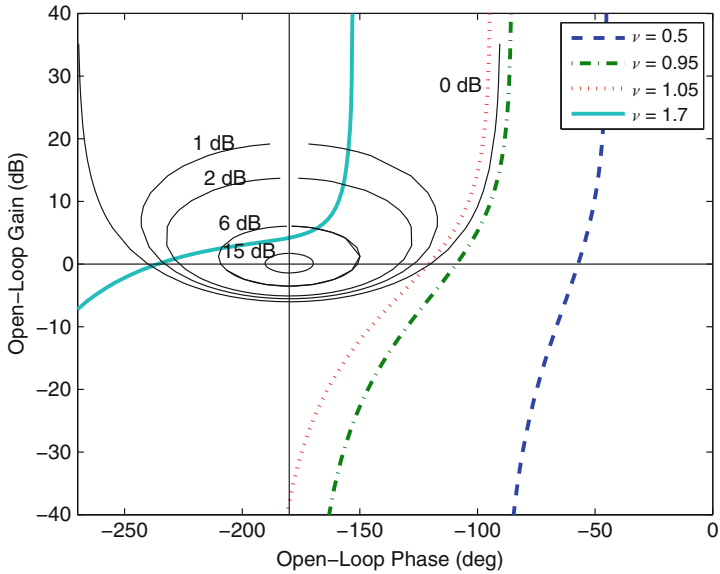


Fig. 8 Nichols charts for $\zeta = +0.7$ and different values of ν

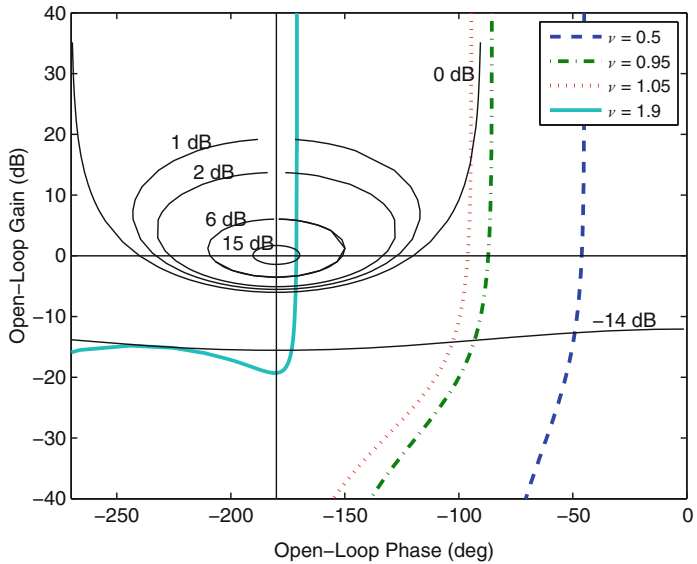


Fig. 9 Nichols charts for $\zeta = 2$ and different values of ν

contour which makes the closed-loop system resonant. For the particular case $\nu = 1.9$, the Nichols chart is tangent to approximately 15 and -14 dB contours which produces a double resonant closed-loop frequency response (see also Example 2, p. 442). The system is unstable if $\nu \geq 2$ as specified in Theorem 1.

5 Numerical Examples

Time-domain simulations of fractional systems in the following examples use Oustaloup's [3] approximation in the appropriate frequency band.

5.1 Example 1

In this example, the particular case $\nu = 0.4$ and $\zeta = -0.7$ is studied. It corresponds to the open-loop transfer function $\beta(s)$ plotted in a dashed-dotted line in Fig. 4. Eq. (36), which reduces to:

$$\Omega^{1.2} - 1.70\Omega^{0.8} + 1.30\Omega^{0.4} - 0.57 = 0, \quad (44)$$

has a single strictly positive real-valued solution and two complex conjugate ones:

$$\Omega_{r1} = 0.97, \quad \Omega_{r2,r3} = 0.36 \pm 0.67j. \quad (45)$$

Only the positive solution is acceptable and corresponds to a resonant frequency, as shown in the Bode diagram of Fig. 10. The amplitudes of the resonant frequency is approximately equal to 15 dB, which corresponds to the 15 dB tangent contour lines of Fig. 7 with $\nu = 0.4$. Moreover, the step and the impulse responses, plotted in Fig. 11, are underdamped as expected due to the resonant frequency.

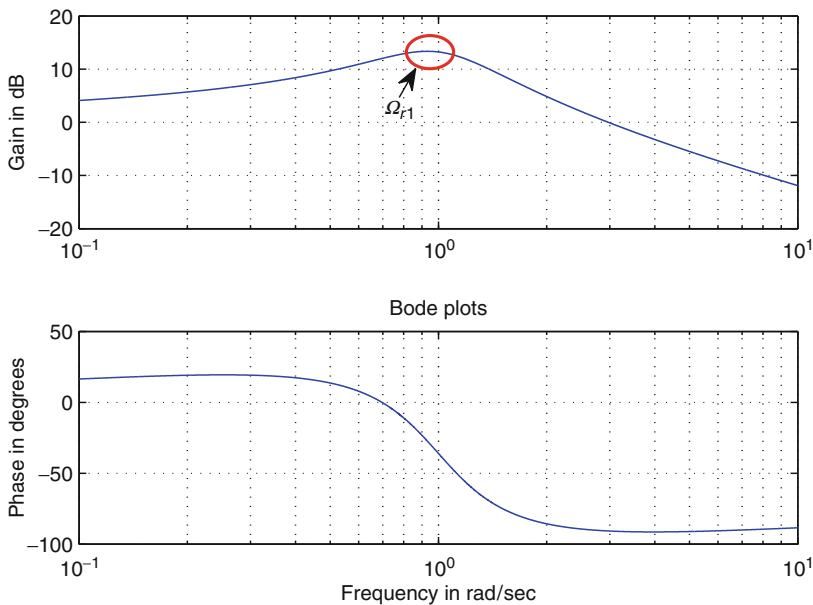


Fig. 10 Bode plot for $\zeta = -0.7$ and $\nu = 0.4$

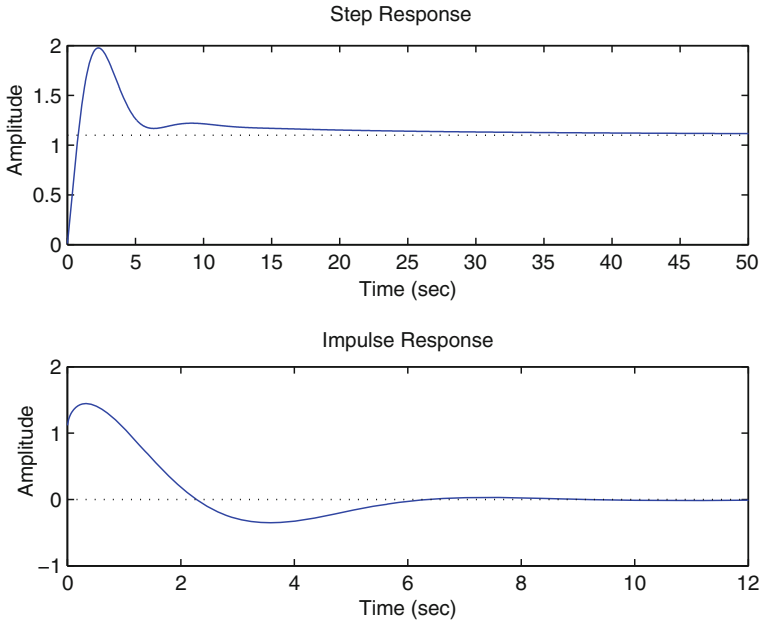


Fig. 11 Step and Impulse responses for $\zeta = -0.5$ and $\nu = 0.5$

5.2 Example 2

Consider now the case $\nu = 1.9$, and $\zeta = 2$, which presents the particularity of having a double resonance as shown in Fig. 1. This case corresponds to the open-loop transfer function $\beta(s)$ plotted in a solid line in Fig. 12. Eq. (36), which reduces to:

$$\Omega^{5.7} - 5.93\Omega^{3.8} + 8.95\Omega^{1.9} - 1.98 = 0, \quad (46)$$

has three real-valued solutions:

$$\Omega_{r1} = 0.50, \quad \Omega_{r2} = 1.47, \quad \Omega_{r3} = 1.96. \quad (47)$$

As shown in Fig. 12, Ω_{r1} corresponds to a resonance, Ω_{r2} to a minimum, and Ω_{r3} to a second resonance. When a system presents two resonant frequencies, it always has a minimum between these two maxima. The amplitudes of the two resonant frequencies are approximately equal to 15 and -14 dB, which correspond to the contour lines of Fig. 12, with $\nu = 1.9$, for which the open-loop transfer function $\beta(s)$ is tangent.

Moreover, the step and the impulse responses, plotted in Fig. 13, are underdamped as expected due to the resonant frequencies.

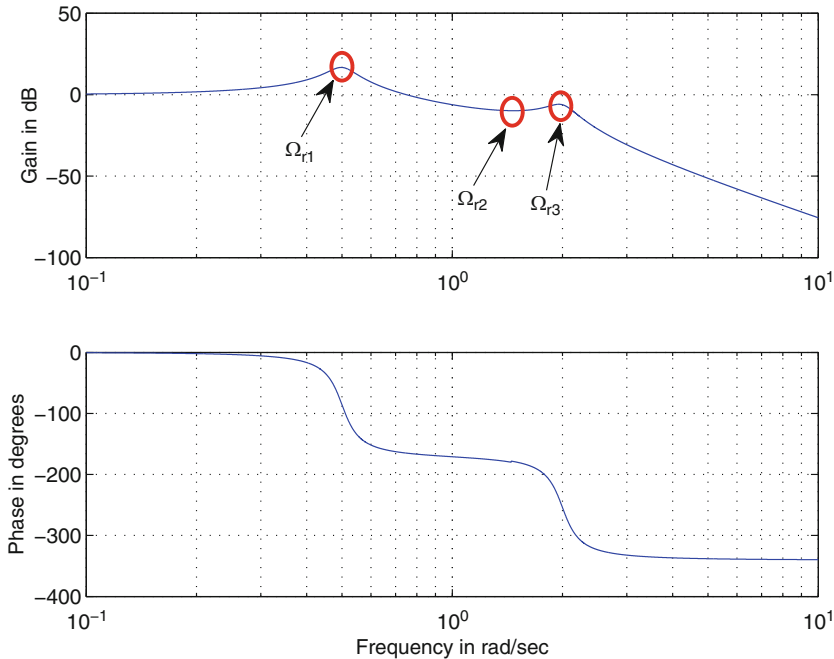


Fig. 12 Bode plot for $\zeta = 2$ and $\nu = 1.9$

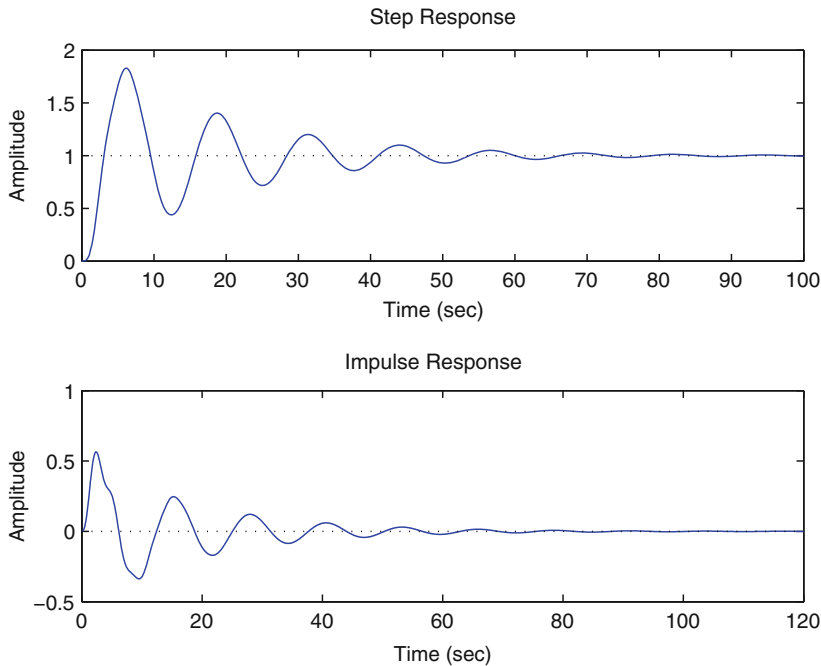


Fig. 13 Step and Impulse responses for $\zeta = 2$ and $\nu = 1.9$

6 Conclusion

The fractional transfer functions of the second kind (4) is studied in this chapter. First, stability conditions are determined in terms of the pseudo-damping factor ζ (it is known that the commensurable order, ν , must satisfy $0 < \nu < 2$) and is summarized in (27). Then, conditions on ζ and ν are determined so that the system is resonant. The resonance conditions are difficult to express analytically. They are computed numerically and plotted in Fig. 1. It is also shown, in Fig. 1, that some combinations of ζ and ν yield two resonant frequencies. All these results are confirmed in the open-loop analysis of the equivalent closed-loop transfer function of the second kind.

References

1. Hartley T, Lorenzo C (1998) A solution to the fundamental linear fractional order differential equation. In: NASA/TP-1998-208693 report. Lewis Research Center
2. Matignon D (1998) Stability properties for generalized fractional differential systems. ESAIM proceedings – Systèmes Différentiels Fractionnaires – Modèles, Méthodes et Applications 5
3. Oustaloup A (1995) La dérivation non-entière. Hermès – Paris
4. Sabatier J, Cois O, Oustaloup A (2003) Modal placement control method for fractional systems: application to a testing bench. In: First symposium on fractional derivatives and their applications at 19th biennial conference on mechanical vibration and noise, ASME-DETC conference (international design engineering technical conferences). Chicago, IL, USA

Synchronization of Fractional-Order Chaotic System via Adaptive PID Controller

Mohammad Mahmoudian, Reza Ghaderi, Abolfazl Ranjbar, Jalil Sadati, Seyed Hassan Hosseinnia, and Shaher Momani

Abstract Chaos in hard spring ϕ^6 -Van der Pol Oscillator, and its modelling and control with fractional-Order Calculus (FOC) is studied in this research. An adaptive PID controller with fractional order adaptation mechanism is proposed to synchronize fractional-order chaotic system. PID coefficients are updated using the gradient method when a proper sliding surface is chosen. Fractional-order ϕ^6 -Van der Pol Oscillator is used as case study to verify the proposed method.

1 Introduction

Recently, due to the growth of applications, investigation on development of new techniques of modelling and analysis, for nonlinear systems has been increased. In recent years, studies have been presented in many theoretical and applications of science and engineering [1,2]. This is a result of better understanding of the potential of fractional calculus revealed by problems such as viscoelasticity and damping, chaos, diffusion, wave propagation, percolation and irreversibility. Furthermore there is a new topic to investigate the control and dynamics of fractional order dynamical systems recently. It is due to the fact that, for some systems, chaotic behaviour could be modelled with FOC better in comparison with the conventional mathematics. There are few reports of investigations on behaviour of nonlinear chaotic systems whit fractional model [3–7]. The task of designing a system (response or slave), whose behaviour mimics another one (drive or master), is called synchronization and is finding fast growing applications in many fields of control and communication. Due

M. Mahmoudian, R. Ghaderi, A. Ranjbar (✉), J. Sadati, and S.H. Hosseinnia
Intelligent system research group, Faculty of Electrical and Computer Engineering,
Noushirvani University of Technology, Babol, P.O. Box 47135-484, Iran
e-mail: mahmoudian52869@gmail.com; r_ghaderi@yahoo.co.uk; a.ranjbar@nit.ac.ir;
jalil.sadati@gmail.com; h.hosseinnia@stu.nit.ac.ir

S. Momani

Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan
e-mail: shahermmm@yahoo.com

to the sensitivity to initial conditions, an important characteristic of chaotic systems, synchronization and control of such systems are rather difficult. Several techniques have been proposed for this problem such as periodic parametric perturbation [8,9], drive-response synchronization [10], adaptive control [11–13], variable structure (or sliding mode) control [14, 15], backstepping control [16], and control [17]. Van der Pol a mathematical structure which has been originally employed to model electrical circuit with a triode valve is one of these cases which serves as a basic model for self-excited oscillations in physics, engineering, electronics, biology, neurology and many other disciplines. In [18], the dynamics behaviour of a ϕ^6 -Van der Pol oscillator subjected to an external disturbance is studied. In this paper ϕ^6 -Van der Pol oscillator dynamics, modelled by FOC and with uncertainty is investigated. The stability range is also spotted. As a novel idea, an adaptive PID controller is developed by fractional adaptation is used for synchronization. The significance of the controller is shown by simulation results. This paper is organized as follows: The control method for the task is presented in Sect. 2. The synchronization task is performed in Sect. 3 using the proposed controller. Finally the work will be concluded at Sect. 4.

2 Adaptive PID Controller Design

Consider the following fractional dynamics:

$$\begin{aligned} D^q x_1 &= x_2, \\ D^q x_2 &= f(X, t) \end{aligned} \quad (1)$$

where, $0 < q \leq 1$ is the order of the fractional time-derivative. $X = [x_1 x_2]^T$ is the state vector. The goal is to synchronize an unknown dynamics expressed in 1. Let's assume the following fractional equation describes the slave dynamics, incorporating $\Delta f(\cdot)$ as uncertainty and $d(t)$ as a disturbance:

$$\begin{aligned} D^q y_1 &= y_2, \\ D^q y_2 &= f(Y, t) + \Delta f(Y, t) + d(t) \end{aligned} \quad (2)$$

where, $Y = [y_1 y_2]^T$ is the state vector of the slave system. $u(t)$ presents the control signal to make synchronization possible. The error is defined as the discrepancy of those master and slave states by $e_i = x_i - y_i, i = 1, 2$. The following adaptive PID controller is used for the synchronization task:

$$u_{PID} = K_P e_1(t) + K_I \int_0^t e_1(\tau) d(\tau) + K_D \frac{de_1(t)}{dt} \quad (3)$$

Coefficients of the PID controller are tuned using a proper gradient based adaptation mechanism. In order to construct the adaptation law, a designation signal y_r is defined as follows:

$$D^q y_r = D^q x_2 + k_2 D^q e_1 + k_1 e_1 \quad (4)$$

The sliding surface will also be defined as

$$S = y_2 - y_r \quad (5)$$

The sliding mode will be activated when, therefore:

$$y_2 = y_r \quad (6)$$

Substitution equation 6 into 4 results as

$$D^{2q} e_1 + k_2 D^q e_1 + k_1 e_1 = 0 \quad (7)$$

Equation 7 can be written in state space format as:

$$\begin{aligned} D^q e_1 &= e_2, \\ D^q e_2 &= -k_1 e_1 - k_2 e_2 \Rightarrow D^q E = AE \end{aligned} \quad (8)$$

where, is the gain (coefficient) matrix for the state error of. Using suitable values for gains k_1, k_2 to meet the argument condition $|\arg(\text{eig}(A))| = q\pi/2$ for the stability of fractional systems the error $e_1(t)$ tends to zero when $t \rightarrow \infty$. To satisfy the sliding condition to provide a stable controller, the following function will be primarily introduced as a Lyapunov function:

$$V = \left(\frac{1}{2}\right) S^2 \quad (9)$$

The sliding condition will be defined using descending trajectory of the energy like Lyapunov function as

$$\dot{V} = S\dot{S} \quad (10)$$

When condition in 10 is satisfied the Lyapunov function will be infinite by the time ends to infinity, i.e. $\lim_{t \rightarrow \infty} S(t) \rightarrow 0$. In order to have a proper adaptation mechanism for tuning three PID control gains a gradient method is used to minimize the sliding condition in 10. The gradient search algorithm is calculated in the direction opposite of the energy flow, the convergence properties of the PID controller tuning can also be obtained. Moreover, it is quite intuitive to choose $S\dot{S}$ as an error function [19]. From (5) and using (2), we have

$$\dot{S} = \dot{y}_2 - \dot{y}_r = D^{1-q} (f(Y, t) + \Delta f(Y, t) + d(t) + u_{PID}) - \dot{y}_r \quad (11)$$

Multiplying both sides of equation 11 to S yields

$$S\dot{S} = S [D^{1-q} (f(Y, t) + \Delta f(Y, t) + d(t) + u_{PID}) - \dot{y}_r] \quad (12)$$

For simplicity the following definition is used:

$$U_{PID} = D^{1-q} (u_{PID}) \quad (13)$$

The following adaptation law updates PID coefficients using the gradient method:

$$\dot{K}_P = -\gamma \frac{\partial S\dot{S}}{\partial K_P} = -\gamma \frac{\partial S\dot{S}}{\partial U_{PID}} \frac{\partial U_{PID}}{\partial K_P} = -\gamma S D^{1-q} (e_1(t)) \quad (14)$$

$$\dot{K}_I = -\gamma \frac{\partial S\dot{S}}{\partial K_I} = -\gamma \frac{\partial S\dot{S}}{\partial U_{PID}} \frac{\partial U_{PID}}{\partial K_I} = -\gamma S D^{-q} (e_1(t)) \quad (15)$$

$$\dot{K}_D = -\gamma \frac{\partial S\dot{S}}{\partial K_D} = -\gamma \frac{\partial S\dot{S}}{\partial U_{PID}} \frac{\partial U_{PID}}{\partial K_D} = -\gamma S D^{2-q} (e_1(t)) \quad (16)$$

where, γ is the learning rate, which will be selected properly by designer based on the dynamics of the system. It should be noted that the selection of the learning rate and initial setting of PID coefficients affects the stability and of course the rate [19]. It should also be noted in Eqs. 14–16 the case $q = 1$ leads to the classical adaptation mechanism of integer derivatives. This Fractional adaptive controller is now applied on a Hard Spring ϕ^6 -Van der Pol Oscillator synchronization procedure.

3 Synchronization of Uncertain Hard Spring ϕ^6 -Van der Pol Oscillator

3.1 System Description

Consider the following system namely ϕ^6 -Van der Pol oscillator [18] as a case study:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x + \alpha x^3 + \lambda x^5 = f_0 \cos(\omega t) \quad (17)$$

The above system for $f_0 = 0$, is a complete oscillatory system. By increasing f_0 chaotic characteristic of the system appears [18]. In Fig. 1, phase plane of the chaotic system namely ϕ^6 -Van der Pol oscillator with $f_0 = 4.5$ has been shown. At the

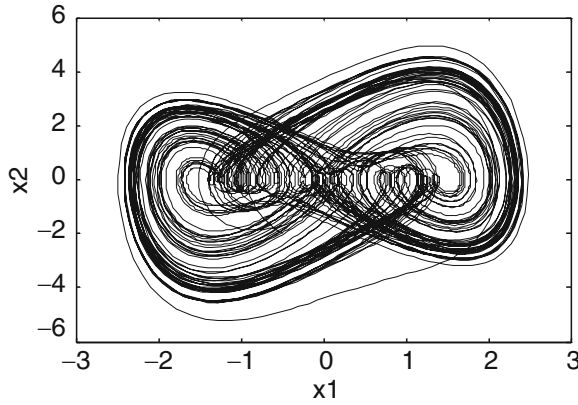


Fig. 1 Phase portrait of ϕ^6 -Van der Pol oscillator

same time, parameters of system have been selected as $\mu = 0.4, \lambda = 0.1, \alpha = 1, \omega_0 = 0.46, \omega = 0.86$. In Fig. 1, chaotic behaviour of the system is shown and f_0 presents uncertainty of the system. This system can be transformed into the following nominal state form:

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= f(x_1, x_2)\end{aligned}\quad (18)$$

where, $f(x_1, x_2) = \mu(1-x_1^2)x_2 + \omega_0^2 x_1 + \alpha x_1^3 + \lambda x_1^5 - f_0 \cos(\omega t)$. The performance of the proposed controller will be emphasized for ϕ^6 -Van der Pol oscillator with the following fractional dynamics:

$$\begin{aligned}D^q x_1 &= x_2, \\ D^q x_2 &= f(x_1, x_2)\end{aligned}\quad (19)$$

The stability condition for such system is obtained as [20]:

$$|\arg(\text{eig}(A))| > q\pi/2 \quad (20)$$

where, A is Jacobian matrix of the nonlinear fractional system at the equilibrium points of $X_e = [0, 0]^T$. The computation has been done assuming $\mu = 0.4, \lambda = 0.1, \alpha = 1, \omega_0 = 0.46, \omega = 0.86$ and $f_0 = 0$.

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & \mu \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.2116 & 0.4 \end{bmatrix} \quad (21)$$

In accordance with Eqs. 19 and 20 the system will be stable for $q < 0.713$ and unstable for $q > 0.713$. This statement is confirmed in Fig. 2 for two cases of

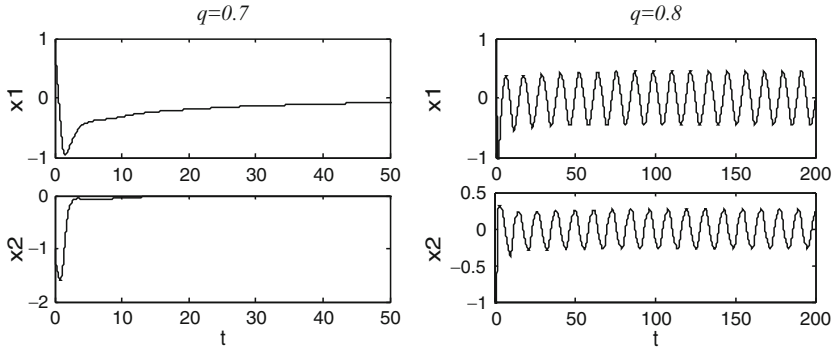


Fig. 2 Time responses of the ϕ^6 -Van der Pol oscillator for two cases of $q = 0.7$ (left) and $q = 0.8$ (right)

$q = 0.7$ and $q = 0.8$. To show the effectiveness of the proposed controller, the procedure is implemented on fractional dynamics of ϕ^6 -Van der Pol oscillator.

3.2 Implementation

The master system is defined by the following dynamics:

$$\begin{aligned} D^q x_1 &= x_2, \\ D^q x_2 &= \mu(1 - x_1^2)x_2 + \omega_0^2 x_1 + \alpha x_1^3 + \lambda x_1^5 \end{aligned} \quad (22)$$

Similarly, the slave system is defined as follows too

$$\begin{aligned} D^q y_1 &= y_2, \\ D^q y_2 &= \mu(1 - y_1^2)y_2 + \omega_0^2 y_1 + \alpha y_1^3 + \lambda y_1^5 - f_0 \cos(\omega t) \end{aligned} \quad (23)$$

The sets of initial conditions of master and slave systems are respectively defined as $[x_1(0), x_2(0)] = [1, -1]$ and $[y_1(0), y_2(0)] = [0.2, -0.2]$. Primary setting of PID coefficients are equal to $K_p(0) = 5$, $K_I(0) = 5$, $K_D(0) = 5$ and the learning rate has been selected as $\gamma = 0.14$. Also, k_1 and k_2 have been assigned as 0.5 and 2, respectively. Figures 3 to 6 show the simulation results. Synchronization and error of synchronization of x_1, x_2 and y_1, y_2 have shown in Figs. 3 and 4, respectively, the control signal has been shown in Fig. 5, and the sliding surface has shown in Fig. 6. It should be noted that in this section, the control signal u_{PID} , has been activated in $t = 10s$. The performance of the controller is significant. Parameters of the system, similar to Sect. 3.1, have been selected as $\mu = 0.4$, $\lambda = 0.1$, $\alpha = 1$, $\omega_0 = 0.46$, $\omega = 0.86$ and $f_0 = 4.5$.

Fig. 3 Synchronization of fractional ϕ^6 -Van der Pol oscillator

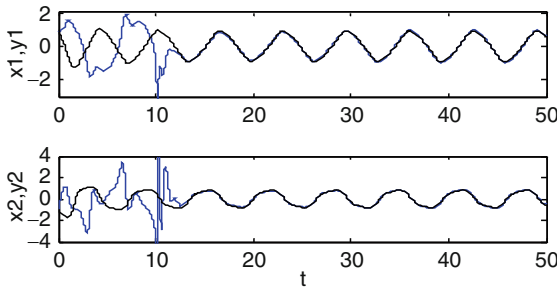


Fig. 4 Error of synchronization

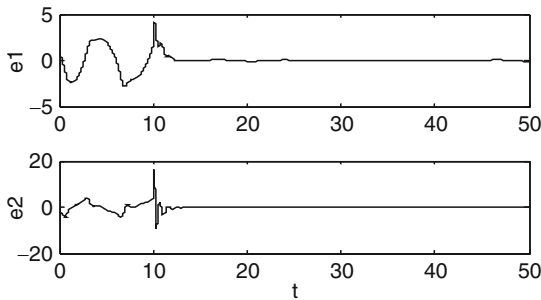


Fig. 5 The control signal

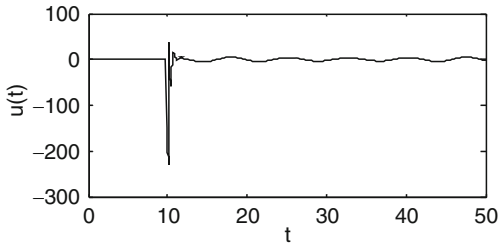
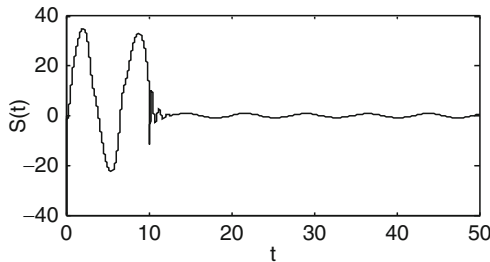


Fig. 6 Sliding surface



4 Conclusion

Stability of Fractional ϕ^6 -Van der Pol oscillator is investigated. It is then used as a Master dynamics in a synchronization task. An adaptive PID has synchronized the slave with the Master. Coefficients of adaptive PID are tuned according to a proper adaptation law. The gradient method is developed for the fractional dynamics and used to construct the adaptation mechanism. The simulation results signify the performance of the proposed technique for synchronization of fractional systems.

References

1. Podlubny I (1999) Fractional differential equations. Academic, New York
2. Hilfer R (2001) Applications of fractional calculus in physics. World Scientific, New Jersey
3. Li C, Chen G (2004) Chaos in the fractional order Chen system and its control. *Chaos Soliton Fract* 22:549–554
4. Wajdi AM, Ahmad MH (2003) On nonlinear control design for autonomous chaotic systems of integer and fractional orders. *Chaos Soliton Fract* 18:693–701
5. Wajdi AM, El-Khazali R, Al-Assaf Y (2004) Stabilization of generalized fractional order chaotic systems using state feedback control. *Chaos Soliton Fract* 22:141–150
6. Wajdi AM (2005) Hyperchaos in fractional order nonlinear systems. *Chaos Soliton Fract* 26:1459–1465
7. Nimmo S, Evans AK (1999) The effects of continuously varying the fractional differential order of chaotic nonlinear systems. *Chaos Soliton Fract* 10:1111–1118
8. Astakhov VV, Anishchenko VS, Kapitaniak T, Shabunin AV (1997) Synchronization of chaotic oscillators by periodic parametric perturbations. *Physics D* 109:11–16
9. Blazejczyk OB, Brindley J, Czolczynski K, Kapitaniak T (2001) Antiphase synchronization of chaos by noncontinuous coupling: two impacting oscillators. *Chaos Soliton Fract* 12: 1823–1826
10. Yang XS, Duan CK, Liao XX (1999) A note on mathematical aspects of drive-response type synchronization. *Chaos Soliton Fract* 10:1457–1462
11. Wang Y, Guan ZH, Wen X (2004) Adaptive synchronization for Chen chaotic system with fully unknown parameters. *Chaos Soliton Fract* 19:899–903
12. Chua LO, Yang T, Zhong GQ, Wu CW (1996) Adaptive synchronization of Chua's oscillators. *Int J Bifur Chaos* 6(1):189–201
13. Liao TL (1998) Adaptive synchronization of two Lorenz systems. *Chaos Soliton Fract* 9:1555–1561
14. Fang JQ, Hong Y, Chen G (1999) Switching manifold approach to chaos synchronization. *Phys Rev E* 59:2523–2526
15. Yin X, Ren Y, Shan X (2002) Synchronization of discrete spatiotemporal chaos by using variable structure control. *Chaos Soliton Fract* 14:1077–1082
16. Wang C, Ge SS (2001) Adaptive synchronization of uncertain chaotic systems via backstepping design. *Chaos Soliton Fract* 12:199–206
17. Suykens JAK, Curran PF, Vandewalle J (1997) Robust nonlinear synchronization of chaotic Lur  systems. *IEEE Trans Circuits Syst I* 44(10):891–904
18. Moukam Kakmeni FM, Bowong S, Tchawoua C, Kaptoum E (2004) Chaos control and synchronization of a ϕ^6 -Van der Pol oscillator. *Phys Lett A* 322:305–323
19. Chang W-D, Yan J-J (2005) Adaptive robust PID controller design based on a sliding mode for uncertain chaotic systems. *Chaos Soliton Fract* 26(1):167–175
20. Matignon D (1996) Stability results for fractional differential equations with applications to control processing. In: Computational engineering in systems applications. Lille, France, IMACS, IEEE-SMC, 2:963–968

On Fractional Control Strategy for Four-Wheel-Steering Vehicle

Ning Chen, Nan Chen, and Ye Chen

Abstract Four-wheel-steering (4WS) system can enhance vehicle cornering ability by steering the rear wheels in accordance with the front wheels steering and vehicle status. In this paper, a fractional control approach for 4WS vehicle with its some design specifications are presented. The simulation results show this approach can improve the stability of vehicle when front wheels are steered too fast in some emergencies. To enhance the robustness of vehicle system, a yaw rate tracking fractional control strategy is also proposed to deal with the uncertainty of actual vehicle.

1 Introduction

Four-wheel-steering (4WS) system is one of three main chassis control systems in vehicle, it can enhance vehicle cornering ability by steering the rear wheels in accordance with the front wheels steering and vehicle status. With such steering control system, it becomes possible to improve the lateral stability and handling performance [2, 4, 6]. In past years, many researchers have proposed a lot of control algorithms and strategies for 4WS vehicle, such as LQR, slide mode variable structure control, fuzzy control, adapting control, H_∞ and μ synthesis robust control and so on. It isn't too hard to realize that, the control method of "front and rear wheels' steering ratio as a function of vehicle speed" [9] has become a prevailing trend in 4WS control. Namely, people try to keep the side-slip angle being

N. Chen (✉)

College of Mechanical Engineering, Southeast University, Nanjing, 210096, China
e-mail: chenning@njfu.com.cn

N. Chen

College of Mechanical Engineering, Southeast University, Nanjing, 210096, China
e-mail: nchen@seu.edu.cn

Y. Chen

College of Mechanical and Electronic Engineering, Nanjing Forestry University,
Nanjing, 210037, China
e-mail: chenye626@yahoo.cn

zero at vehicle's gravity center. However, zero side-slip angle approach seems to be infeasible since the driver might have a feeling of discomfort caused by its motion different from that of the conventional vehicle. So keeping the angle to a small is a more practical control algorithm as compared to zero. On the other hand, the most dangerous moment for vehicle often occurs when front steering wheel angle is too large, or the vehicle turns too fast and a larger overshoot of lateral acceleration happen. In this research, we propose a new rear wheel steering control approach for 4WS vehicle by introducing the fractional derivative theory so that the effects of front wheels steering angle and steering angular velocity are taken into consideration. And a yaw rate tracking control is also presented to enhance the robustness of vehicle control. Fractional calculus has a 300-year-long history with, the theory of fractional-order derivative developed mainly in the nineteenth century. Applying fractional calculus to dynamic system control is increasingly a focus of people's interest. Regardless of its physical meanings are not as clear as ordinary integer order calculus, the fractional calculus as a class of integral transformation has been used frequently to improve the performance and stabilities of some control systems in the last two decades. The PID control is one of typical examples [5, 8, 10].

This paper is organized as follows. In Sect. 2, the two degrees of freedom dynamic model of 4WS vehicle and a new control approach is proposed for 4WS vehicle by using the fractional derivative theory. In Sect. 3 the feedback control of yaw rate is introduced to fractional control approach to improve the performances of vehicles. And a yaw rate tracking control to enhance the robustness of vehicle control is also given. In Sect. 4, illustrative simulations are given to demonstrate the effectiveness of these control approaches. Section 5 concludes this paper with some remarks on future research.

2 Improvement of Feedforward Control for 4WS Vehicle by Introducing of Fractional Derivative

2.1 Two Degrees of Freedom Vehicle Model

To model the steering behavior of the vehicle, the so-called bicycle model is used with two degrees of freedom. The vehicle body is assumed to be a rigid beam and the left and right tires are combined into a single one. Furthermore, it is assumed that all motions occur in the plane of the vehicle model. This model describes the vehicle in-plane behavior sufficiently well at high speeds; however, it is unable to consider the effect of load transfer between the left and right side wheels. The schematic diagram of the model is shown in Fig. 1.

When the front and rear steering angles δ_f and δ_r are small, the linear equations of motion may be written as follows:

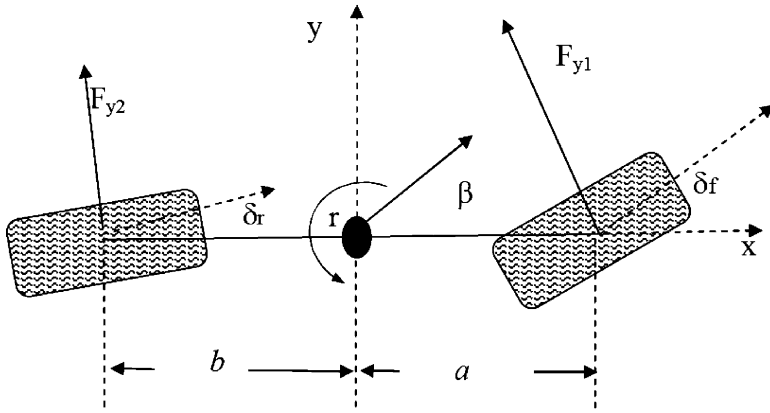


Fig. 1 Two degrees of freedom dynamic model of 4WS vehicle

$$\begin{cases} mV(\dot{\beta} + r) = -(C_f + C_r)\beta - \frac{aC_f - bC_r}{V}r + C_f\delta_f + C_r\delta_r, \\ J_z\dot{r} = -(aC_f - bC_r)\beta - \frac{a^2C_f + b^2C_r}{V}r + aC_f\delta_f + bC_r\delta_r. \end{cases} \quad (1)$$

where C_f and C_r are the central equivalent tire cornering stiffness of front and rear axis, respectively; V is the longitudinal components of vehicle gravity center speed based on the vehicle reference frame; m is the mass of the vehicle; δ_f and δ_r are the front and rear wheel steering angles respectively; a and b are the distance of gravity center between front and rear axes; J_z is the moment of inertia of the vehicle around the z -axis; r is the gravity center yaw rate; and β is the gravity center side-slip angle.

2.2 Fractional Order Feedforward Control Strategy

There have been many studies about 4WS vehicle control methods changing the vehicle's dynamics [2, 4, 6, 9]. Typically, the strategies associated with 4WS vehicle can be categorized into feedforward and feedback approaches. With the feedback approaches, the rear steer angle is usually set according to the vehicle statuses which are hard to be determined and the controller is complicate. With the feedforward approaches, the rear steer angle is usually set to be a function of the front steer angles. For example as follows:

$$\delta_r(t) = K\delta_f(t), \quad (2)$$

where the values of the constant K may be determined via Linear Optimal Control [1]. The goal of the filtering is usually to alleviate the reverse vehicle side-slip during transient responses. The case $K = 0$ can be regarded as a traditional two-wheel-steering vehicle (2WS). Because it is easier to realize, feedforward control strategies

have become the prevailing current of 4WS vehicle control strategies. But there are the same disadvantages in feedforward approaches, such as understeering at high speed and the effect of steering angle velocity is not considered. A new feedforward approaches is proposed here, that is

$$\delta_r(t) = KD^\alpha \delta_f(t - \tau), \quad 0 \leq \alpha < 1, \quad (3)$$

where K can be taken as a constant or a function of vehicle speed V . τ is the delay time what is introduced for enhancing the dynamic stability of 4WS vehicle. The $D^\alpha()$ represents the operator of fractional derivative, α is real number denoting the order of fractional derivative.

The Riemann–Liouville(RL) definition of the fractional differentiation is used here

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau) d\tau}{(t-\tau)^\alpha} \quad 0 \leq \alpha < 1, \quad (4)$$

where $\Gamma()$ is the Gamma function. The RL derivatives have many interesting properties. By direct calculation, we observe that the RL derivative of a constant is not zero, namely

$${}_0 D_t^\alpha C = C \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad (5)$$

and $\alpha = 0$ we have

$$D^0 f(t) = f(t). \quad (6)$$

Above means the approach (2) is only the specific case we proposed in approach (3). When $K = 0$, the 4WS vehicle also becomes a conventional 2WS vehicle. Substitution of Eq. 3 into Eq. 1 and using Laplace transformation, two transfer functions $G_\beta(s)$ and $G_r(s)$ from front steering input δ_f to side-slip angle and yaw rate are yielded.

$$G_\beta(s) = \frac{b_{12}Ke^{-\tau s}s^{1+\alpha} + b_{11}s + (a_{12}b_{22} - a_{22}b_{12})Ke^{-\tau s}s^\alpha + a_{12}b_{21} - a_{22}b_{11}}{s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21}} \quad (7)$$

$$G_r(s) = \frac{b_{22}Ke^{-\tau s}s^{1+\alpha} + b_{21}s + (a_{21}b_{12} - a_{11}b_{22})Ke^{-\tau s}s^\alpha + a_{21}b_{11} - a_{11}b_{21}}{s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21}} \quad (8)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -\frac{C_f + C_r}{mV} & -1 - \frac{aC_f - bC_r}{mV^2} \\ -\frac{aC_f - bC_r}{J_z} & -\frac{a^2C_f + b^2C_r}{J_zV} \end{bmatrix}, \quad (9)$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} \frac{C_f}{mV} & \frac{C_r}{mV} \\ \frac{aC_f}{J_z} & \frac{bC_r}{J_z} \end{bmatrix}, \quad (10)$$

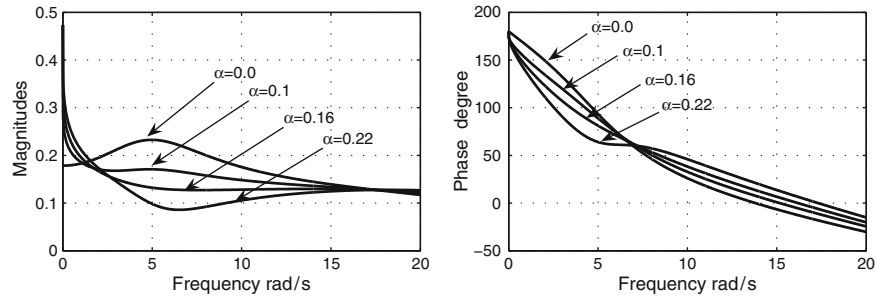


Fig. 2 Frequency response of vehicle side-slip angle β

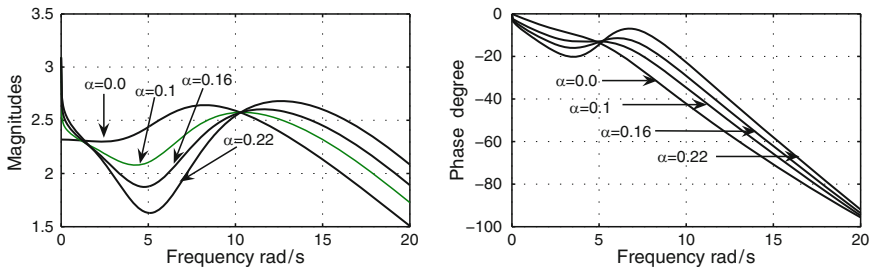


Fig. 3 Frequency response of vehicle yaw rate r

Table 1 Vehicle parameters

Parameter	Physical meaning	Value	Unit
m	Vehicle mass	1310	Kg
J_z	Inertia moment of vehicle yaw	2,352	Kgm^2
a	Distance from front axle to C.G	0.986	m
b	Distance from rear axle to C.G	1.596	m
C_f	Cornering stiffness of front tire	46,600	N/rad
C_r	Cornering stiffness of rear tire	65,350	N/rad

The response diagrams of magnitude and phase of the transfer functions $G_\beta(j\omega)$ and $G_r(j\omega)$ are given in Figs. 2 and 3 (vehicle parameters in Table 1). The larger the is, the larger magnitude of the $G_\beta(j\omega)$ and $G_r(j\omega)$ are at low frequencies ($\omega < 1$ rad/s). At high frequencies, as the value α increases, the magnitude of the $G_\beta(j\omega)$ ($1 \text{ rad/s} < \omega < \text{about } 18 \text{ rad/s}$ shown in Fig. 2) and $G_r(j\omega)$ ($1 \text{ rad/s} < \omega < \text{about } 10 \text{ rad/s}$ shown in Fig. 3) gets smaller. The peak value of magnitude curve is decreased when the $\alpha \neq 0$, this contributes to improve transient response of vehicle at high speed turning. The result shows the effects of steering angle velocity on performance of vehicle are considered by adopting the fractional control method.

3 Yaw Feedback Control for 4WS Vehicle

3.1 Fractional Order Control Strategy with Yaw Feedback

In order to improve the stability performances of 4WS vehicle, many states feedback control strategies for 4WS vehicle are researched in amount of papers [1, 2]. Owing to the side-slip angle is difficult to be determined, the yaw rate is often used as the output feed-back variable [3]. So the yaw rate is also taken as the variable of feedback control into Eq. 3, the close-loop control law is

$$\delta_r(t) = KD^\alpha \delta_f(t - \tau) + K_r r, \quad 0 \leq \alpha < 1, \quad (11)$$

where K_r is the gain of feedback control. Also, the transfer functions of side-slip angle and yaw rate are

$$G_\beta = \frac{N_\beta(s)}{D(s)}; \quad G_r = \frac{N_r(s)}{D(s)}, \quad (12)$$

where

$$\begin{aligned} N_\beta(s) &= Ke^{-\tau s} [b_{12}s^{1+\alpha} + (a_{12}b_{22} - a_{22}b_{12})s^\alpha] + b_{11}s + (b_{12}b_{21} - b_{11}b_{22})K_r \\ &\quad + a_{12}b_{21} - a_{22}b_{11}, \\ N_r(s) &= Ke^{-\tau s} [-b_{22}s^{1+\alpha} + (a_{11}b_{22} - a_{21}b_{12})s^\alpha] - b_{21}s + a_{11}b_{21} - a_{21}b_{11}, \\ D(s) &= s^2 - (a_{11} + a_{22} + b_{22}K_r)s + (a_{12}b_{22} - a_{21}b_{12})K_r + a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$

There are many design specifications to determine those parameters K_r , K , and α . A simple design specification is given here:

Firstly, according to the denominator (12.c) of transfer function (11), the K_r could be chosen to make the damping ratio of the system being 0.707. Thus, the K_r satisfies the equation

$$b_{22}K_r^2 + 2(a_{21}b_{12} + a_{22}b_{22})K_r + a_{11}^2 + a_{22}^2 + 2a_{12}a_{21} = 0. \quad (13)$$

After the choice of K_r , the poles of system or the nature frequencies ω_n are determined.

Secondly, the delay time and the gain K can be regarded as two tuning parameters to get the maximum magnitude of $|G_\beta(j\omega)|$ (when $\alpha = 0$) minimized. Thus, and K may be determined or be estimated by using the optimization method. The frequency ω_p is also determined when $|G_\beta(j\omega)|$ is at peak point.

Lastly, we choose the order α to keep $|G_\beta(j)|$ attenuate in a range of frequency $(0, \omega_p)$, it means that the slope of $|G_\beta(j\omega)|$ is negative and the following requirement is better to be fulfilled

$$|G_\beta(j\omega)|_{\omega=\omega_1, \alpha \neq 0} = |G_\beta(j\omega)|_{\omega=\omega_1, \alpha=0} \quad (14)$$

The frequency ω_1 is the half of or one-third of ω_p . On that condition the could be directly determined.

3.2 Yaw Rate Tracking Control Strategy for 4WS Vehicle

It is useful to improve the performance of 4WS vehicle by introducing the fractional order control method and yaw feedback control. But there are many parameters are uncertain or changing in actual vehicle system. So how to keep the yaw rate responses with robustness is an importance problem to vehicle dynamic control. Here, a yaw rate tracking control strategy is presented and it's control block diagram is shown in Fig. 4.

From Eq. 1, the relationship between state vector of actual vehicle system with front and rear tires steering inputs can be written as

$$\begin{pmatrix} \beta \\ r \end{pmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{pmatrix} \delta_f \\ \delta_r \end{pmatrix} \quad (15)$$

As above, these $\bar{G}_{ij} (i, j = 1, 2)$ stand for the transfer function matrix elements of the nominal vehicle model. From the control system in Fig. 4, we can obtain the yaw rate output as follow

$$r = \left[\frac{1}{1 + G_{22}P} \frac{G_{21}}{\bar{G}_{21}} + \frac{G_{22}P}{1 + G_{22}P} \right] \bar{G}_{21}\delta_f + \frac{G_{22}P}{1 + G_{22}P} \bar{G}_{22}\delta_r \quad (16)$$

where P is a fractional order $PI^\lambda D^\mu$ controller

$$P = K + K_i s^\lambda + K_d s^\mu \quad (17)$$

The five parameters of the fractional order $PI^\lambda D^\mu$ (17) can be determined by the tuning method proposed by Monje et al. [10] (P is the controller and G_{22} is the plant here).

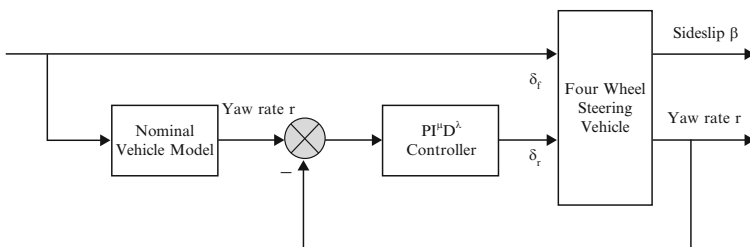


Fig. 4 The block diagram of yaw tracking control

4 Computational Simulation

To check the validity of the proposed control approach, numerical simulations are carried out under various steering inputs. The parameters of the vehicle and the tires adopted in simulation are summarized in Table 1.

For simulation of a fractional system in MATLAB, the Oustaloup approximation of fractional integration [7, 11] is adopted in simulation. Now suppose there are three kinds of vehicles running at same speed $V = 30\text{m/s}$ with different rear-wheels-steering control approach (shown in Table 2).

In order to investigate the effects of front wheels steering angle velocity on dynamic performances of vehicle, these kinds of vehicles are subjected to a steering angle input $\delta_f(t)$ vary as a sine function. First input frequency of $\delta_f(t)$ is 0.2 Hz and second's is 0.8 Hz. Simulation results of the side-slip angles and the yaw rates with time are shown in Figs. 5 and 6.

When the steering input frequency is small, the dynamic performances of Vehicle-I are a bit worse than that of Vehicle-II but better than that of Vehicle-III. On the contrary, if the steering input frequency is higher than 0.8 Hz, the dynamic performances of Vehicle-I are better than that of the others. If the frequency of $\delta_f(t)$ is lower and trends to zero, the dynamic performances of Vehicle-I have a tendency to that of 4WS vehicle only with yaw rate feedback control.

When vehicle parameters such as mass,cornering stiffness of tire, etc. vary, the yaw rate of 4WS vehicle should be different if the front wheels are steered same angle (see Fig. 7). This is a serious problem to driver. But, the vehicle with yaw rate tracking approach can be dealt with such problem most effectively (see Fig. 8). Here, the rear-wheel-steering control approach of vehicle which it's simulation

Table 2 The control laws of rear-wheel-steering for vehicles in simulation

Vehicle type	control law of rear-wheel-steering
Vehicle-I	$\delta_r(t) = 0.18D^{(0.23)}\delta_f(t - 0.12) + 0.06864r$
Vehicle-II	$\delta_r(t) = 0.18\delta_f(t) + 0.06864r$
Vehicle-III	$\delta_r(t) = 0$

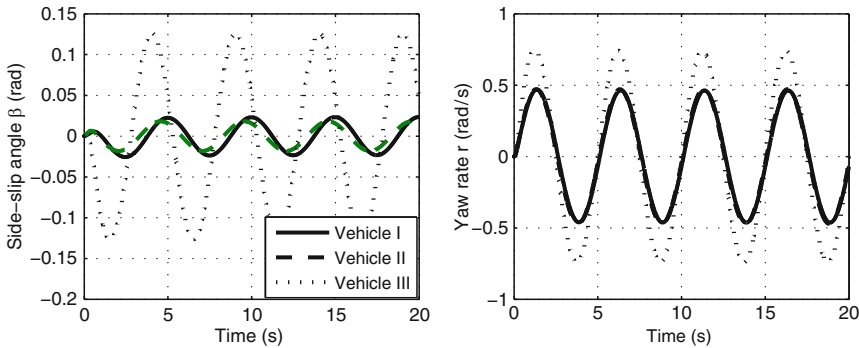


Fig. 5 Vehicle responses when front wheel steering $\delta_f = 0.2\sin(1.2t)$

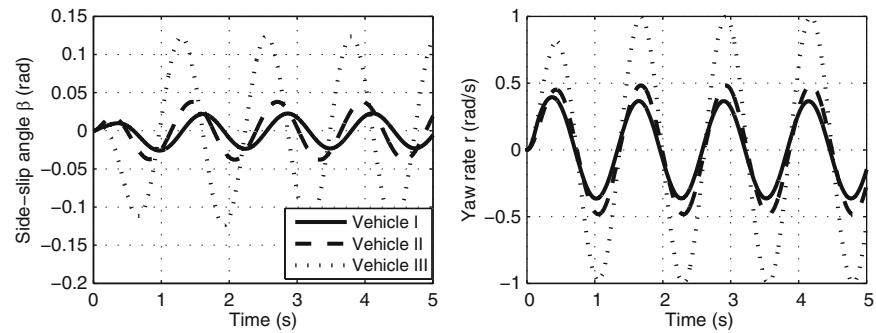


Fig. 6 Vehicle responses when front wheel steering $\delta_f = 0.2 \sin(5.2t)$

Fig. 7 The Step responses of yaw rates without yaw tracking control

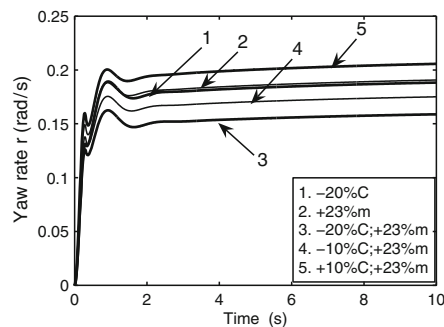
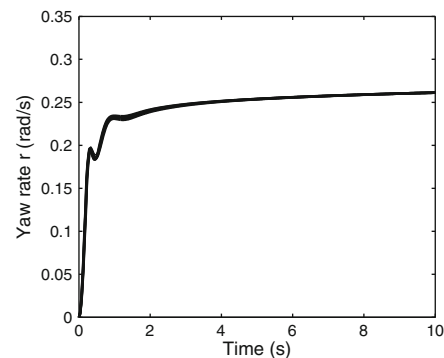


Fig. 8 The Step responses of yaw rates with yaw tracking control



results are shown in Fig. 7 is the same as that of the following model which be applied in yaw rate tracking control. Each parameter of $PI^\lambda D^\mu$ is: $K_p = -0.6313$, $K_i = -2.5811$, $K_d = 0.2684$, $\lambda = 0.9996$, $\mu = 0.01563$. The step response results of yaw rate are presented in Fig. 8 and they illustrate that the yaw rate tracking control approach can enhance the robustness of 4WS vehicle control.

5 Conclusion

A fractional control approach for four-wheel-steering vehicle, that is, the rear wheels steering angle is related to the fractional derivative of front wheels steering angle and vehicle's yaw rate, is proposed in the article. A simple design specification for this approach is also provided. From simulation results, this control strategy was certified that the effects of front wheels steering angular velocity has also been taken into count. Therefore, the dynamic performances of 4WS vehicle with this control approach can be enhanced when front wheels are steered too fast in some emergencies. Based on it, a yaw rate tracking control strategy with $PI^\lambda D^\mu$ control is proposed too. The simulation results certify this control approach can enhance the robustness of vehicle system. To sum up, the fractional control approach is a versatile, valuable and alternative control strategy that deserves more research.

Acknowledgement This work was supported by the Ford-China Research and Development Foundation (No. 50122153) and College Natural Science Foundation of Jiangsu Province, China (08KJD130002).

References

1. Cho YH, Kim J (1995) Design of optimal four-wheel steering system. *Vehicle Syst Dyn* 24:661–682
2. Guo KH (1998) Vehicle 4WS control strategy. *J Jilin Ind Univ* 92:1–4 (in Chinese)
3. Hu HY, Wu Z (2000) Stability and Hopf bifurcation of a four-wheel-steering vehicle involving driver's delay. *Nonlinear Dynam* 22:361–374
4. Iguchit SB (1989) Development of "Superhicas": a new rear wheel steering system with phase reversal control. SAE Paper No. 89197:327–332
5. Monje CA, Vinagre BM, Chen YQ, Feliu V, Lanusse P, Sabatier J (2004) Proposals for fractional PID tuning. In *First IFAC Workshop on Fractional Differentiation and its Applications*, Bordeaux
6. Namio I, Junsuke B (1990) 4WS technology and the prospect for improvement of vehicle dynamics. SAE Paper No. 901167:544–561
7. Oustaloup A (1995) *La drivation non entire: theorie, synthse et applications*. Hermès, Paris
8. Podlubny I (1999) Fractional-order systems and PID controllers. *IEEE Trans Automat Contr* 44:208–214
9. Sanos RS (1986) Four wheel steering system with rear wheel steer angle controlled as a function of steering wheel angle. SAE Paper No. 860625:363–365
10. Shunji Manabe (2002) A suggestion of fractional-order controller for flexible spacecraft attitude control. *Nonlin Dynam* 29:251–268
11. Tenreiro Machado JA (1997) Analysis and design of fractional-order digital control systems. *J Syst Anal Model Simul* 27:107–122

Fractional Order Sliding Mode Controller Design for Fractional Order Dynamic Systems

Mehmet Önder Efe

Abstract Sliding mode control, also called variable structure control, has been elaborated in this work. After adopting the reaching law approach, it is shown how integer order fractional variable structure control (FVSC) is achieved and the results are extended to the case of fractional order plants. Few examples are shown and relevant stability issues are discussed.

1 Introduction

Although the concept of Variable Structure Control (VSC) and the theory of fractional systems are not new, their integration, the FVSC, is an interesting field of research dwelt on this paper with some applications. The motivation of this research stands on two driving forces: First, most systems in reality display behavior characterized best in the domain of fractional order operators, second, the uncertainties on the process dynamics can appropriately be alleviated by utilizing the VSC technique. This is particularly because of the fact that the feedback control system is designed based upon a representative model which always introduces a plant-model mismatch entailing robustness. In the sequel, a brief summary of the relevant literature is presented to position the merit and effectiveness of the presented FVSC approach.

The approximation of the fractional derivative has been a core issue addressed recently by [12] with an in depth discussion. A comparison with Crone controllers as well as the placement of poles and zeros and step and impulse responses are discussed. A particularly focussed section of the fractional order control is the design of PID controllers having noninteger order integration of order λ and differentiation of order μ , i.e. $PI^\lambda D^\mu$ setting. In [11], tuning of controller gains and noninteger differentiation orders are discussed with a set of Ziegler–Nichols based tuning rules.

M.Ö. Efe (✉)

TOBB Economics and Technology University, Söğütözü Cad. No. 43, TR-06560 Söğütözü
e-mail: onderefe@etu.edu.tr

Determining the values as a result of an optimization process subjected to several design specifications on the frequency response is considered in [8], where the design exploits the utilities of Laplace transform. A comparison of several controller designs are presented in [14], thorough investigations considering the continuous time case is dealt with in [9] and in [5]. Applications focusing on adaptive control can be found in [7] while a brief treatment of state space models with discretization is presented in [2]. Control of switched fractional order systems is discussed in the framework of generalized PI sliding control in [10], where an electric radiator system is used to validate the analytical claims.

The notion of variable structure control dates back to the pioneering work [4]. Philosophically, the system behavior is driven towards a predefined subspace of the state space (or the phase space), which is an attractor guiding the trajectories on it toward the origin of the state space [6, 13, 16]. The whole course of motion is comprised of two phases, namely, the reaching mode and the sliding mode. The motion in the phase space is attracted by the sliding hyperplane, and after reaching this particular locus, the trajectories are confined to this subspace while sliding towards the origin of the phase space. The confinement to a lower dimensional subspace is the fact giving the name VSC. Frequently, the name Sliding Mode Control (SMC) is used to mean VSC referring to the latter dynamic behavior. Although the design task is a well-established topic for integer order systems, to our best knowledge, there are no attempts on fractional variable structure control for fractional order systems.

This paper is organized as follows: The second section describes integer order VSC for integer order systems, the third section gives the fractional VSC for fractional order systems. The fourth section presents some examples and the conclusions are presented at the end of the paper.

2 Integer Order VSC for Integer Order Systems

Consider the system

$$\begin{aligned}\dot{x}_i &= x_{i+1}, \quad i = 1, 2, \dots, n-1 \\ \dot{x}_n &= f(\cdot) + g(\cdot)u, \quad g(\cdot) \neq 0\end{aligned}\tag{1}$$

where $f(\cdot)$ and $g(\cdot)$ are functions of the state variables, and consider a given reference trajectory $r(t)$, possessing the derivatives $\dot{r}, \ddot{r}, \dots, r^{(n)}$ all being finite. Define $d_i := r^{(i-1)}$, $i = 1, 2, \dots, n$ and the state tracking errors $e_i := x_i - d_i$, $i = 1, 2, \dots, n$. Choose a sliding hypersurface $s := e_n + \sum_{i=1}^{n-1} \lambda_i e_i$ such that the dynamics described by $s = 0$ is stable. The goal of the design is to make sure that $\dot{s} = -\zeta \text{sgn}(s)$ is satisfied. This particularly emphasizes that the design forces to reach where $s = 0$ and to maintain this value, while this is satisfied, due to the stability requirement on the coefficients λ_i , the errors tend towards the origin. More explicitly we have

$$\begin{aligned}\dot{s} &:= \dot{e}_n + \sum_{i=2}^n \lambda_{i-1} e_i = f(\cdot) + g(\cdot)u - r^{(n)} + \sum_{i=2}^n \lambda_{i-1} e_i \\ &:= -\zeta \operatorname{sgn}(s)\end{aligned}\quad (2)$$

and solving u from the equality of the last two lines, the control law given by $u = g(\cdot)^{-1}(r^{(n)} - f(\cdot) - \sum_{i=2}^n \lambda_{i-1} e_i - \zeta \operatorname{sgn}(s))$ is obtained. With this control action, a hitting in finite time occurs and this time is bounded as given by $t_h \leq \frac{|s(0)|}{\zeta}$ sec., and the motion for $t > t_h$ takes place in the vicinity of the sliding hypersurface. The crux of the presented design is the fact that one can choose ζ such that if there are uncertainties on the functions embodying the system dynamics, the VSC scheme can respond robustly against such design difficulties.

In the literature, several modifications are proposed to eliminate the chattering arising due to the dependence on the sign of a quantity that is very close to zero. A common choice is to adopt the following approximation $\operatorname{sgn}(s) \approx \frac{s}{|s|+\delta}$ with $\delta > 0$ determining the slope around the origin.

In what follows, we present the design of a fractional VSC scheme for fractional systems described in the state space form.

3 Fractional Order VSC for Fractional Order State Space Systems

Consider the fractional order state space system

$$\begin{aligned}x_i^{(\beta_i)} &= x_{i+1}, \quad i = 1, 2, \dots, n-1 \\ x_n^{(\beta_n)} &= f(\cdot) + g(\cdot)u\end{aligned}\quad (3)$$

where $0 < \beta_i < 1$ are the fractional differentiation orders, $f(\cdot)$ and $g(\cdot)$ are functions of the state variables. Consider a given reference trajectory $d_1 = r$, possessing the fractional derivatives $d_i = r^{(Q_i)}$, $Q_i = \sum_{k=1}^{i-1} \beta_k$, $i = 2, 3, \dots, n$ all being finite. Define the state tracking errors $e_i := x_i - d_i$. Choose a sliding hypersurface $s := e_n + \sum_{i=1}^{n-1} \lambda_i e_i$ such that the dynamics described by $s = 0$ is stable. Now differentiate s at order β_n . This yields

$$\begin{aligned}s^{(\beta_n)} &:= e_n^{(\beta_n)} + \sum_{i=1}^{n-1} \lambda_i e_i^{(\beta_n)} = f(\cdot) + g(\cdot)u - d_n^{(\beta_n)} + \sum_{i=1}^{n-1} \lambda_i e_i^{(\beta_n)} \\ &:= -\zeta \operatorname{sgn}(s)\end{aligned}\quad (4)$$

Solving the control signal would let us have

$$u = \frac{d_n^{(\beta_n)} - f(\cdot) - \sum_{i=1}^{n-1} \lambda_i e_i^{(\beta_n)} - \zeta \operatorname{sgn}(s)}{g(\cdot)}\quad (5)$$

Indeed, the application of this signal forces the reaching dynamics $s^{(\beta_n)} = -\zeta \operatorname{sgn}(s)$, which enforces $ss^{(\beta_n)} = -\zeta|s| < 0$, $s \neq 0$. Obtaining $s^{(\beta_n)}(t)s(t) < 0$ can arise in the following cases. In the first case, $s(t) > 0$ and the integral $\int_0^t \frac{s(\xi)}{(t-\xi)^\beta} d\xi$ is monotonically decreasing. In the second case $s(t) < 0$ and the integral $\int_0^t \frac{s(\xi)}{(t-\xi)^\beta} d\xi$ is monotonically increasing. In both cases, the signal $|s(t)|$ is forced to converge the origin faster than $t^{-\beta}$. A natural consequence of this is to observe a very fast reaching phase as the signal $t^{-\beta}$ is a very steep function around $t \approx 0$. In conventional sense, one can have the following equalities to see the closed loop stability [13].

$$s^{(\beta_n)} = -\zeta \operatorname{sgn}(s) \quad (6)$$

Defining the fractional differintegration operator of order β by $D^{(\beta)}$, integrating both sides by order β_n yields (7), and differentiating once at order unity gives (8).

$$s = -\zeta D^{(-\beta_n)} \operatorname{sgn}(s) \quad (7)$$

$$\dot{s} = -\zeta D^{(1-\beta_n)} \operatorname{sgn}(s) \quad (8)$$

According to [13], $\operatorname{sgn}(D^{(1-\beta_n)} \operatorname{sgn}(s)) = \operatorname{sgn}(s)$ and this proves that the chosen form of the control signal causes $s\dot{s} \leq 0$. This result practically tells us that the locus described by $s = 0$ is an attractor, and when confined to this subspace, the errors tend towards the origin and the closed loop systems displays certain degrees of robustness to uncertainties and becomes insensitive to disturbances entering the system through control channels.

4 Two Application Examples

In this section, we consider a second order and a third order system to demonstrate the temporal results of FVSC scheme.

4.1 A Second Order System (Two State Variables)

The system considered in this example is a linear one described by

$$\begin{pmatrix} \dot{x}_1^{(\beta_1)} \\ \dot{x}_2^{(\beta_2)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \quad (9)$$

where $\beta_1 = 0.4$ and $\beta_2 = 0.8$.

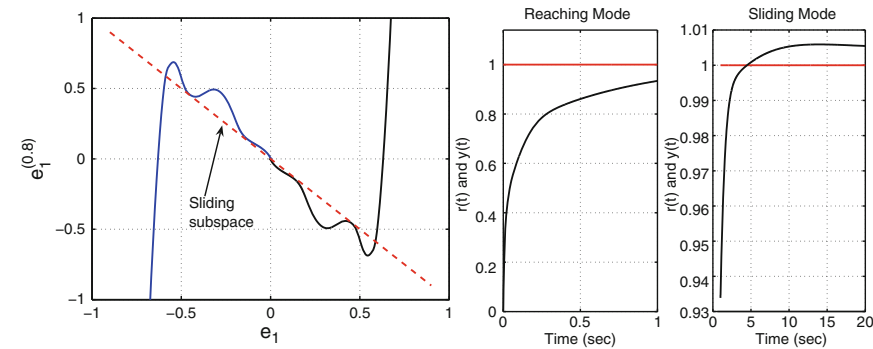


Fig. 1 Left: phase space behavior for $r(t) = 1$ and $r(t) = -1$. In this case $n = 2$, $\lambda_1 = 1$ and $\zeta = 1$. Right: reaching and sliding modes when $r(t) = 1$. In this case $n = 2$, $\lambda_1 = 1$ and $\zeta = 1$

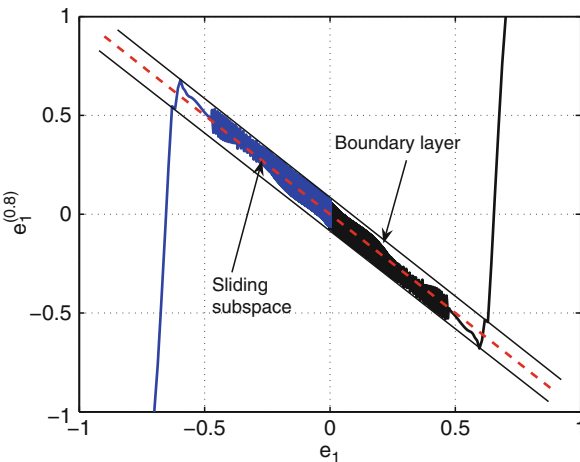


Fig. 2 Reaching and sliding modes with $\zeta = 10$, $r(t) = 1$ and $r(t) = -1$

For the results shown in Fig. 1, $\lambda_1 = 1$, $\zeta = 1$ and $\delta = 0.05$. A rough look at these figures could stipulate that the systems output rises up very quickly, gets very close to the final value and slows down during the near target region. Such a behavior supposedly entails considerable control efforts during the reaching phase.

Keeping all other variables the same and setting $\zeta = 10$ would let us obtain the result depicted in Fig. 2, where the boundary layer is much visible in this case and the cost of this is excessively high frequency components introduced into the control signal.

The results seen in the figures demonstrate that a sliding mode response can emerge after a fast reaching phase and the trajectories in the phase space lie on this particular line, called *switching surface* in the related literature. Once confined to

the locus described by $s = 0$, the error dynamics is governed by $e_2 + \lambda_1 e_1 = 0$ or more explicitly,

$$e_1^{(0.8)}(t) = -e_1(t), \quad e_1(0) = x_1(0) - r_1(0). \quad (10)$$

This is a stable dynamics as it satisfies the condition $|\arg(-\lambda_1)| = \pi > 0.8\frac{\pi}{2}$. The reader is referred to [1] for details.

4.2 A Third Order System (Three State Variables)

In this section, we consider a nonlinear and uncertain model to validate our claims. A similar model considered in this example was studied several times previously in [15] and in [3] with integer order derivatives. In order to demonstrate the reaching and sliding phases in three dimensions, we consider the plant with the following details.

$$\begin{aligned} x_i^{(\beta_1)} &= x_i + 1, \quad i = 1, 2 \\ x_3^{(\beta_3)} &= f(\cdot) + \Delta f(\cdot) + g(\cdot)u + \xi \end{aligned} \quad (11)$$

where $f(x_1, x_2, x_3) = -0.5x_1 - 0.5x_2^3 - 0.5x_3|x_3|$ is the known nominal part of the nonlinear part, $g(t) = 1 + 0.1 \sin(\frac{\pi t}{3})$ is a known nonzero function,

$$\begin{aligned} \Delta f(x_1, x_2, x_3) &= (-0.05 + 0.25 \sin(5\pi t))x_1 + (-0.03 + 0.3 \cos(5\pi t))x_2^3 \\ &\quad + (-0.05 + 0.25 \sin(7\pi t))x_3|x_3| \end{aligned} \quad (12)$$

is the uncertainty that is not available for the design and $\xi = 0.2 \sin(4\pi t)$ is the disturbance entering the control system. The fractional differentiation orders are $\beta_1 = \beta_2 = \beta_3 = \beta = 0.5$ and we choose $\lambda_2 = 2\sigma > 0$ and $\lambda_1 = \sigma^2$ making the dynamics in the sliding regime

$$\left(\frac{d^\beta}{dt^\beta} + \sigma \right)^2 e_1 = 0 \quad (13)$$

Clearly the implication of this is

$$e_1^{2\beta} + 2\sigma e_1^\beta + \sigma^2 e_1 = e_3 + \lambda_2 e_2 + \lambda_1 e_1 = s = 0 \quad (14)$$

Unsurprisingly, the choice in (13) ensures that the dynamics characterized by $s = 0$ is stable as $|\arg(-\sigma)| = \pi > \beta\frac{\pi}{2}$.

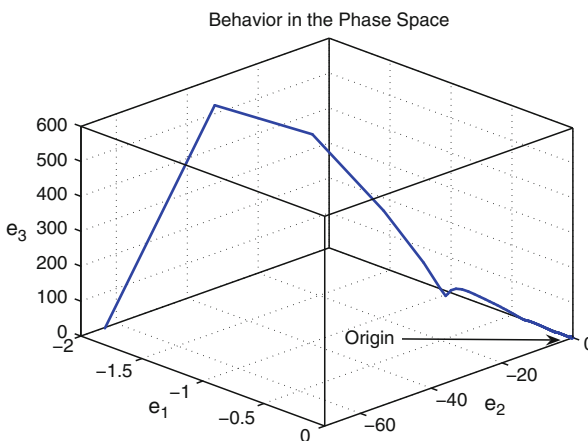


Fig. 3 Phase space behavior with $\zeta = 1$, $r(t) = 1 + \cos(t)$

In Fig. 3, the phase space behavior is illustrated. After a very fast reaching phase, the shown trajectory converges the origin of the phase space indicating that the three error components are maintained in the vicinity of the origin.

5 Concluding Remarks

This paper considers the fractional variable structure control of fractional order systems described in the controller canonical form. Two application examples are considered. The first example is a linear one having two state variables and the sliding subspace is a one dimensional locus, indeed a line, in the two dimensional phase space. Two exemplar cases are illustrated on a single figure emphasizing that the sliding subspace is an attractor guiding the phase space trajectories to the origin. The latter part of this conclusion is due to the stability of the locus described by $s = 0$.

The second example is involved with a nonlinear and uncertain plant with disturbances. The process has three state variables and the sliding subspace is a plane in the entire phase space. Satisfactorily successful results are obtained in this case too. Robustness observed against disturbances and uncertainties is a prominent feature to emphasize.

The purpose of this paper is to demonstrate that the fractional sliding mode control might be a practical alternative when the process under investigation is represented in fractional orders of derivative operator and when the performance specifications excessively stringent for manipulations in integer order. The approach and the examples are in good compliance with each other. Briefly, the two essential phases of the design is to show that all trajectories are attracted by the sliding subspace, second, the behavior on the sliding subspace is stable.

Acknowledgements The Matlab toolbox *Ninteger* v.2.3¹ is used and the efforts of its developer, Dr. Duarte Valério, are gratefully acknowledged. This work is supported by Turkish Scientific Council (TÜBİTAK) Contract 107E137.

References

1. Chen YQ, Ahn H-S, Podlubny I (2006) Robust stability check of fractional order linear time invariant systems with interval uncertainties. *Signal Process* 86:2611–2618
2. Dorcak L, Petras I, Kostial I, Terpak J (2002) Fractional-order state space models. *International Carpathian Control Conference*. Malenovice, Czech Republic, 27–30 May, pp 193–198
3. Efe MÖ (2002) A novel error critic for variable structure control with an adaline. *Trans Inst Meas Control* 24:403–415
4. Emelyanov SV (1967) *Variable structure control systems*. Moscow, Nauka
5. Hartley TT, Lorenzo CF (2002) Control of initialized fractional-order systems. *NASA Technical Report*
6. Hung JY, Gao W, Hung JC (1993) Variable structure control: a survey. *IEEE Trans Ind Electron* 40:2–22
7. Ladaci S, Charef A (2006) On fractional adaptive control. *Nonlinear Dynam* 43:365–378
8. Monje CA, Vinagre BM, Chen YQ, Feliu V, Lanusse P, Sabatier J (2004) Proposals for fractional $\pi^{\lambda}d^{\mu}$ tuning. The first IFAC symposium on fractional differentiation and its applications. Bordeaux, France, 19–20 July
9. Ortigueira MD (2000) Introduction to fractional linear systems part 1: continuous-time case. *IEE Proc-Vis Image Sign* 147:62–70
10. Sira-Ramirez H, Feliu-Batlle V (2006) On the gpi-sliding mode control of switched fractional order systems. *Proceedings of the 2006 workshop on variable structure systems*. Alghero, Italy, 5–7 June, 310–315
11. Valerio D (2006) Tuning of fractional pid controllers with ziegler-nichols type rules. *Signal Process* 86:2771–2784
12. Valerio D, Sa da Costa J (2005) Time-domain implementation of fractional order controllers. *IEE Proc-Contr Theor App* 152:539–552
13. Vinagre BM, Calderón AJ (2006) On fractional sliding mode control. *Proceedings of the 7th Portuguese conference on automatic control (CONTROLO'2006)*. Lisbon, Portugal, 11–13 September
14. Xue D, Chen YQ (2002) A comparative introduction of four fractional order controllers. *Proceedings of the 4th world congress on intelligent control and automation*. Shanghai, China, 10–14 June, 3228–3235
15. Yilmaz C, Hurmuzlu Y (2000) Eliminating the reaching phase from variable structure control. *J Dynam Syst Meas Contr-Trans ASME* 122:753–757
16. Young KD, Utkin VI, Özgüner Ü (1999) A control engineer's guide to sliding mode control. *IEEE Trans Contr Syst Technol* 7:328–342

¹ <http://mega.ist.utl.pt/dmov/ninteger/ninteger.htm>

A Fractional Order Adaptation Law for Integer Order Sliding Mode Control of a 2DOF Robot

Mehmet Önder Efe

Abstract The aim of adaptive sliding mode control is to maintain some robustness with a set of performance indications, and to observe non-drifting evolution of tunable parameters. This work demonstrates that a fractional adaptation law can achieve this better than its integer order counterpart. A 2DOF Scara robot is utilized to justify the claims.

1 Introduction

Fractional calculus and dynamics described by fractional differential equations are becoming more and more popular as the underlying facts about the differentiation and integration is significantly different from the integer order counterparts and beyond this, many real life systems are described better by fractional differential equations, e.g. heat equation, telegraph equation and a lossy electric transmission line are all involved with fractional order operators. Two of the highly valuable references for this subject field are [8, 9], where some discussion is devoted to fractional order PID controller, i.e. $PI^\lambda D^\mu$, is presented to some extent. A majority of works published so far has concentrated on fractional variants of the PID controller implemented for the control of linear systems, for which the issues of parameter selection, tuning, stability and performance are rather mature concepts (see [6]) than those involving the nonlinear models and nonlinearities in the approaches (see [7]).

Parameter tuning in adaptive control systems is a central part of the overall mechanism alleviating the difficulties associated with the changes in the parameters that influence the closed loop performance. Numerous remarkable studies are reported in the past and the field of adaptation has become a blend of techniques of dynamical systems theory, optimization and heuristics (intelligence). Today, the advent of very high speed computers and networked computing facilities, even within

M.Ö. Efe (✉)

TOBB Economics and Technology University, Söğütözü Cad. No 43, TR-06560 Söğütözü
e-mail: onderefe@etu.edu.tr

microprocessor based systems, tuning of system parameters based upon some set of observations and decisions has greatly been facilitated. In [1], an in depth discussion for parameter tuning in continuous and discrete time is presented. Particularly for gradient descent rule for model reference adaptive control, which is considered in the integer order in [1], has been implemented in fractional order by [10], where the integer order integration is replaced with an integration of fractional order 1.25 and by [5] where the good performance in noise rejection is emphasized.

The contribution of this work is to present a nonlinear adaptation law in fractional order and to emphasize the advantages like the elimination of drifts in the adjustable parameters and better system response. In the next section, we introduce the plant dynamics. In the third section, the sliding mode control through fractional order adaptation is given. Simulation results and the concluding remarks constitute the last part of the paper.

2 Robot Dynamics and the Control Problem

The dynamics of the system under control is given by $M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) = \tau - \eta$, where $\theta = (\theta_1 \ \theta_2)^T$ vector of angular positions and $\dot{\theta} = (\dot{\theta}_1 \ \dot{\theta}_2)^T$ is the vector of angular velocities. In above, $\tau = (\tau_1 \ \tau_2)^T$ is the vector of control inputs (torques) and $\eta = (\eta_1 \ \eta_2)^T$ is the vector of friction forces. The terms M and V are given below:

$$M = \begin{pmatrix} p_1 + 2p_3 \cos(\theta_2) & p_2 + p_3 \cos(\theta_2) \\ p_2 + p_3 \cos(\theta_2) & p_2 \end{pmatrix}, \quad V = \begin{pmatrix} -\dot{\theta}_2(2\dot{\theta}_1 + \dot{\theta}_2)p_3 \sin \theta_2 \\ \dot{\theta}_1^2 p_3 \sin \theta_2 \end{pmatrix} \quad (1)$$

where $p_1 = 3.31655 + 0.18648M_p$, $p_2 = 0.1168 + 0.0576M_p$ and $p_3 = 0.16295 + 0.08616M_p$. Here, M_p denotes the payload mass. The details of the plant model can be found in [2, 3]. The constraints regarding the plant dynamics are $|\tau_1| \leq 245\text{N}$, $|\tau_2| \leq 39.2\text{N}$, and the friction terms are $\eta_1 = 4.9\text{sgn}(\dot{\theta}_1)$ and $\eta_2 = 1.67\text{sgn}(\dot{\theta}_2)$.

The control problem is to force the system states to a predefined and differentiable trajectories within the workspace of the robot. More explicitly $e_1 = \theta_1 - r_1$, $e_2 = \theta_2 - r_2$ and the (integer order) time derivatives of these error terms are desired to converge the origin of the phase space.

3 Sliding Mode Control Through a Fractional Order Adaptation Scheme

Theorem 1. *Let r_1 and r_2 be continuous and differentiable reference trajectories. Let the switching function for each link be defined by*

$$s_{p,i} = \dot{e}_i + \lambda_i e_i, \quad i = 1, 2, \quad \lambda_i > 0 \quad (2)$$

Let

$$\tau_i = \phi_i^T u_i, \quad i = 1, 2 \quad (3)$$

be the controller of the i -th link with $\phi_i = (\phi_{i,1} \ \phi_{i,2} \ \phi_{i,3})^T$ being the adjustable parameter set for the i -th controller and $u_i = (e_i \ \dot{e}_i \ 1)^T$ being the vector signal exciting the i -th controller. Define $\tau_{d,i}$ as a control signal forcing the desired system response at i -th link and $\forall i \in \{1, 2\}$ let

$$\left| \sum_{k=1}^{\infty} \frac{\Gamma(1+\beta)}{\Gamma(1+k)\Gamma(1-k+\beta)} (\phi_i^{(\beta-k)})^T u_i^{(k)} \right| \leq \mathcal{B}_1 \quad (4)$$

$$|\tau_{d,i}^{(\beta)}| \leq \mathcal{B}_2, \quad \forall i \in \{1, 2\} \quad (5)$$

$$|\tau_i| \leq \mathcal{B}_3, \quad \forall i \in \{1, 2\} \quad (6)$$

The adaptation law

$$\phi_i^{(\beta)} = -\frac{u_i}{u_i^T u_i} K_i \text{sgn}(\sigma_i) \quad (7)$$

with $\sigma_i := \tau_i - \tau_{d,i}$ drives the parameters of the i -th controller to values such that the plant under control enters the sliding mode, hitting in finite time satisfying

$$\frac{\mathcal{K} - \mathcal{B}_1}{\Gamma(1+\beta)} t_{h,i}^\beta \leq \frac{|\sigma_i^{(\beta-1)}(0)| + |\tau_{d,i}^{(\beta-1)}(0)|}{\Gamma(\beta)} t_{h,i}^{\beta-1} + |\tau_{d,i}(t_{h,i})| \quad (8)$$

is observed if $\mathcal{K} > \mathcal{B}_1 + \mathcal{B}_2$ is satisfied.

Proof. The block diagram of the control system is depicted in Fig. 1.

Remark 1. The integer order version of this problem is studied in [4], where the crux of the approach is to extract a quantified error on the applied control signal utilizing the available measurements. In this reference, the map $\Psi(\cdot)$ is a monotonically increasing function of its argument and a common choice for it is a unit function, i.e. $\sigma_i = s_{p,i}$. The practical interpretation of this choice is the adoption of the distance from the switching line as a measure to penalize the control action. That is to say, set of all control signals driving the error vector toward the sliding hypersurface is denoted by $\tau_{d,i}$ and the error on the control signal (see [4]) described by $\tau_i - \tau_{d,i}$ is a monotonically increasing function along the $s_{p,i}$ axis. Such a selection with a tuning mechanism minimizing the value of $s_{p,i}$ naturally forces the emergence of sliding mode in the conventional sense.

Define $\gamma_i := \sum_{k=1}^{\infty} \frac{\Gamma(1+\beta)}{\Gamma(1+k)\Gamma(1-k+\beta)} (\phi_i^{(\beta-k)})^T u_i^{(k)}$ and check whether the quantity $\sigma_i^{(\beta)} \sigma_i$ for every i is negative or not. With these expressions, we have

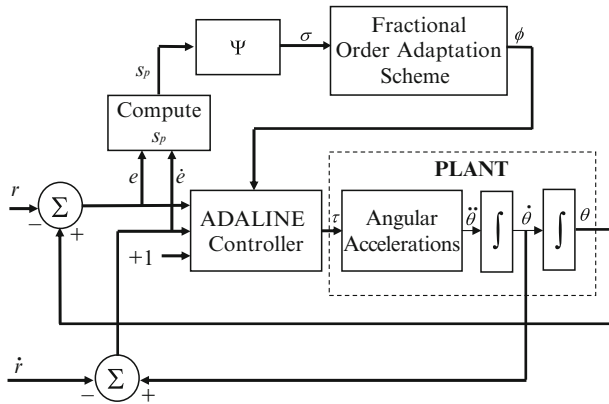


Fig. 1 Block diagram of the control system

$$\begin{aligned}
 \sigma_i^{(\beta)} \sigma_i &= (\tau_i^{(\beta)} - \tau_{d,i}^{(\beta)}) \sigma_i = ((\phi_i^{(\beta)})^T u_i) \sigma_i + (\gamma_i - \tau_{d,i}^{(\beta)}) \sigma_i \\
 &= -\mathcal{K} \operatorname{sgn}(\sigma_i) \sigma_i + (\gamma_i - \tau_{d,i}^{(\beta)}) \sigma_i \leq -\mathcal{K} |\sigma_i| + |\gamma_i| |\sigma_i| + |\tau_{d,i}^{(\beta)}| |\sigma_i| \\
 &\leq (-\mathcal{K} + \mathcal{B}_1 + \mathcal{B}_2) |\sigma_i| \leq 0 \quad \text{Since } \mathcal{K} > \mathcal{B}_1 + \mathcal{B}_2
 \end{aligned} \tag{9}$$

This proves that the trajectories in the phase space are attracted by the subspace described by $\sigma_i = 0$. Since ${}_0\mathbf{D}_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-\xi)^{-\beta} f(\xi) d\xi = f^{(\beta)}(t)$, claiming $\sigma_i^{(\beta)} \sigma_i < 0$ for stability is equivalent to the following

$$\sigma_i^{(\beta)}(t) \sigma_i(t) = \frac{\sigma_i(t)}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{\sigma_i(\xi)}{(t-\xi)^\beta} d\xi \tag{10}$$

Obtaining $\sigma_i^{(\beta)}(t) \sigma_i(t) < 0$ can arise in the following cases. In the first case, $\sigma_i(t) > 0$ and the integral $\int_0^t \frac{\sigma_i(\xi)}{(t-\xi)^\beta} d\xi$ is monotonically decreasing. In the second case $\sigma_i(t) < 0$ and the integral $\int_0^t \frac{\sigma_i(\xi)}{(t-\xi)^\beta} d\xi$ is monotonically increasing. In both cases, the signal $|\sigma_i(t)|$ is forced to converge the origin faster than $t^{-\beta}$. A natural consequence of this is to observe a very fast reaching phase as the signal $t^{-\beta}$ is a very steep function around $t \approx 0$.

When plotted, it is seen that the reaching force read along the vertical axis is excessive initially and as $|\sigma|$ approaches zero, this force gradually decreases and this property leads to very fast reaching and large control signals during the early instants of control applications. Now we must prove that first hitting to the switching function occurs in finite time, denoted by $t_{h,i}$. Evaluate $\sigma_i^{(\beta)}$ utilizing (7) as given below.

$$\sigma_i^{(\beta)} = -\mathcal{K} \operatorname{sgn}(\sigma_i) + \gamma_i - \tau_{d,i}^{(\beta)} \tag{11}$$

Applying the fractional integration defined as ${}_0\mathbf{I}_t^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\xi)^{\beta-1} f(\xi) d\xi$ with final time $t = t_{h,i}$ to both sides of (11) one gets

$$\sigma_i(t_{h,i}) - \sigma_i^{(\beta-1)}(0) \frac{t_{h,i}^{\beta-1}}{\Gamma(\beta)} = \frac{-\mathcal{K} \operatorname{sgn}(\sigma_i(0))}{\Gamma(1+\beta)} t_{h,i}^\beta + {}_0\mathbf{I}_{t_{h,i}}^\beta \left(\gamma_i - \tau_{d,i}^{(\beta)} \right) \quad (12)$$

Noting that $\sigma_i(t) = 0$ when $t = t_{h,i}$, multiplying both sides of (12) by $\operatorname{sgn}(\sigma_i(0))$, we have

$$\begin{aligned} -\sigma_i^{(\beta-1)}(0) \operatorname{sgn}(\sigma_i(0)) \frac{t_{h,i}^{\beta-1}}{\Gamma(\beta)} &= \frac{-\mathcal{K}}{\Gamma(1+\beta)} t_{h,i}^\beta \\ + {}_0\mathbf{I}_{t_{h,i}}^\beta (\operatorname{sgn}(\sigma_i(0)) \gamma_i) - {}_0\mathbf{I}_{t_{h,i}}^\beta (\operatorname{sgn}(\sigma_i(0)) \tau_{d,i}^{(\beta)}) \end{aligned} \quad (13)$$

Due to the above definition of fractional order integration, we have

$${}_0\mathbf{I}_{t_{h,i}}^\beta (\operatorname{sgn}(\sigma_i(0)) \gamma_i) \leq {}_0\mathbf{I}_{t_{h,i}}^\beta |\gamma_i| \leq {}_0\mathbf{I}_{t_{h,i}}^\beta \mathcal{B}_1 = \mathcal{B}_1 \frac{t_{h,i}^\beta}{\Gamma(1+\beta)} \quad (14)$$

Similarly

$$\begin{aligned} {}_0\mathbf{I}_{t_{h,i}}^\beta \left(\operatorname{sgn}(\sigma_i(0)) \tau_{d,i}^{(\beta)} \right) &= \operatorname{sgn}(\sigma_i(0)) {}_0\mathbf{I}_{t_{h,i}}^\beta \tau_{d,i}^{(\beta)} \\ &= \operatorname{sgn}(\sigma_i(0)) \left(\tau_{d,i}(t_{h,i}) - \tau_{d,i}^{(\beta-1)}(0) \frac{t_{h,i}^{\beta-1}}{\Gamma(\beta)} \right) \end{aligned} \quad (15)$$

Substituting the results in (14) and (15) into (13), we obtain an inequality given as

$$\begin{aligned} -\sigma_i^{(\beta-1)}(0) \operatorname{sgn}(\sigma_i(0)) \frac{t_{h,i}^{\beta-1}}{\Gamma(\beta)} &\leq \frac{-\mathcal{K}}{\Gamma(1+\beta)} t_{h,i}^\beta + \mathcal{B}_1 \frac{t_{h,i}^\beta}{\Gamma(1+\beta)} \\ -\operatorname{sgn}(\sigma_i(0)) \tau_{d,i}(t_{h,i}) + \tau_{d,i}^{(\beta-1)}(0) \operatorname{sgn}(\sigma_i(0)) \frac{t_{h,i}^{\beta-1}}{\Gamma(\beta)} \end{aligned} \quad (16)$$

Straightforward manipulations will lead to the inequality in (8). Clearly, the left hand side of the inequality in (8) is a monotonically increasing function of $t_{h,i}$. On the other hand, the right hand side of the inequality is a monotonically decreasing function of $t_{h,i}$. With these facts, the inequality is satisfied on the interval $t_{h,i} \in (0, \alpha]$, where α is the point of intersection of the two expressions lying on the left and right hand sides of (8). According to this discussion, one can see that $t_{h,i} \leq \alpha$ and particularly for $\beta = 0.5$, we have the following value: $\alpha = \left(\frac{b + \sqrt{b^2 + 4ac}}{2a} \right)^2$, where $a = \frac{\mathcal{K} - \mathcal{B}_1}{\Gamma(1+\beta)}$, $b = |\tau_{d,i}(t_{h,i})|$, and $c = \frac{|\sigma_i^{(\beta-1)}(0)| + |\tau_{d,i}^{(\beta-1)}(0)|}{\Gamma(\beta)}$.

Now we turn our attention to the assumptions we made in (4) through (6). Obviously, the assumptions are rather demanding and stringent. The control system presented here would be globally stable if these conditions hold true for the entire course of operation, however, imposing such bounds make the presented design valid only within a local region.

4 Simulation Results

The presented approach is implemented for the plant introduced in the second section. The system run for 20 s of time and the reference trajectories shown in Fig. 2 are used. The solid curves represent the reference trajectories while the dashed ones stand for the response of the robot. During the operation, a 5 kg of payload is grasped when $t = 2$ s and released when $t = 5$ s and this is repeated when the robot is motionless at $t = 9$ s and $t = 12$ s. The manipulator is desired to stay motionless after $t = 15$ s.

It should be noted that the payload scenario is a significant disturbance changing the dynamics of the plant suddenly. Another difficulty is the initial conditions that the controllers are supposed to alleviate. Initially, $\theta_1(0) = \frac{\pi}{3}$ and $\theta_2(0) = -\frac{\pi}{2}$, which are large enough to test the performance of a controller. The discrepancies between the reference profiles and the system response are seen to convergence zero exponentially.

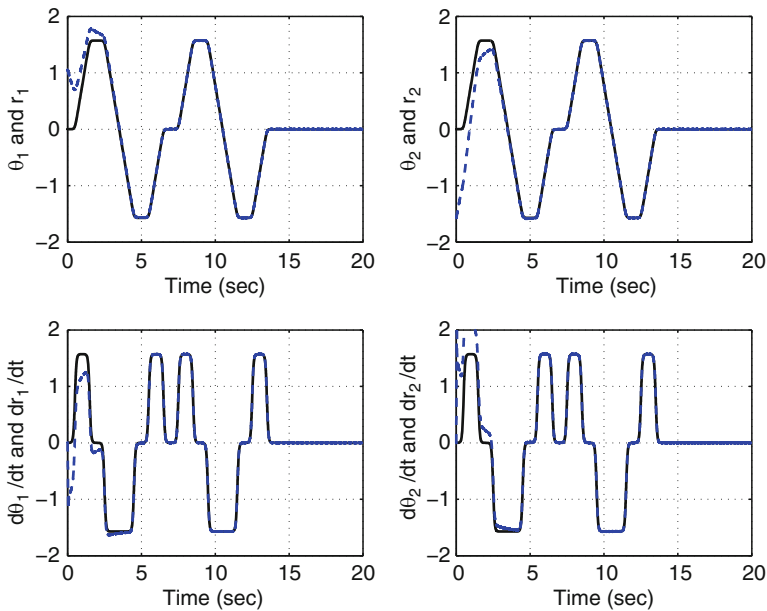


Fig. 2 Reference trajectories and the response of the robot

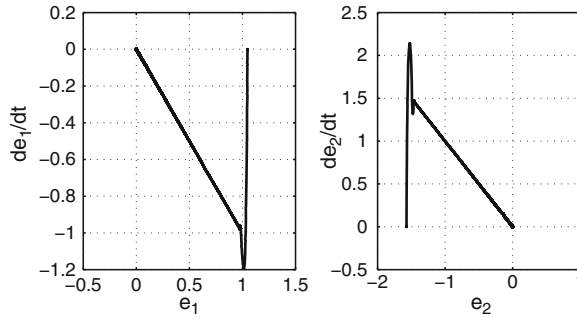


Fig. 3 Behavior in the phase space

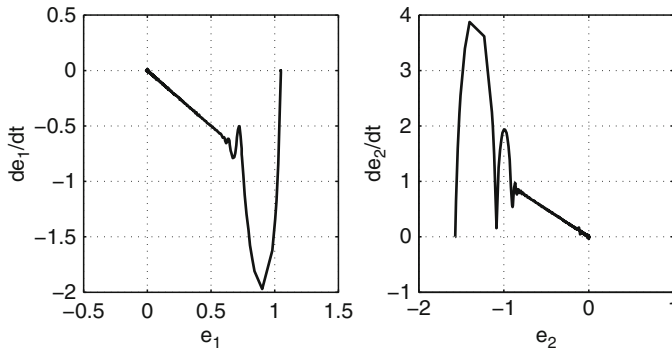


Fig. 4 The behavior in the phase space for the integer case, i.e. $\beta = 1$

The behavior in the phase space illustrated in Fig. 3 is another evidence of robustness of the control system and insensitivity to variations in the plant dynamics.

The time evolution of the controller parameters, which are all started from zero, are seen to display a fast transient, the parameters settle down to constant values shortly. If we remember the reference profiles, the system is desired to be motionless after $t > 15$ s, this means that the tuning activity during this time is subject to the effects of noise. That is to say, the system is at a desired state but we would like to figure out what how parameter tuning mechanism functions during this period. Any possible undesired drift in the controller parameters are suppressed appropriately yet these are not included here due to the space limit.

In Fig. 4, we demonstrate the results obtained with integer order version of the same tuning scheme. In this case the phase space behavior is worse and the parameters do not evolve bounded.

In all results involving the fractional order integration operator, we utilize a 25th order approximation to the operator over the frequency range 0.01 rad/s and 1,000 rad/s. The order of the approximation may be seen as a disadvantage at the first glance, however, the controller has few parameters to tune, therefore the computational load of the overall scheme due to the fractional operations is affordable even with average speed microprocessors.

5 Conclusions

In this paper, we propose a fractional order parameter tuning scheme, which was utilized with integer order operators in the past literature. A two degrees of freedom planar robot is utilized to justify the claims and a comparison with the integer order version is presented. The presented form of the adaptation law provides better parametric evolution that displays no drifts, better tracking capabilities and better robustness and disturbance rejection capabilities than its integer order counterpart, which is only computationally simple. Briefly, the fractional order tuning law outperforms the tuning mechanisms exploiting integer order operators.

Acknowledgements The Matlab toolbox *Ninteger* v.2.3¹ is used and the efforts of its developer, Dr. Duarte Valério, are gratefully acknowledged. This work is supported by Turkish Scientific Council (TÜBİTAK) Contract 107E137.

References

1. Åström KJ, Wittenmark B (1995) Adaptive control, 2nd edn. Addison
2. Direct Drive Manipulator R&D Package User Guide (1992) Integrated Motions Incorporated, 704 Gillman Street, Berkeley, CA 94710, USA
3. Efe MÖ, Kaynak O (2000) A comparative study of soft computing methodologies in identification of robotic manipulators. *Robot Auton Syst* 30:221–230
4. Efe MÖ (2002) A novel error critic for variable structure control with an adaline. *Trans Inst Meas Contr* 24:403–415
5. Ladaci S, Charef A (2006) On fractional adaptive control. *Nonlinear Dynam* 43:365–378
6. Matignon D (1998) Stability properties for generalized fractional differential systems. *ESAIM: Proceedings* 5:145–158
7. Momani S, Hadid S (2004) Lyapunov stability solutions of fractional integrodifferential equations. *Int J Math Math Sci* 2004:2503–2507
8. Oldham KB, Spanier J (1974) The fractional calculus. Academic, New York, USA
9. Podlubny I (1998) Fractional differential equations, 1st edn. Elsevier& Technology Books
10. Vinagre BM, Petráš I, Podlubny I, Chen YQ (2002) Using fractional order adjustment rules and fractional order reference models in model-reference adaptive control. *Nonlinear Dynam* 29:269–279

¹ <http://mega.ist.utl.pt/dmov/ninteger/ninteger.htm>

Synchronization of Chaotic Nonlinear Gyros Using Fractional Order Controller

Hadi Delavari, Reza Ghaderi, Abolfazl Ranjbar, and Shaher Momani

Abstract In this paper, a fractional sliding mode controller is proposed to synchronize chaotic fractional-order gyroscope systems in a master–slave structure. The dynamic has been controlled by a fractional controller in a fractional modelled dynamic gyro. A fuzzy control is applied to reduce the chattering phenomenon in the proposed sliding mode controller. A genetic algorithm identifies parameters of the fuzzy sliding mode controller. Numerical simulation verifies the significance of the proposed technique.

1 Introduction

Synchronization, the task of designing a system (response or slave) whose behaviour mimics another one (drive or master), has a close relationship with control. Synchronization of chaotic systems is attractive in many applications such as chemical reactions, power converters, biological systems, information processing and secure communication. It has been shown that in many cases, such systems can be modelled best with Fractional Order Calculus (FOC). Over the past few years, fractional operators have been applied with satisfactory results in modelling and control of complex process [1] and synchronization of chaos has been investigated in many papers [2,3]. In [4] the problem of chaos control of three types of fractional order systems using simple state feedback gains is studied. Numerically investigation of the chaotic behaviours of the fractional-order Lü system, and its conditions (the lowest order) to model chaotic behaviour is addressed in [5]. A fractional-order controller to stabilize unstable fixed points in an unstable open-loop system is proposed in [6].

H. Delavari, R. Ghaderi, and A. Ranjbar (✉)
Babol(Noushivani) University of Technology, Faculty of Electrical and Computer
Engineering, Babol, Iran, P.O. Box 47135-484
e-mail: hdelavary@gmail.com; r.ghaderi@nit.ac.ir; a.ranjbar@nit.ac.ir

S. Momani
Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan
e-mail: shahermmm@yahoo.com

In [7] several alternative methods of FOC control of power electronic buck converters are presented. Single Input Single Output (SISO) switched fractional order systems (SFOS) have been investigated from the viewpoints of the Generalized Proportional Integral (GPI) feedback control and sliding mode control based on $\Sigma - \Delta$ modulation implementation [8]. In [9] numerically investigation of hyper chaotic behaviour of autonomous nonlinear system (fractional order Arneodo's system addressed in [10]) has been studied. The dynamics of fractional order Newton-Leipnik system is studied in [11]. In [12, 13] algorithms based on active sliding mode controller has been employed to synchronize different chaotic systems. Chaotic behavior of fractional order unified system are numerically investigated in [14] and in [15] synchronizations of two identified autonomous generalized van der Pol chaotic systems are obtained by replacing their corresponding exciting terms by the same function of chaotic states of a third non autonomous or autonomous generalized van der Pol system. In this research, synchronization of gyroscope which has great utility in many scientific and engineering application fields (such as optics, aeronautics and navigation) has been aimed. In some conditions the gyro system may show chaotic behaviour. The main innovation has been based on the fact that modelling with FOC, would give a compact mathematical structure with better controllability by fewer parameters (say just commensurate factor) which ease performance analysis and design procedure. Sliding mode control (SMC) strategy would be applied to set a fractional PD for synchronization because of its good robustness. To overcome the chattering, a genetic fuzzy modification is proposed and simulation results show the effectiveness of the proposed method.

This paper is organized as follows: Sect. 2 presents the gyroscope system description. Synchronization for fractional chaotic systems is discussed in Sect. 3. Fuzzy controller is studied in Sect. 4. Finally, concluding remarks are drawn in Sect. 5. The significance of the proposed method is shown on the following chaotic gyro system.

2 System Description

The dynamic of symmetrical gyro with linear-plus-cubic damping of angle θ can be expressed as:

$$\ddot{\theta} + a^2 \frac{(1 - \cos\theta)}{\sin^3\theta} - b\sin\theta + c_1\dot{\theta} + c_2\dot{\theta}^3 - \beta\sin(\omega t)\sin(\theta) = 0 \quad (1)$$

where, $\beta \sin(\omega t)$ represents a parametric excitation, $c_1\dot{\theta}$ and $c_2\dot{\theta}^3$ are linear and nonlinear damping terms, respectively, and $a^2 \frac{(1 - \cos\theta)}{\sin^3\theta} - b\sin\theta$ is a nonlinear resilience force. Given the states $x_1 = \theta$, $x_2 = \dot{\theta}$ and $f(x_1, x_2) = -a^2 \frac{(1 - \cos x_1)}{\sin^3 x_1} - b\sin x_1 - c_1 x_2 - c_2 x_2^3 + (b + \beta\sin(\omega t))\sin(x_1)$ this system can be transformed into the following nominal state form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, x_2) \end{aligned} \quad (2)$$

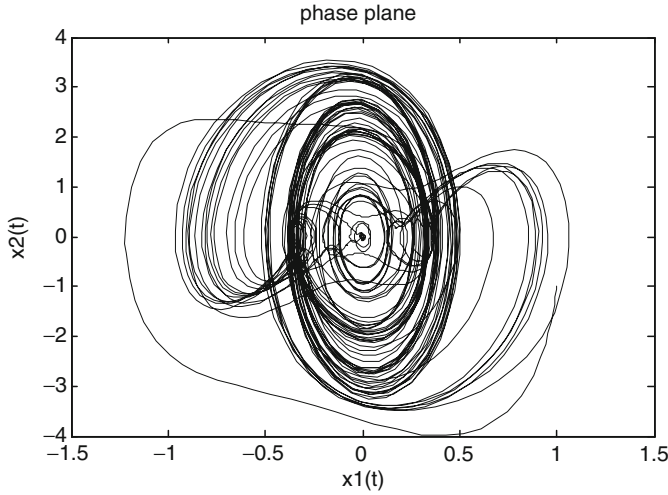


Fig. 1 Phase portrait of x_1 vs. x_2

This Gyro system demonstrates complex dynamics. The behavior has been studied by Chen [18] for variety of β in the range $32 < \beta < 36$ and constant values of $a = 10, b = 1, c_1 = 0.5, c_2 = 0.05$ and $\omega = 2$, the chaotic attractor with initial condition of $[x_1(0), x_2(0)] = [1, -1]$ has shown in Fig. 1.

3 Synchronization of Fractional Modelled System by Fractional PD SMC

The classical gyro in (2) is imaginary generalized to fractional dynamic (3) as the Master:

$$\begin{aligned} D_t^\alpha x_1 &= x_2 \\ D_t^\alpha x_2 &= f(x_1, x_2) \end{aligned} \quad (3)$$

where α is the fractional order, $0 < \alpha \leq 1$. The slave (4) considered to be essentially the same as the Master (3), perturbed with uncertainty and disturbance.

$$\begin{aligned} D_t^\alpha y_1 &= y_2 \\ D_t^\alpha y_2 &= f(y_1, y_2) + \Delta f(y_1, y_2) + d(t) + u(t) \end{aligned} \quad (4)$$

where $\Delta f(y_1, y_2) = -0.1 \sin(y_1)$ and $d(t) = 0.2 \cos(\pi t)$ are uncertainty and disturbance, respectively. Initial conditions for master and slave systems are chosen

as: $(x_1, x_2) = (1, -1)$ and $(y_1(0), y_2(0)) = (1.6, 0.8)$. with synchronization errors $(e_1 = y_1 - x_1, e_2 = y_2 - x_2)$, the error dynamic can be defined as

$$\begin{aligned} D_t^\alpha e_1 &= e_2 \\ D_t^\alpha e_2 &= g(x_1, y_1) + \Delta f(y_1, y_2) + d(t) + u(t) \end{aligned} \quad (5)$$

The control goal is to design the control law $u(t)$ to synchronize the slave with the master. It means $\|e(t)\| \rightarrow 0$ as $t \rightarrow \infty$. The appropriate fractional switching surface and its slope would be as

$$S = K_p e_1 + K_d D^\lambda e_1 \quad (6)$$

$$\dot{S} = K_p \dot{e}_1 + K_d D^{1+\lambda-\alpha}(D^\alpha e_1) = K_p \dot{e}_1 + K_d D^{1+\lambda-2\alpha}(D^\alpha e_2) \quad (7)$$

Forcing $\dot{S} = 0$, the control signal would be obtained:

$$u(t) = -(K_p/K_d)D^{2\alpha-\lambda}e_1 - g(x_1, y_1) - \Delta f(y_1, y_2) - d(t) - K \operatorname{sgn}(S) \quad (8)$$

Simulation results with $K_p = 1$, $K_d = 0.6$, $\lambda = 0.97$, and $\alpha = 0.98$ has been shown in Fig. 2. (activation at $t = 10$ S). It shows that the designed sliding mode controllers produce chattering phenomena. To reduce the chattering a saturation function is used instead of the signum function [17]. The alternative control signal

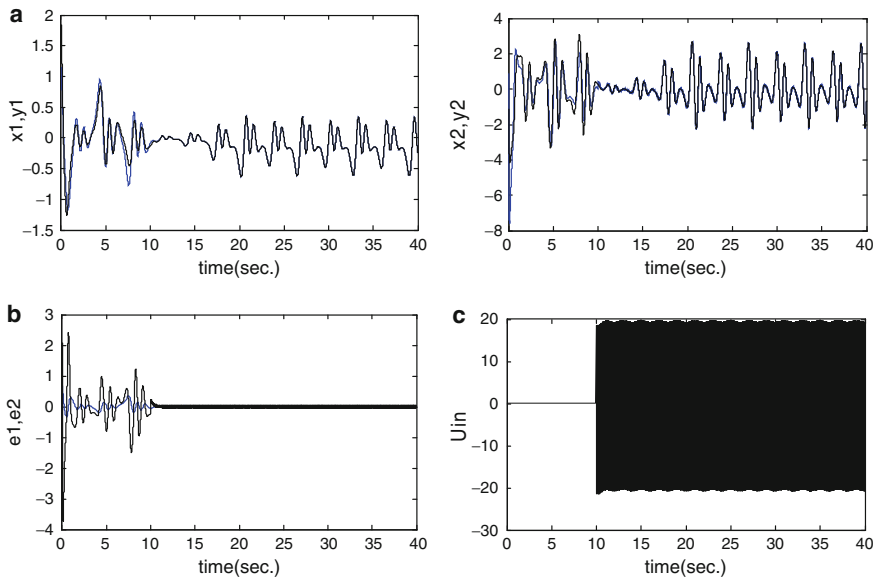


Fig. 2 (a) Time response of controlled chaotic gyro synchronization system, (b) synchronization error, (c) control signal $u(t)$, with signum function for master and slave system

of (8) would be as

$$u(t) = -(K_p/K_d)D^{2\alpha-\lambda}e_1 - g(x_1, y_1) - \Delta f(y_1, y_2) - d(t) - K \text{sat}(S/\Phi) \quad (9)$$

In (9) Φ is the width of boundary layer and K is a positive switch gain. The saturation function reduces the chattering, but to have a satisfactory compromise between small chattering and good tracking precision in the presence of uncertainties, a fuzzy logic control is proposed.

4 Fuzzy Controller

The combination of fuzzy logic with SMC has been proposed to preserve advantages of previous approaches to develop the control law in (9). The IF-THEN rules of fuzzy sliding mode controller are described as:

$$\begin{aligned} R_1 : & \text{If } \Psi \text{ is NB then } K \text{ is PB} \\ R_2 : & \text{If } \Psi \text{ is NM then } K \text{ is PM} \\ R_3 : & \text{If } \Psi \text{ is NS then } K \text{ is PS} \\ R_4 : & \text{If } \Psi \text{ is ZE then } K \text{ is ZE} \\ R_5 : & \text{If } \Psi \text{ is PS then } K \text{ is NS} \\ R_6 : & \text{If } \Psi \text{ is PM then } K \text{ is NM} \\ R_7 : & \text{If } \Psi \text{ is PB then } K \text{ is NB} \end{aligned} \quad (10)$$

where NB, NM, NS, ZE, PS, PM, PB are the linguistic terms of antecedent fuzzy set. They mean Negative Big, Negative Medium, Negative Small, Zero, Positive Medium Positive Small and Positive Big, respectively. A fuzzy membership function for each fuzzy term should be a proper design factor in the fuzzy control problem [16, 17]. A general form is used to describe these fuzzy rules as:

$$R_i : \text{If } \Psi \text{ is } A_i, \text{ then } K \text{ is } B_i \quad (11)$$

where A_i has a triangle membership function (depicted in Fig. 3) and B_i is a fuzzy singleton. The modified controller invites an idea to restrict the width of boundary layer Φ , which uses a continuous function to smoothen the control action. There-

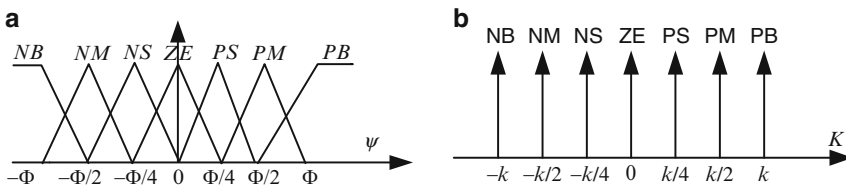


Fig. 3 (a) The input membership function of the FSMC, (b) the output membership FSMC

fore, the problem of the discontinuousness of the signum function can be treated, and the chattering phenomena will be decreased. From the control point of view, the parameters of structures should be automatically modified by evaluating the results of fuzzy control in (10). The hitting time and chattering phenomenon are two important factors that influence the performance of the proposed controller. The width of boundary layer Φ , influences the chattering magnitude of the control signal, whilst the gain K , will influence speed of synchronization. The reaching time can be reduced via a suitable selection of parameter K, Φ . GA is used to search for a best fit for these parameters in (9). The tracking error and the chattering of the controlled response are chosen as a performance index to select the parameters. The cost function is defined in such a way that the selected parameters to minimize the error:

$$\int [W_1(S^2) + W_2(e_1^2 + e_2^2)] dt \quad (12)$$

where W_1 and W_2 are the weighting factors. The simulation results for novel genetic based fuzzy fractional sliding mode controller are shown in Fig. 4. It can be seen that the control signal (Fig. 4.c) is smoother than the conventional one (Fig. 2.c). In addition, the tracking error (Fig. 4.b) is less than the conventional SMC (Fig. 2.b).

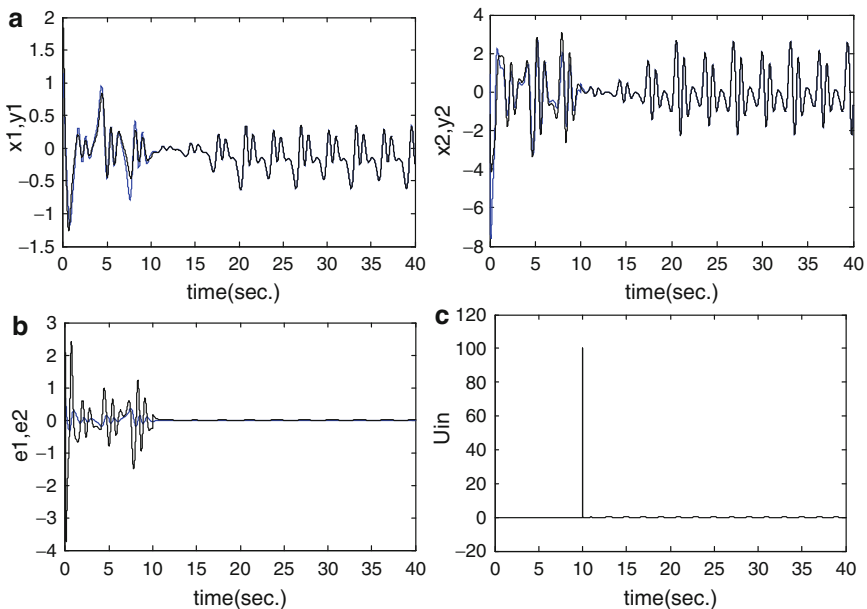


Fig. 4 (a) Time response of controlled chaotic gyro synchronization system, (b) synchronization error, (c) control signal $u(t)$, with genetic based fuzzy fractional SMC for master and slave system

5 Conclusion

In this paper synchronization of fractional type order of chaotic system with uncertainties and disturbance have been investigated. The performance will be improved when the sliding surface is chosen fractional. More improvement has been achieved when the signum function is replaced with a fuzzy controller. The work has been progressed to find best fit parameters of the fuzzy controller through Genetic Algorithm. The simulation results have confirmed the potentiality of the proposed scheme to synchronize two chaotic systems through a single control signal. This controller is found robust in the presence of uncertainties and disturbances. The application of the proposed controller is promising.

References

1. Jafari H, Momani S (2007) Solving fractional diffusion and wave equations by modified homotopy perturbation method. *Phys Lett A* 370(5–6):388–396
2. Yau H-T (2004) Design of adaptive sliding mode controller for chaos synchronization with uncertainties. *Chaos Soliton Fract* 22:341–347
3. Yau H-T, Chen C-L (2006) Chattering-free fuzzy sliding-mode control strategy for uncertain chaotic systems. *Chaos Soliton Fract* 30:709–718
4. Wajdi AM, El-Khazali R, Al-Assaf Y (2004) Stabilization of generalized fractional order chaotic systems using state feedback control. *Chaos Soliton Fract* 22:141–150
5. Lu JG (2006) Chaotic dynamics of the fractional order L system and its synchronization. *Phys Lett A* 354(4):305–311
6. Tavazoei MS, Haeri M (2008) Chaos control via a simple fractional-order controller. *Phys Lett A* 372(6):798–807
7. Caldern AJ, Vinagre BM, Feliu V (2006) Fractional order control strategies for power electronic buck converters. *Signal Process* 86:2803–2819
8. Sira-Ramirez H, Feliu-Batlle V (2006) On the GPI-sliding mode control of switched fractional order systems. *International Workshop on Variable Structure Systems*. Italy, pp 310–315
9. Wajdi AM (2005) Hyperchaos in fractional order nonlinear systems. *Chaos Soliton Fract* 26:1459–1465
10. Lu JG (2005) Chaotic dynamics and synchronization of fractional order Arneodo's systems. *Chaos Soliton Fract* 26(4):1125–1133
11. Sheu LJ, Chen HK, Chen JH, Tam LM, Chen WC, Lin KT, Kang Y (2008) Chaos in the Newton-Leipnik system with fractional order. *Chaos Soliton Fract* 36:98103
12. Tavazoei MS, Haeri M (2007) Determination of active sliding mode controller parameters in synchronizing different chaotic systems. *Chaos Soliton Fract* 32:583–591
13. Tavazoei MS, Haeri M (2008) Synchronization of chaotic fractional-order systems via active sliding mode controller. *Physica A* 387(1):57–70
14. Wu X, Li J, Chen G (2008) Chaos in the fractional order unified system and its synchronization. *J Franklin Inst* 345:392–401
15. Ge ZM, Hsu MY (2008) Chaos excited chaos synchronizations of integral and fractional order generalized van der Pol systems. *Chaos Soliton Fract* 36:592–604
16. Delavari H, Ranjbar A (2007) Robust intelligent control of coupled tanks. *WSEAS International Conferences*. Istanbul, pp 1–6
17. Delavari H, Ranjbar A (2007) Genetic-based fuzzy sliding mode control of an interconnected twin-tanks. *IEEE Region 8 EUROCON 2007 conference*. Poland, pp 714–719
18. Chen HK (2002) Chaos and chaos synchronization of a symmetric gyro with linear-plus-cubic damping. *J Sound Vib* 255:719–740

Nyquist Envelope of Fractional Order Transfer Functions with Parametric Uncertainty

Nusret Tan, M. Mine Ozyetkin, and Celaleddin Yeroglu

Abstract The paper presents a method for computation of the Nyquist envelope of fractional order interval control systems (FOICS). The given method is based on the computation of the value sets of fractional order interval polynomials (FOIP). The results obtained will be useful for estimating the frequency domain specifications such as robust gain and phase margins, and for designing robust controller for FOICS.

1 Introduction

A system represented by differential equations where the orders of derivatives can take any real number not necessarily integer number can be considered as a fractional order system. The significance of fractional order representation is that fractional order differential equations are more adequate to describe real world systems than those of integer order models [1, 2]. Therefore, in recent years considerable attention has been given to the fractional order control systems (FOCS) due to the better understanding of fractional calculus [3] and the emergence of a new electrical circuit called “fractance” [4] which make the implementation of a fractional order controller feasible [5]. As a result, some important studies dealing with the applications of the fractional calculus to the control systems have been done in [6–16]. This field of research is still new and there is not much work dealing with the robustness analysis of FOCS with parametric uncertainty [17, 18].

The frequency domain analysis of systems is an important topic in control theory. There are some powerful graphical tools in classical control, such as the Nyquist plot, Bode plots and Nichols charts, which are widely used to evaluate the frequency domain behaviours of systems. Motivated by the results especially the Kharitonov

N. Tan (✉), M.M. Ozyetkin, and C. Yeroglu
Inonu University, Engineering Faculty, Department of Electrical and Electronics Engineering,
44280, Malatya, Turkey
e-mail: ntan@inonu.edu.tr; mmyetkin@inonu.edu.tr; cyeroglu@inonu.edu.tr

and the Edge theorems [19, 20] obtained in the parametric robust control, there have been several studies [21–25] on the computation of the frequency responses of control systems under parametric uncertainty. However, these results are related to the integer order control systems with parametric uncertainty. Therefore, extensions of these results to FOCS with parametric uncertainty will be very important.

The purpose of this paper is to present a method for computation of the Nyquist envelopes of fractional order interval transfer functions (FOITF). The numerator and denominator polynomials of a FOITF are FOIP of the form

$$P(s, q) = q_0 s^{\alpha_0} + q_1 s^{\alpha_1} + q_2 s^{\alpha_2} + q_3 s^{\alpha_3} + \cdots + q_n s^{\alpha_n} \quad (1)$$

where $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ are generally real numbers, $q = [q_0, q_1, \dots, q_n]$ is the uncertain parameter vector and the uncertainty box is $\mathcal{Q} = \{q : q_i \in [\underline{q}_i, \bar{q}_i], i = 0, 1, \dots, n\}$. Here $\underline{q}_i$ and $\bar{q}_i$ are specified lower and upper bounds of i^{th} perturbation q_i , respectively. Thus, a FOITF can be represented as

$$G(s, a, b) = \frac{N(s, b)}{D(s, a)} = \frac{b_0 s^{\alpha_0} + b_1 s^{\alpha_1} + b_2 s^{\alpha_2} + b_3 s^{\alpha_3} + \cdots + b_m s^{\alpha_m}}{a_0 s^{\beta_0} + a_1 s^{\beta_1} + a_2 s^{\beta_2} + a_3 s^{\beta_3} + \cdots + a_n s^{\beta_n}} \quad (2)$$

where $\alpha_0 < \alpha_1 < \cdots < \alpha_m$ and $\beta_0 < \beta_1 < \cdots < \beta_n$ are generally real numbers, $a = [a_0, a_1, \dots, a_n]$ and $b = [b_0, b_1, \dots, b_m]$ are uncertain parameter vectors, $A = \{a : a_i \in [\underline{a}_i, \bar{a}_i], i = 0, 1, \dots, n\}$ and $B = \{b : b_i \in [\underline{b}_i, \bar{b}_i], i = 0, 1, \dots, m\}$ are uncertainty boxes. It is first shown that the value set of the family of polynomial of Eq. 1 can be constructed using the upper and lower values of uncertain parameters. Then, using the geometric structure of the value set a procedure for computing the Nyquist envelopes of FOITF represented by Eq. 2 is given.

2 Construction of the Value Set of FOIP

For FOIP of Eq. 1, substituting $s = j\omega$ gives,

$$\begin{aligned} P(j\omega, q) &= q_0(k_{0r} + jk_{0i})\omega^{\alpha_0} + q_1(k_{1r} + jk_{1i})\omega^{\alpha_1} + \cdots + q_n(k_{nr} + jk_{ni})\omega^{\alpha_n} \\ &= (q_0 k_{0r} \omega^{\alpha_0} + \cdots + q_n k_{nr} \omega^{\alpha_n}) + j(q_0 k_{0i} \omega^{\alpha_0} + \cdots + q_n k_{ni} \omega^{\alpha_n}) \end{aligned} \quad (3)$$

where k_{lr} and k_{li} , $l = 1, 2, \dots, n$ are constant. From Eq. 3, it is clear that the uncertain parameters appearing both in the real and imaginary parts are linearly dependent to each other. The value set of such a polynomial in the complex plane is a polygon. Thus the corresponding polytope of a family of Eq. 1 in the coefficient space has $2^{(n+1)}$ vertices and $(n+1)2^n$ exposed edges since the polynomial family has $(n+1)$ uncertain parameters. For example, the uncertainty box in the parameter space and image of the exposed edges in the complex plane for a polynomial of the form of Eq. 1 with Eq. 3 uncertain parameters are shown in Fig. 1.

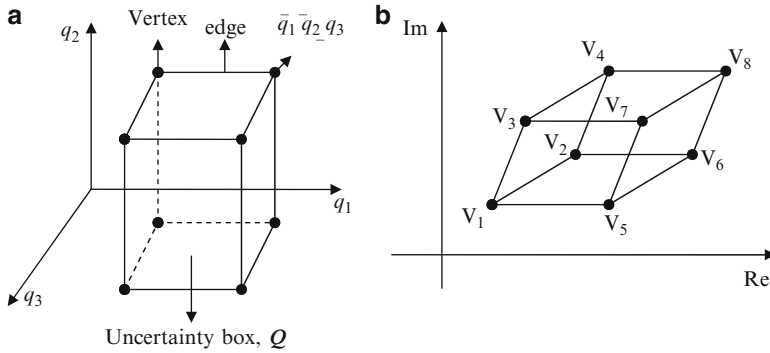


Fig. 1 For a polynomial of the form of Eq. 1 with Eq. 3 uncertain parameters: (a) uncertainty box in the parameter space, (b) images of exposed edges in the complex plane

Using the upper and lower values of the uncertain parameters, all the $2^{(n+1)}$ vertex polynomials of $P(s, q)$ can be written in the following pattern

$$\begin{aligned}
 v_1(s) &= \underline{q_0}s^{\alpha_0} + \underline{q_1}s^{\alpha_1} + \underline{q_2}s^{\alpha_2} + \dots + \underline{q_n}s^{\alpha_n} \\
 v_2(s) &= \overline{q_0}s^{\alpha_0} + \underline{q_1}s^{\alpha_1} + \underline{q_2}s^{\alpha_2} + \dots + \underline{q_n}s^{\alpha_n} \\
 &\vdots \\
 v_{2^{(n+1)}}(s) &= \overline{q_0}s^{\alpha_0} + \overline{q_1}s^{\alpha_1} + \overline{q_2}s^{\alpha_2} + \dots + \overline{q_n}s^{\alpha_n}
 \end{aligned} \tag{4}$$

From these vertex polynomials the exposed edges can be obtained. For example, the vertex polynomial $v_1(s)$ and $v_2(s)$ have the same structure except the parameter q_0 is its lower value ($\underline{q_0}$) in $v_1(s)$ and its upper value ($\overline{q_0}$) in $v_2(s)$. Thus one of the exposed edges can be expressed as

$$e(v_1, v_2) = (1 - \lambda)v_1(s) + \lambda v_2(s) \tag{5}$$

where $\lambda \in [0, 1]$. Similarly, the remaining exposed edges can be constructed. Define the sets which contain all the vertex polynomials and exposed edges as,

$$P_V = \{v_1, v_2, \dots, v_{2^{(n+1)}}\} \quad \text{and} \quad P_E = \{e_1, e_2, \dots, e_{(n+1)2^n}\} \tag{6}$$

Theorem 1. $\partial P(j\omega, q) \subset P_E(j\omega)$, where ∂ denotes the boundary and P_E is defined in Eq. 6.

Proof. From Eq. 3, it can be seen that the uncertain parameters in the real and imaginary parts are related to each other linearly. This type of polynomial family is called polytopic family [21]. Therefore, a linear map of the uncertainty box Q from the parameter space to the complex plane is a polygon whose vertices and exposed

edges can be obtained by mapping the vertices and exposed edges of Q as shown in Fig. 1a, b. As a result, the boundary of the value set of $P(s, q)$ at $s = j\omega^*$ can be obtained from the images of exposed edges. Therefore, $\partial P(j\omega, q) \subset P_E(j\omega)$ for all real ω . $\square$

3 Nyquist Envelope of FOITF

Nyquist plot of a transfer function is important in classical control theory for the analysis and design. For example, the frequency domain specifications such as the gain and phase margins can be obtained using the Nyquist plot of a transfer function.

The numerator and denominator polynomials of FOITF of Eq. 2 are in the form of $P(s, q)$ of Eq. 1. Therefore, the results given in the previous section can be used to obtain the Nyquist envelope of FOITF. Consider the transfer function given in Eq. 2, and let $n_1, n_2, \dots, n_{2(m+1)}$ and $d_1, d_2, \dots, d_{2(n+1)}$ be the vertex polynomials of $N(s, b)$ and $D(s, a)$ polynomials, respectively. Define the sets N_V and N_E , which contain the vertices and edges of $N(s, b)$ as

$$N_V(s) = \{n_1, n_2, \dots, n_{2(m+1)}\} \quad \text{and} \quad N_E(s) = \{ne_1, ne_2, \dots, ne_{(m+1)2^m}\} \quad (7)$$

similarly define D_V and D_E , for the $D(s, a)$ as

$$D_V(s) = \{d_1, d_2, \dots, d_{2(n+1)}\} \quad \text{and} \quad D_E(s) = \{de_1, de_2, \dots, de_{(n+1)2^n}\} \quad (8)$$

Define the extremal system as,

$$G_E(s) = \frac{N_V(s)}{D_E(s)} \cup \frac{N_E(s)}{D_V(s)} \quad (9)$$

where N_V , N_E , D_V and D_E are defined in Eqs. 7 and 8.

Theorem 2. At $s = j\omega$, $\partial G(j\omega, a, b) \subset G_E(j\omega)$, where G_E is defined in Eq. 9 and ∂ denotes the boundary.

Proof. Let A_1 and A_2 be the two complex plane polygons with vertex sets V_{A_1} and V_{A_2} , and edge sets E_{A_1} and E_{A_2} , respectively. Then, from the complex plane geometry, $\partial(A_1/A_2) \subset (E_{A_1}/V_{A_2}) \cup (V_{A_1}/E_{A_2})$ is known [21]. Since the value sets of the numerator and the denominator of Eq. 2 are independent polygons, one can write $V_{A_1} = N_V$, $V_{A_2} = D_V$, $E_{A_1} = N_E$ and $E_{A_2} = D_E$. Thus, $\partial G(j\omega, a, b) \subset G_E(j\omega) = (N_V/D_E) \cup (N_E/D_V)$. $\square$

Gain and phase margins are two important frequency domain specifications. In this section, we deal with the calculation of the robust gain and phase margins for systems with an uncertain transfer function of the form of Eq. 2.

Suppose that a closed-loop system with an uncertain plant of the form of Eq. 2 is stable. The robust gain margin is then the largest value of the gain K greater than 1 for which the stability of $KG(s, a, b)$ is preserved, and the robust phase margin is the largest value of phase θ for which the uncertain system with $e^{-j\theta}G(s, a, b)$ is robustly stable. Thus, the worst case gain margin K^* and phase margin θ^* can be stated as $K^* = \inf_{G(s) \in G(s, a, b)} K_G$ and $\theta^* = \inf_{G(s) \in G(s, a, b)} \theta_G$, where K_G is the gain margin of $G(s)$ and θ_G is the phase margin of $G(s)$. Using Theorem 3, the values of K^* and θ^* can be computed from the extremal system $G_E(s)$.

Theorem 3. Suppose a unity feedback system with $G(s, a, b)$ is stable. The robust gain and phase margins are then $K^* = \inf_{G(s) \in G_E(s)} K_G$ and $\theta^* = \inf_{G(s) \in G_E(s)} \theta_G$, where $G_E(s)$ defined in Eq. 9.

Proof. Let A_1 and A_2 be the two complex plane polygons with vertex sets V_{A_1} and V_{A_2} , and edge sets E_{A_1} and E_{A_2} respectively. From the complex plane geometry, $\partial(A_1 + A_2) \subset (E_{A_1} + V_{A_2}) \cup (V_{A_1} + E_{A_2})$ is known. Now, for the calculation of the gain margin, one needs to find the maximum value of K greater than 1 for which $\Delta(s) = KN(s, b) + D(s, a)$ is stable. The multiplication of a convex polygon with a fixed K is still a convex polygon. Thus, for fixed value of K , we can write $V_{A_1} = KN_V$, $V_{A_2} = D_V$, $E_{A_1} = KN_E$ and $E_{A_2} = D_E$. Therefore, it can be written that $\Delta(j\omega) \subset \Delta_E(j\omega) = (KN_E + D_V) \cup (KN_V + D_E)$. Thus, the stability of $\Delta_E(s)$ implies the stability of $\Delta(s)$. For the phase margin calculation, the gain K will be a complex gain and the same proof will be valid. $\square$

The following procedure can be used for computation of the Nyquist envelope of a given FOITF:

1. Construct the vertices and edges of $N(s, b)$ and $D(s, a)$ using Eqs. 4 and 5.
2. Obtain the vertex and edge sets of Eqs. 7 and 8.
3. For each $s = j\omega$, using Theorem 2, obtain the Nyquist envelope.
4. From the Nyquist envelopes and using Theorem 3, compute the robust gain and phase margins.

4 A Numerical Example

Consider a negative unity feedback control system with the following FOITF

$$G(s, a, b) = \frac{N(s, b)}{D(s, a)} = \frac{1}{a_0 + a_1 s^{0.9} + a_2 s^{2.2}} \quad (10)$$

where, $a_0 \in [0.8, 1.2]$, $a_1 \in [0.5, 0.9]$ and $a_2 \in [0.6, 1]$. It can be seen that $N(s, b) = 1$ which is constant and $D(s, a)$ has 3 uncertain parameters. Therefore, there are $2^3 = 8$ vertex polynomials and $3 \times 2^2 = 12$ exposed edges for $D(s, a)$

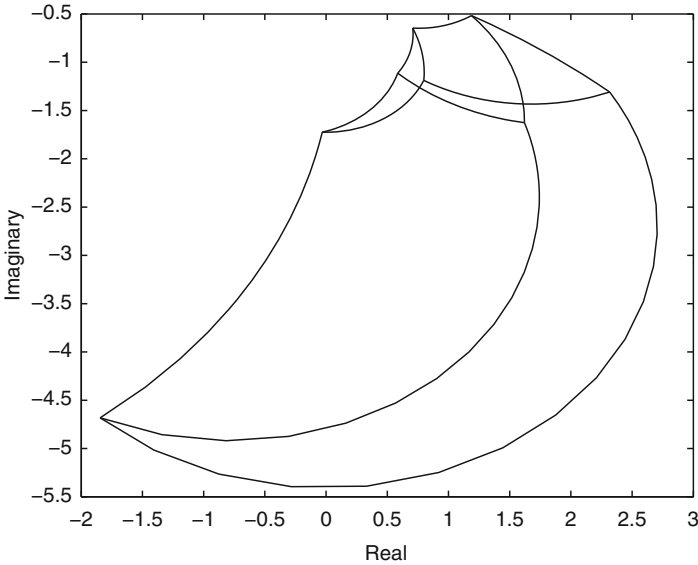


Fig. 2 Nyquist template at $\omega = 1$ rad/s

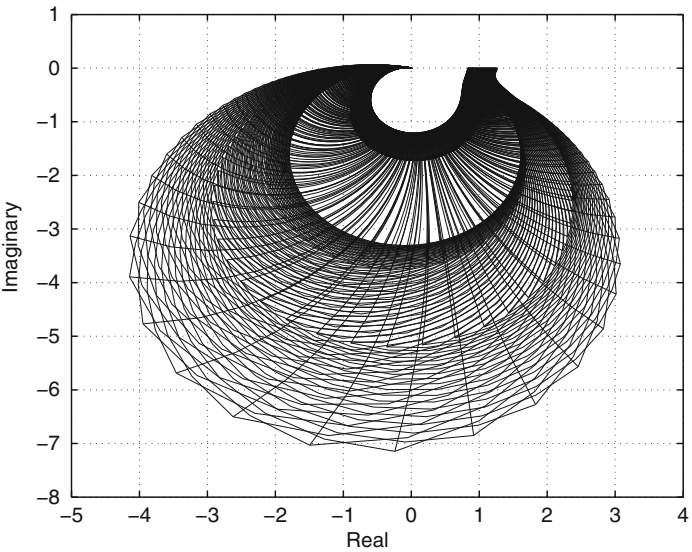


Fig. 3 Nyquist envelope

which can be obtained using Eqs. 4 and 5. The Nyquist template at $\omega = 1$ rad/sec. and the Nyquist envelope of $G(s, a, b)$ are shown in Figs. 2 and 3. From Fig. 3, one can say that the given FOICS is not robust BIBO(Bounded Input Bounded Output) stable.

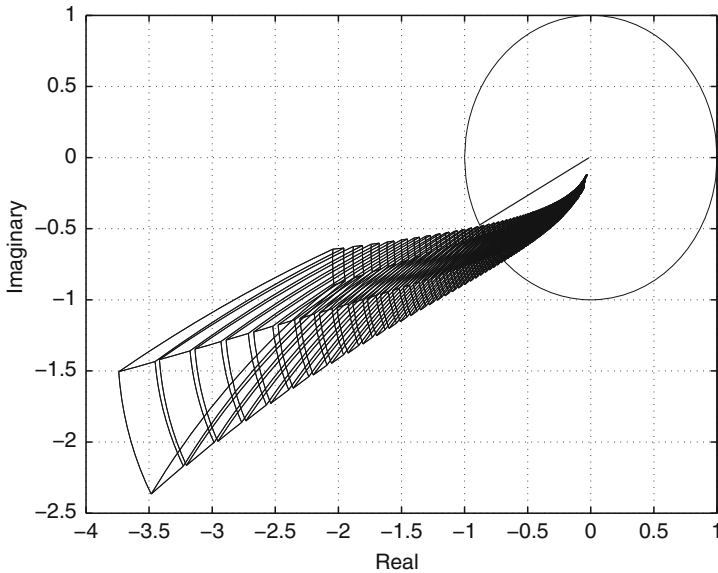


Fig. 4 Nyquist envelope of $C(s)G(s, a, b)$

Now, consider that a fractional order PD^μ controller namely, $C(s) = K_p + K_d s^\mu = 20 + 3s^{1.15}$ is connected in the forward path of the given control system. In this case, the Nyquist envelope of $C(s)G(s, a, b)$ is shown in Fig. 4 where it can be calculated that the robust gain margin is equal to ∞ and the robust phase margin is about 30° . Therefore, the given system is now robust stable.

5 Conclusions

In this paper, a method has been presented for the computation of the Nyquist envelopes of fractional order control systems with interval uncertainty structure. The results obtained are basically extensions of some results developed in the parametric robust control to the FOICS. It has been shown that the value set of a FOIP family can be constructed using the exposed edges. Using the geometric structure of the value set of a FOIP, an effective algorithm has been proposed for the computation of the Nyquist envelope of a FOITF whose numerator and denominator polynomials are FOIP. The results obtained will be very important for the robust stability analysis and design of FOCS with parametric uncertainty.

References

1. Nonnenmacher TF, Glockle WG (1991) A fractional model for mechanical stress relaxation. *Philos Mag Lett* 64(2):89–93
2. Westerlund S (1994) Capacitor theory. *IEEE Trans Dielec Elec Insul* 1(5):826–839
3. Podlubny I (1999) Fractional differential equations. Academic, San Diego
4. Nakagava N, Sorimachi K (1992) Basic characteristics of a fractance device. *IEICE Trans Fund* E75-A(12):1814–1818
5. Hwang C, Cheng YC (2006) A numerical algorithm for stability testing of fractional delay systems. *Automatica*, 42:825–831
6. Sabatier J, Poullain S, Latteux P, Thomas JL, Oustaloup A (2004) Robust speed control of a low damped electromechanical system based on CRONE control: application to a four mass experimental test bench. *Nonlinear Dynam* 38:383–400
7. Petras I (1999) The fractional order controllers: methods for their synthesis and application. *J Elec Eng* 50:284–288
8. Podlubny I (1999) Fractional-order systems and $PI^\lambda D^\mu$ controllers. *IEEE Trans Autom Control* 44(1):208–214
9. Manabe S (2003) Early development of fractional order control. *Proceedings of ASME 2003 Design Engineering Technical Conference*. Chicago, Illinois
10. Valrio D, Da Costa JS (2005) Time domain implementation of fractional order controllers. *IEE Proceedings of Contr Theor Appl* 152(5):539–552
11. Valrio D, Da Costa JS (2005) Ziegler-Nichols type tuning rules for fractional PID controllers. *Proceedings of IDETC/CIE 2005*. Long Beach, California
12. Machado J (2001) Discrete-time fractional-order controllers. *Fract Calc Appl Anal* 4(1):47–66
13. Monje CA, Vinagre BM, Feliu V, Chen YQ (2008) Tuning and auto-tuning of fractional order controllers for industry applications. *Contr Eng Pract* 16:798–812
14. Brin IA (1962) On the stability of certain systems with distributed and lumped parameters. *Automation and Remote Control*, 23, pp. 798–807
15. Ozturk N, Uraz A (1984) An analytic stability test for a certain class of distributed parameter systems with a distributed lag. *IEEE Trans Autom Control* 29:368–370
16. Cheng YC, Hwang C (2006) Stabilization of unstable first-order time-delay systems using fractional-order PD controllers. *J Chinese Inst Eng* 29:241–249
17. Chen YQ, Ahn HS, Podlubny I (2006) Robust stability check of fractional order linear time invariant systems with interval uncertainties. *Signal Process* 86:2611–2618
18. Chen YQ, Moore KL (2002) Analytical stability bound for a class of delayed fractional-order dynamic systems. *Nonlinear Dynam* 29:191–200
19. Kharitonov VL (1979) Asymptotic stability of an equilibrium position of a family of systems of linear differential equations. *Diff Equat* 14:1483–1485
20. Bartlett AC, Hollot CV, Lin H (1988) Root location of an entire polytope of polynomials: it suffices to check the edges. *Math Control Signal syst* 1:61–71
21. Bhattacharyya SP, Chappellat H, Keel LH (1995) Robust control: the parametric approach. Prentice Hall
22. Chappellat H, Bhattacharyya SP (1989) A generalization of Kharitonov's theorem: robust stability of interval plants. *IEEE Trans Autom Control* 34:306–311
23. Soylemez MT, Munro N (1997) Robust pole assignment in uncertain systems. *Proc IEE Part D* 144(3):217–224
24. Tan N, Atherton DP (2000) Frequency response of uncertain systems: a 2q-convex parpolygonal approach. *IEE Proc Contr Theor Appl* 147(5):547–555
25. Tan N (2002) Computation of the frequency response of multilinear affine systems. *IEEE Trans Autom Control* 47:1691–1696

Synchronization of Gyro Systems via Fractional-Order Adaptive Controller

Seyed Hassan Hosseinnia, Reza Ghaderi, Abolfazl Ranjbar, Jalil Sadati, and Shaher Momani

Abstract Chaos in fractional Gyro system is studied in this paper. An adaptive fractional-order controller has been designed to synchronize two identical chaotic systems. This controller is a fractional PID controller, which the coefficients will be tuned according to a proper adaptation mechanism. The adaptation law will be constructed from a sliding surface via gradient method. The simulation results show the efficiency of the proposed controller.

1 Introduction

Fractional calculus is an old mathematical topic since seventeenth century. Although it has a long history, its applications to physics and engineering are just a recent focus of interest. Many systems are known to display fractional order dynamics, such as earthquake oscillation [1], Riccati [2, 3], wave equation [4], and chaotic equations in control engineering [5]. There is a new topic to investigate the control and dynamics of fractional order dynamical system. The behaviour of nonlinear chaotic systems when their models become fractional have widely been investigated [6–10]. Sensitive dependence on initial conditions is an important characteristic of chaotic systems. Therefore, chaotic systems are difficult to be synchronized or controlled. A chattering-free fuzzy sliding-mode control (FSMC) strategy for uncertain chaotic systems has been proposed in [11]. In [12] the authors proposed an active sliding mode control method for synchronizing two chaotic systems perturbed by parametric uncertainty. An algorithm to determine parameters of the active sliding

S.H. Hosseinnia, R. Ghaderi, A. Ranjbar (✉), and J. Sadati
Intelligent system research group, Faculty of Electrical and Computer Engineering, Noushervani
University of Technology, P.O. Box: 47135-484, Babol, Iran
e-mail: h.hosseinnia@stu.nit.ac.ir; r_ghaderi@yahoo.co.uk; a.ranjbar@nit.ac.ir;
jalil.sadati@gmail.com

S. Momani
Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan
e-mail: shahermmm@yahoo.com

mode controller in synchronizing different chaotic systems has been studied by [13]. In [11] an adaptive sliding mode controller is presented for a class of master-slave chaotic synchronization systems with uncertainties. The design of adaptive sliding mode controller for chaos synchronization with via backstepping design was discussed in [14]. In [15] backstepping control has been proposed to synchronize the chaotic systems. Even though, synchronization has been implemented in many chaotic systems with integer derivatives, but a few works are reported on fractional order chaotic system. It is because; proof of stability of the fractional order is more complex than the system with integer order. In this paper, an adaptive fractional controller has been proposed as a novel idea to control systems with fractional order dynamic. This controller is in essence a PID controller but fractional characteristics. PID coefficients K_P , K_I and K_D will be updated according to a proper gradient-based adaptation mechanism. This paper is organized as follows: Primarily, the proposed fractional controller will be presented in Sect. 2, to control such similar systems. The performance of the controller will be investigated in Sect. 3 when it is used to synchronize a gyro dynamic. Ultimately, the work will be concluded at Sect. 4.

2 Fractional Adaptive Controller Design

The following model represents a chaotic system with fractional order dynamic:

$$\begin{aligned} D^q x_1 &= x_2, \\ D^q x_2 &= f(X, t) \end{aligned} \quad (1)$$

where, $0 < q \leq 1$ and $X = [x_1 x_2]^T$ is the state vector. Consider the model in (1) as a master. A secondary goal is to synchronize a usually simpler dynamic, called slave, to follow a known system, called Master. From point of view of the slave, function $f(\cdot)$ in Eq. 1 is an unknown nonlinear function. A fractional dynamic of slave can be generally represented as

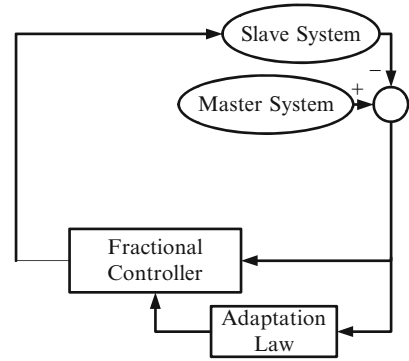
$$\begin{aligned} D^q y_1 &= y_2, \\ D^q y_2 &= f(Y, t) + \Delta f(Y, t) + d(t) \end{aligned} \quad (2)$$

where, $Y = [y_1 y_2]^T$ is state of the slave dynamic, $\Delta f(\cdot)$ stands for uncertainty, $d(t)$ is disturbance and $u(t)$ is the control signal to synchronize the slave with the master. It is suggested to synchronize via an adaptive fractional controller. The synchronization error is defined by $e_i = x_i - y_i$, $i = 1, 2$ where $i = 1, 2$. Schematic diagram of the closed loop system together with the proposed adaptive fractional controller is shown in Fig. 1.

Supposed that PID controller is of the following form:

$$u_{PID} = K_P D^{\alpha_1} e_1(t) + K_I D^{\alpha_2} e_1(t) + K_D D^{\alpha_3} e_1(t) \quad (3)$$

Fig. 1 Schematic diagram of a synchronization mechanism



It should be noted that the controller would be of the classic one if $\alpha_1 = 0$, $\alpha_2 = -1$ and $\alpha_3 = 1$. The reason behind the selection is that this kind of controller is most popular in the literature. Furthermore, the fractional controller provides the stability with more degree of freedom. To have a fractional order, parameters are chosen as $0 \leq \alpha_1 < 1$, $-1 \leq \alpha_2 < -2$ and $1 \leq \alpha_3 < 2$. Parameters of PID controller, i.e. K_P , K_I and K_D will be updated via a proper gradient-based adaptation mechanism to provide a robust synchronizing controller [16]. The following fractional order differential equation describes a follower dummy output state by y_r :

$$D^q y_r = D^q x_2 + k_2 D^q e_1 + k_1 e_1 \quad (4)$$

The sliding surface will also be defined as the error between two outputs, which is as follows:

$$S = y_2 - y_r \quad (5)$$

When the sliding mode is activated, i.e. $S = 0$, therefore we have:

$$y_2 = y_r \quad (6)$$

Since $e_2 = y_2 - x_2$ and $\dot{e}_1 = e_2$, replacing Eq. 6 in Eq. 4 immediately results as:

$$D^{2q} e_1 + k_2 D^q e_1 + k_1 e_1 \quad (7)$$

Or in a state space format:

$$\begin{aligned} D^q e_1 &= e_2, \\ D^q e_2 &= -k_1 e_1 - k_2 e_2 \Rightarrow D^q E = AE \end{aligned} \quad (8)$$

where, is the gain (coefficient) matrix for the state error of . Using suitable values for gains k_1, k_2 to meet the argument condition $|\arg(\text{eig}(A))| = q\pi/2$ for the stability

of fractional systems the error $e_1(t)$ tends to zero when $t \rightarrow \infty$. Let us candidate the following function as a Lyapunov function in term of the sliding surface:

$$V = \left(\frac{1}{2}\right) S^2 \quad (9)$$

The sliding condition will be of the form:

$$\dot{V} = S \dot{S} \quad (10)$$

When Eq. 10 is met, unboundedness of the sliding surface will be guaranteed when time tends to infinity. This means $S(t) \rightarrow 0$ when $t \rightarrow \infty$. The gradient search algorithm is calculated in the direction opposite to the energy flow. Moreover, it is quite intuitive to choose $S \dot{S}$ as an error function. From Eq. 5 and using Eq. 2, we have

$$\dot{S} = \dot{y}_2 - \dot{y}_r = D^{1-q} (f(Y, t) + \Delta f(Y, t) + d(t) + u_{PID}) - \dot{y}_r \quad (11)$$

Pre multiplying both sides of Eq. 11 by S yields

$$S \dot{S} = S [D^{1-q} (f(Y, t) + \Delta f(Y, t) + d(t) + u_{PID}) - \dot{y}_r] \quad (12)$$

Let us define the following equation:

$$U_{PID} = D^{1-q} (u_{PID}) \quad (13)$$

PID coefficients will be obtained if one uses the gradient of the adaptation law [16], which are as follows:

$$\dot{K}_P = -\gamma \frac{\partial S \dot{S}}{\partial K_P} = -\gamma \frac{\partial S \dot{S}}{\partial U_{PID}} \frac{\partial U_{PID}}{\partial K_P} = -\gamma S D^{1-q} (D^{\alpha_1} e_1(t)) \quad (14)$$

$$\dot{K}_I = -\gamma \frac{\partial S \dot{S}}{\partial K_I} = -\gamma \frac{\partial S \dot{S}}{\partial U_{PID}} \frac{\partial U_{PID}}{\partial K_I} = -\gamma S D^{1-q} (D^{\alpha_2} e_1(t)) \quad (15)$$

$$\dot{K}_D = -\gamma \frac{\partial S \dot{S}}{\partial K_D} = -\gamma \frac{\partial S \dot{S}}{\partial U_{PID}} \frac{\partial U_{PID}}{\partial K_D} = -\gamma S D^{1-q} (D^{\alpha_3} e_1(t)) \quad (16)$$

Substitution of $\alpha_1 = q - 1$, $\alpha_2 = q - 2$ and $\alpha_3 = q$ in Eqs. 14-16 results the following form:

$$\dot{K}_P = -\gamma S e_1(t) \quad (17)$$

$$\dot{K}_I = -\gamma S \int_0^t e_1(\tau) d(\tau) \quad (18)$$

$$\dot{K}_D = -\gamma S \frac{de_1(t)}{dt} \quad (19)$$

where, γ is the learning rate. It should be noted that γ and K_P , K_I and K_D should be carefully selected to maintain the convergence [16]. The proposed controller is applied on a fractional gyro dynamic in a synchronization task to show the performance of the technique.

3 Synchronization of Uncertain Fractional-Order Gyro System

3.1 System Description

According to the study by Chen [17], dynamics of a symmetrical Gyro with linear-plus-cubic damping of the angle can be expressed as [18]:

$$\ddot{\theta} + a^2 \frac{(1 - \cos\theta)}{\sin^3\theta} - b\sin\theta + c_1\dot{\theta} + c_2\dot{\theta}^3 - \beta\sin(\omega t)\sin(\theta) \quad (20)$$

where, $\beta \sin(\omega t)$ represents a parametric excitation, $c_1\dot{\theta}$ and $c_2\dot{\theta}^3$ are linear and nonlinear damping terms, respectively, and $a^2 \frac{(1 - \cos\theta)}{\sin^3\theta} - b\sin\theta$ is a nonlinear resilience force. Given the states $x_1 = \theta$, $x_2 = \dot{\theta}$ and, $f(\theta, \dot{\theta}) = -a^2 \frac{(1 - \cos\theta)}{\sin^3\theta} - b\sin\theta - c_1\dot{\theta} - c_2\dot{\theta}^3 + (b + \beta\sin(\omega t))\sin(\theta)$ this system can be transformed into the following nominal state form:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f(x_1, x_2) \end{aligned} \quad (21)$$

This Gyro system demonstrates complex dynamics. The behavior has been studied by Chen [17] for variety of β in the range $32 < \beta < 36$ and constant values of $a = 10$, $b = 1$, $c_1 = 0.5$, $c_2 = 0.05$ and $\omega = 2$. Let us consider the fraction Gyro dynamic in the following state space format:

$$\begin{aligned} D^q x_1 &= x_2, \\ D^q x_2 &= f(x_1, x_2) \end{aligned} \quad (22)$$

Figure 2 shows the phase portrait of gyro chaotic system with fractional derivative in presence of $\beta = 35.5$, initial $(x_1, x_2) = (1, -1)$ conditions of and $q = 0.97$. To show the effectiveness of the proposed controller, the procedure is implemented on fractional Gyro dynamic.

Fig. 2 Phase portrait of fractional Gyro chaotic system

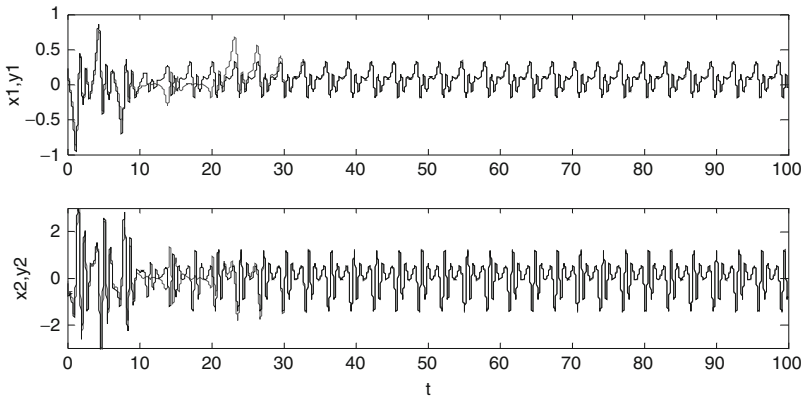
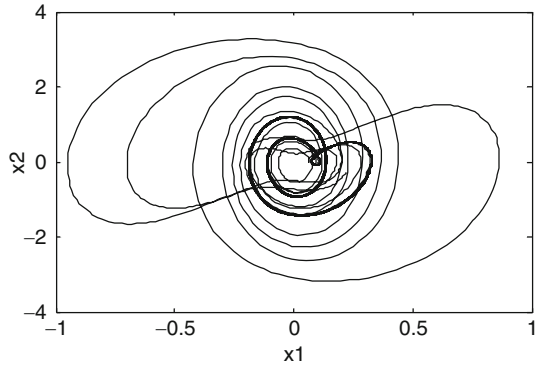


Fig. 3 Synchronization of fractional Gyros system

3.2 Implementation

Consider system 23 as a master, which is perturbed with such an uncertainty. A slave system may be defined as the following equation:

$$\begin{aligned} D^q y_1 &= y_2, \\ D^q y_2 &= f(y_1, y_2) + \Delta f(y_1, y_2) + d(t) + u(t) \end{aligned} \quad (23)$$

The sets of initial conditions of master and slave systems are respectively defined as $[x_1(0), x_2(0)] = [1, -1]$ and $[y_1(0), y_2(0)] = [1.6, 0.8]$. In order to chose an uncertainty and disturbance, $\Delta f(y_1, y_2) = -0.1 \sin(y_1)$ and $d(t) = 0.2 \cos(\pi t)$ are assigned, respectively. Primarily setting of PID coefficients are chosen equal to K_P , K_I and K_D and the learning rate has been selected as $\gamma = 1$. Furthermore k_1 and k_2 are selected as 0.5 and 1, respectively. Simulation results have shown in Figs. 3 to 6. In Fig. 3, synchronization of x_1, y_1 and x_2, y_2 are made perfect.

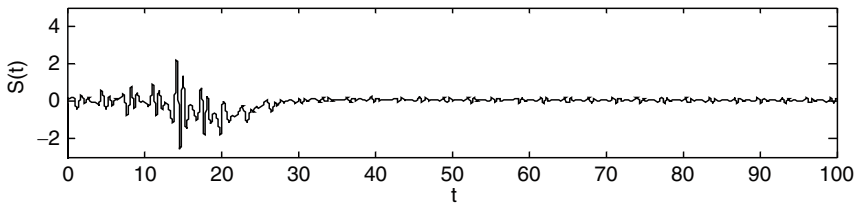


Fig. 4 Sliding surface

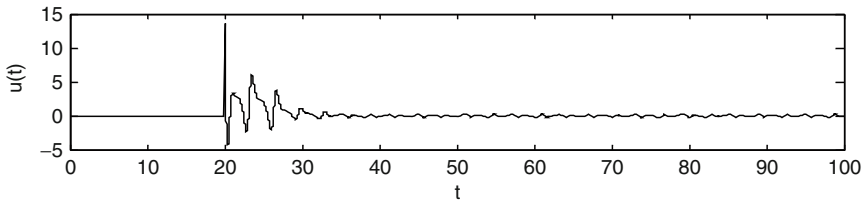


Fig. 5 The control signal

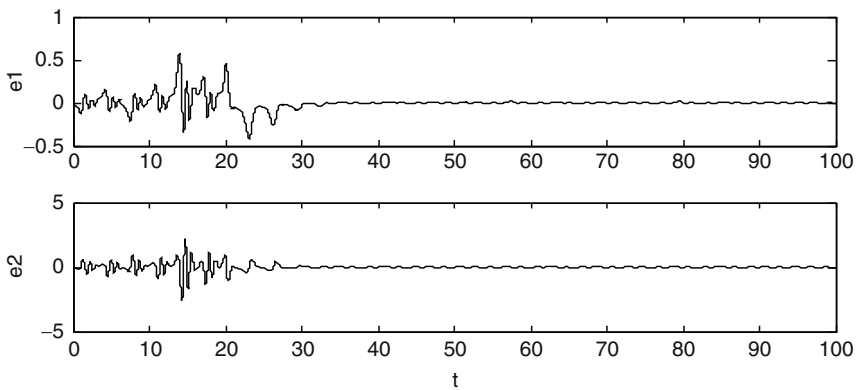


Fig. 6 Error of synchronization

The sliding surface and the control input are shown in Figs. 4 and 5, respectively, whereas Fig. 3 shows the synchronization error. It should be noted that the control signal, $u(t)$ has been activated in $t = 20$ s.

Primary setting of PID coefficients are equal to $K_p(0) = 5$, $K_I(0) = 5$, $K_D(0) = 5$ and the learning rate has been selected as $\gamma = 0.14$. Also, k_1 and k_2 have been assigned as 0.5 and 2, respectively. Figures 3 to 6 show the simulation results. Synchronization and error of synchronization of x_1 , x_2 and y_1 , y_2 have shown in Figs. 3 and 4, respectively, the control signal has been shown in Fig. 5, and the sliding surface has shown in Fig. 6. It should be noted that in this section, the control signal u_{PID} , has been activated in $t = 10$ s. The performance of the controller is significant. Parameters of the system, similar to Sect. 3.1, have been selected as $\mu = 0.4$, $\lambda = 0.1$, $\alpha = 1$, $\omega_0 = 0.46$, $\omega = 0.86$ and $f_0 = 4.5$.

4 Conclusion

An adaptive fractional controller is proposed to synchronize a chaotic system. Coefficients and parameters of the controller are updated using a gradient-based adaptation mechanism. The controller has successfully been applied on the dynamic of fractional Gyro system. The simulation results verify the significance of the proposed controller.

References

1. He JH (1998) Nonlinear oscillation with fractional derivative and its applications. International Conference on Vibrating Engineering'98. Dalian, China, pp 288–291
2. Odibat Z, Momani S (2008) Modified homotopy perturbation method: Application to quadratic Riccati differential equation of fractional order. *Chaos Soliton Fract* 36(1):167–174
3. Cang J, Tan Y, Xu H, Liao SJ (2007) Chattering-free fuzzy sliding-mode control strategy for uncertain chaotic systems. *Chaos Solitons and Fractals* 30:709–718
4. Jafari H, Momani S (2007) Solving fractional diffusion and wave equations by modified homotopy perturbation method. *Phys Lett A* 370(5–6):388–396
5. Ge Z-M, Ou C-Y (2008) Chaos synchronization of fractional order modified duffing systems with parameters excited by a chaotic signal. *Chaos Soliton Fract* 35(4):705–717
6. Li C, Chen G (2004) Chaos in the fractional order Chen system and its control. *Chaos Soliton Fract* 22:549–554
7. Wajdi AM, Ahmad MH (2003) On nonlinear control design for autonomous chaotic systems of integer and fractional orders. *Chaos Soliton Fract* 18:693–701
8. Wajdi AM, El-Khazali R, Al-Assaf Y (2004) Stabilization of generalized fractional order chaotic systems using state feedback control. *Chaos Soliton Fract* 22:141–150
9. Wajdi AM (2005) Hyperchaos in fractional order nonlinear systems. *Chaos Soliton Fract* 26:1459–1465
10. Nimmo S, Evans AK (1999) The effects of continuously varying the fractional differential order of chaotic nonlinear systems. *Chaos Soliton Fract* 10:1111–1118
11. Yau H-T, Chen C-L (2006) Chattering-free fuzzy sliding-mode control strategy for uncertain chaotic systems. *Chaos Soliton Fract* 30:709–718
12. Zhang H, Ma X-K, Liu W-ZC (2004) Synchronization of chaotic systems with parametric uncertainty using active sliding mode control. *Chaos Soliton Fract* 21:1249–1257
13. Tavazoei MS, Haeri M (2007) Determination of active sliding mode controller parameters in synchronizing different chaotic systems. *Chaos Soliton Fract* 32:583–591
14. Yau H-T (2004) Design of adaptive sliding mode controller for chaos synchronization with uncertainties. *Chaos Soliton Fract* 22:341–347
15. Wang C, Ge SS (2001) Adaptive synchronization of uncertain chaotic systems via backstepping design. *Chaos Soliton Fract* 12:199–206
16. Chang W-D, Yan J-J (2005) Adaptive robust PID controller design based on a sliding mode for uncertain chaotic systems. *Chaos Soliton Fractals* 26(1):167–175
17. Chen HK (2002) Chaos and chaos synchronization of a symmetric gyro with linear-plus-cubic damping. *J Sound Vib* 255:719–740
18. Yau H-T (2008) Chaos synchronization of two uncertain chaotic nonlinear gyros using fuzzy sliding mode control. *Mech Sys Signal Proc* 22(2):408–418

Controllability and Minimum Energy Control Problem of Fractional Discrete-Time Systems

Jerzy Klamka

Abstract In the paper minimum energy control problem of infinite-dimensional fractional-discrete time linear systems is addressed. Necessary and sufficient conditions for the exact controllability of the system are established. Sufficient conditions for the solvability of the minimum energy control of the infinite-dimensional fractional discrete-time systems are given. A procedure for computation of the optimal sequence of inputs minimizing the quadratic performance index is proposed.

1 Introduction

Controllability is one of the fundamental concepts in modern mathematical control theory. This is qualitative property of control systems and is of particular importance in control theory. Systematic study of controllability was started at the beginning of 1960s in twentieth century and theory of controllability is based on the mathematical description of the dynamical system.

Many dynamical systems are such that the control does not affect the complete state of the dynamical system but only a part of it. On the other hand, very often in real industrial processes it is possible to observe only a certain part of the complete state of the dynamical system. Therefore, it is very important to determine whether or not control of the complete state of the dynamical system is possible. Roughly speaking, controllability generally means, that it is possible to steer dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls.

Controllability plays an essential role in the development of the modern mathematical control theory. There are important relationships between controllability, stability and stabilizability of linear control systems. Controllability is also strongly

J. Klamka (✉)

Institute of Control Engineering, Silesian Technical University Akademicka 16,
44–100 Gliwice, Poland
e-mail: jerzy.klamka@polsl.pl

connected with the theory of minimal realization of linear time-invariant control systems. Moreover, it should be pointed out that there exists a formal duality between the concepts of controllability and observability. Moreover, controllability is strongly connected with so-called minimum energy control problem [1].

It should be pointed out that in the literature there are many results concerning controllability and minimum energy control, which depend on the type of dynamical control system [1]. The reachability and controllability to zero of positive fractional linear systems have been investigated in [2–4]. The minimum energy control problem has been solved for different classes of linear systems in [1]. In this paper the minimum energy control problem will be addressed for infinite-dimensional fractional discrete-time linear systems.

The paper is organized as follows. In Sect. 2 the solution of the difference state equation the infinite-dimensional fractional systems is recalled. Necessary and sufficient conditions for the exact controllability of the infinite-dimensional fractional systems are established in Sect. 3. The main result of the paper is presented in Sect. 4, in which the minimum energy control problem is formulated and solved. Concluding remarks are given in Sect. 5.

To the best knowledge of the author the minimum energy control problem for the infinite-dimensional fractional discrete-time linear systems have not been considered yet.

2 Fractional Systems

The set of nonnegative integers will be denoted by Z_+ . Let X and U be the separable generally infinite-dimensional Hilbert spaces and $x_k \in X, u_k \in U, k \in Z_+$. In finite-dimensional case $X = R^n$ and $U = R^m$.

In this paper extending definition of the fractional difference of the form

$$\Delta^\alpha x_k = \sum_{j=0}^k (-1)^j \binom{\alpha}{j} x_{k-j}, \quad (1)$$

$$n-1 < \alpha < n \in N = \{1, 2, \dots\}, k \in Z_+$$

will be used, where $\alpha \in R$ is the order of the fractional difference and

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0 \\ \frac{\alpha(\alpha-1) \cdots (\alpha-j+1)}{j} & \text{for } j = 1, 2, \dots \end{cases} \quad (2)$$

Let us consider the fractional discrete linear system, described by the infinite-dimensional state-space equations

$$\Delta^\alpha x_{k+1} = Ax_k + Bu_k, \quad k \in Z_+ \quad (3)$$

where $x_k \in X, u_k \in U$ are the state and input and $A: X \rightarrow X, B: U \rightarrow X$ are given linear and bounded operators. In finite dimensional case A and B are $n \times n$ and $n \times m$ constant matrices, respectively.

Using definition (1) we may write the Eq. (3) in the form

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k, \quad k \in \mathbb{Z}_+ \quad (4)$$

Lemma 1. The solution of Eq. 4 with initial condition $x_0 \in X$ is given by

$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} Bu_i \quad (5)$$

where linear and bounded operators $\Phi_k: X \rightarrow X$ are determined by the equation

$$\Phi_{k+1} = (A + I_n \alpha) \Phi_k + \sum_{i=2}^{k+1} (-1)^{i+1} \binom{\alpha}{i} \Phi_{k-i+1} \quad (6)$$

with $\Phi_0 = I$, where I is the identity operator.

Remark 1. In finite-dimensional case operators $\Phi_k: R^n \rightarrow R^n$ are constant $n \times n$ dimensional matrices.

3 Controllability

First of all, in order to define controllability concepts let us introduce the notion of reachable set in q steps for infinite-dimensional discrete-time fractional control system (4).

Definition 1. For fractional system (4) reachable set in q steps from $x_0 = 0$ is defined as follows

$$K_q = \{x \in X : x \text{ is a solution of Eq. (4) for } k = q \text{ and for sequence of controls } u_0, u_1, \dots, u_k, \dots, u_{q-1}\} \quad (7)$$

Remark 2. It should be pointed out, that in infinite-dimensional case it is necessary to distinguish between exact and approximate controllability.

Definition 2. The fractional system (4) is exactly controllable in q -steps if

$$K_q = X \quad (8)$$

Definition 3. The fractional system (4) is approximately controllable in q -steps if

$$cl(K_q) = X \quad (9)$$

where $cl(K_q)$ means the closure of the set K_q .

Theorem 1. The fractional system (4) is exactly controllable in q steps if and only if the image ImR_q of controllability operator

$$R_q := [B, \Phi_1 B, \dots, \Phi_{q-1} B] \quad (10)$$

is the whole space X .

Proof. Using (5) for $k = q$ and $x_0 = 0$ we obtain

$$x_f = x_q = \sum_{i=0}^{q-1} \Phi_{q-i-1} B u_i = R_q \begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_0 \end{bmatrix} \quad (11)$$

From Definition 2 and (11) it follows that for every final state $x_f \in X$ there exists a input sequence $u_i \in U, i = 0, 1, \dots, q-1$ if and only if the image of controllability operator ImR_q is the whole space X .

Corollary 1. The fractional system (4) is exactly controllable in q steps if and only if $R_q R_q^*$ is invertible operator; i.e. there exist linear and bounded operator $(R_q R_q^*)^{-1}$.

Corollary 2. The fractional system (4) is approximately controllable in q steps if and only if $cl(ImR_q)$ of controllability operator (10) is the whole state space X , or equivalently if and only if the reachable set in q steps K_q is dense in the Hilbert space X .

Since for $X = R^n$ approximate controllability in q -steps and exact controllability in q -steps coincide, we say shortly controllability in q -steps. Therefore, taking into account Theorem 1 we have the following Corollary.

Corollary 3. The fractional finite-dimensional system (4) is controllable in q steps if and only if $n \times nm$ dimensional controllability matrix

$$R_q := [B, \Phi_1 B, \dots, \Phi_{q-1} B] \quad (12)$$

has full row rank n .

4 Minimum Energy Control

Consider the fractional infinite-dimensional system (4). If the system is exactly controllable in q steps then generally there exist many different input sequences that steer the initial state of the system from $x_0 = 0$ to the final state $x_f \in X$.

Among these input sequences we are looking for the sequence $u_i \in U, i = 0, 1, \dots, q-1, i \in \mathbb{Z}_+$ that minimizes the quadratic performance index

$$I(u) = \sum_{j=0}^{q-1} u_j^* Q u_j \quad (13)$$

where $Q: {}^U\mathbb{R}^U$ is a selfadjoint positive definite operator, q is a given number of steps in which the state of the system is transferred from $x_0 = 0$ to $x_f \in X$ and $u^* \in U$ denotes adjoint element, which in finite dimensional case denotes vector transposition.

The minimum energy control problem for the infinite-dimensional fractional system (4) can be stated as follows. For a given linear bounded operators A, B and the order α of the fractional system (4), the number of steps q , final state $x_f \in X$ and the selfadjoint operator Q of the performance index (13), find a sequence of inputs $u_i \in U, i = 0, 1, \dots, q-1$, that steers the state of the system from initial state $x_0 = 0$ to $x_f \in X$ and minimizes the quadratic performance index (13).

In order to solve the minimum energy problem we define selfadjoint operator

$$W(q, Q) = R_q \overline{Q} R_q^* \quad (14)$$

where R_q is controllability operator defined by (10) and selfadjoint operator

$$\overline{Q} : \underbrace{U \times U \times \dots \times U}_{q\text{-times}} \rightarrow \underbrace{U \times U \times \dots \times U}_{q\text{-times}}$$

is defined as follows

$$\overline{Q} = \text{blockdiag}[Q^{-1}, Q^{-1}, \dots, Q^{-1}] \quad (15)$$

From (15) it follows that operator $W(q, Q)$ is invertible if and only if $R_q R_q^*$ is invertible operator, i.e. there exist linear and bounded operator $(R_q R_q^*)^{-1}$ and therefore, fractional system (4) is exactly controllable in q steps.

If the condition of Theorem 1 is met then the system is exactly controllable in q steps. In this case we may define for a given $x_f \in X$ the following sequence of inputs

$$\begin{bmatrix} \widehat{u}_{q-1} \\ \widehat{u}_{q-2} \\ \vdots \\ \widehat{u}_0 \end{bmatrix} = \overline{Q} R_q^* W^{-1}(q, Q) x_f \quad (16)$$

Theorem 2. *Let the fractional system (4) be exactly controllable in q steps. Moreover let $\bar{u}_i \in U, i = 0, 1, \dots, q-1$ be a sequence of inputs that steers the state of the system from $x_0 = 0$ to $x_f \in X$. Then the sequence of inputs $\hat{u}_i \in U, i = 0, 1, \dots, q-1$ defined by (16) also steers the state of the system from $x_0 = 0$ to $x_f \in X$ and minimizes the performance index (13), i.e.*

$$I(\hat{u}) \leq I(\bar{u}) \quad (17)$$

The minimal value of performance index (13) for the minimum energy control (16) is given by

$$I(\bar{u}) = x_f^* W^{-1}(q, Q)x_f \quad (18)$$

Proof. If the fractional system (4) is exactly controllable in q steps, then for $x_f \in X$ we shall show that the sequence of controls given by equality (16) steers the state of the system (4) from initial state $x_0 = 0$ to final state $x_f \in X$.

Using equality (5) for $k = q, x_0 = 0$ and (12), (16) we obtain

$$x_q = R_q \hat{u}_{0q} = R_q \bar{Q} R_q^* W^{-1}(q, Q)x_f = x_f \quad (19)$$

since

$$R_q \bar{Q} R_q^* W^{-1}(q, Q) = I$$

The both sequences of inputs $\bar{u}_{0q}$ and $\hat{u}_{0q}$ steer the state of the system from $x_0 = 0$ to the same final state x_f . Hence $x_f = R_q \hat{u}_{0q} = R_q \bar{u}_{0q}$ and

$$R_q [\hat{u}_{0q} - \bar{u}_{0q}] = 0 \quad (20)$$

Using (20) and (20) we shall show that

$$[\hat{u}_{0q} - \bar{u}_{0q}]^T \hat{Q} \hat{u}_{0q} = 0 \quad (21)$$

where $\hat{Q} = \text{block diag } [Q, \dots, Q]$.

Therefore, (21) yields

$$(\hat{u}_{0q} - \bar{u}_{0q})^* R_q^* = 0 \quad (22)$$

Postmultiplying the equality by $W^{-1}(q, Q)x_f$ we obtain

$$(\hat{u}_{0q} - \bar{u}_{0q})^* R_q^* W^{-1}(q, Q)x_f = 0 \quad (23)$$

Using (16) and (23) we obtain (24) since

$$\begin{aligned} & (\hat{u}_{0q} - \bar{u}_{0q})^* \hat{Q} \hat{u}_{0q} \\ &= (\hat{u}_{0q} - \bar{u}_{0q})^* \hat{Q} \bar{Q} R_q^* W^{-1}(q, Q)x_f \\ &= (\hat{u}_{0q} - \bar{u}_{0q})^* R_q^* W^{-1}(q, Q)x_f = 0 \end{aligned}$$

and $\hat{Q} \bar{Q} = I$.

Using (25) it is easy to verify that

$$\bar{u}_{0q}^T \bar{Q} \hat{u}_{0q} = \hat{u}_{0q}^T \hat{Q} \hat{u}_{0q} + [\bar{u}_{0q} - \hat{u}_{0q}]^T \hat{Q} [\bar{u}_{0q} - \hat{u}_{0q}] \quad (24)$$

From (24) it follows that the inequality (21) holds, since

$$[\bar{u}_{0q} - \hat{u}_{0q}]^T \hat{Q} [\bar{u}_{0q} - \hat{u}_{0q}] \geq 0$$

In order to find the minimal value of the performance index we substitute (16) into (15) and we use (16). Then we obtain

$$\begin{aligned} I(\bar{u}) &= \bar{u}_{0q}^* \bar{Q} \bar{u}_{0q} \\ &= (\bar{Q} R_q^* W^{-1}(q, Q) x_f)^* \bar{Q} (\bar{Q} R_q^* W^{-1}(q, Q) x_f) \\ &= x_f^* W^{-1}(q, Q) R_q \bar{Q} R_q^* W^{-1}(q, Q) x_f \\ &= x_f^* W^{-1}(q, Q) x_f \end{aligned}$$

since $\bar{Q} \bar{Q} = I$ and $W^{-1}(q, Q) R_q \bar{Q} R_q^* = I$.

5 Concluding Remarks

The minimum energy control problem of infinite-dimensional fractional discrete linear systems has been addressed. Necessary and sufficient conditions for the exact controllability in q steps of the systems have been established.

Under assumption on exact controllability in q steps solvability of the minimum energy control of the infinite-dimensional fractional discrete-time linear systems have been given and a procedure for computation of the optimal sequence of inputs minimizing the quadratic performance index has been proposed.

Finally, it should be mentioned, that the considerations can be extended for infinite-dimensional fractional discrete-time linear systems with delays both in control and state variables and for infinite-dimensional fractional continuous-time linear systems with constant parameters.

Acknowledgements This work was supported by the Ministry of Science and High Education of Poland under grant NN 514 415834.

References

1. Klamka J (1991) Controllability of dynamical systems. Kluwer, Dordrecht
2. Kaczorek T (2002) Positive 1D and 2D systems. Springer, London
3. Kaczorek T (2007) Reachability and controllability to zero of cone fractional linear systems. Arch Control Sci. 17(3):357–367
4. Klamka J (2002) Positive controllability of positive systems. In Proceedings of American Control Conference, ACC-2002. Anchorage, CD-ROM

Control of Chaos via Fractional-Order State Feedback Controller

Seyed Hassan Hosseinnia, Reza Ghaderi, Abolfazl Ranjbar, Farzad Abdous, and Shaher Momani

Abstract In this paper, a fractional-order state feedback controller has been proposed to stabilize chaos. The fractional controller converts the system behaviour with integer derivatives into a system with fractional derivatives. This increases the degree of freedom of the system by means of providing the stability without need for such a pole placement technique. In addition, an integer state feedback controller is used to increase the rate of convergence. The proposed controller uses the benefit of both integer and fractional order controllers at the same time. The performance of the controller is shown via simulation on chaotic Genesis-Tesi system.

1 Introduction

Fractional calculus has a 300-year mathematical history. Nowadays, it has been found that some fractional order differential systems demonstrate chaotic behaviour. Chua circuit [1], Duffing system [2], jerk model [3], Chen dynamic [4], characterization [5], Rossler system [6], Arneodo model [7] and Newton-Leipnik formulation [8] are examples of known systems that display chaotic behaviour. Highly sensitivity to initial conditions is a main characteristic of chaotic systems. Therefore, these systems are difficult for synchronization or control. In [9], a fractional order controller has proposed to control unstable systems. Chaos, control and synchronization of a fractional order rotational mechanical system with a centrifugal governor are studied in [10]. A chattering-free fuzzy sliding-mode control (FSMC) strategy for uncertain chaotic systems has also been proposed in [11]. In [12] the authors pro-

S.H. Hosseinnia, R. Ghaderi, A. Ranjbar (✉), and F. Abdous
Intelligent system research group, Faculty of Electrical and Computer Engineering,
Noushirvani University of Technology, P.O. Box: 47135-484, Babol, Iran
e-mail: h.hosseinnia@stu.nit.ac.ir; r_ghaderi@yahoo.co.uk; a.ranjbar@nit.ac.ir;
f_abdous@yahoo.com

S. Momani
Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan
e-mail: shahermmm@yahoo.com

posed an active sliding mode control method for synchronizing two chaotic systems with parametric uncertainty. An algorithm to determine parameters of an active sliding mode controller in synchronizing to different chaotic systems has been studied in [13]. In [14] an adaptive sliding mode controller is presented for a class of master-slave chaotic synchronization systems with uncertainties. In [15] back-stepping control has proposed to synchronize the chaotic systems. In this paper, by combining two controllers of a fractional controller [9] and a state feedback controller, a new configuration of the controller was proposed to control the chaotic systems. This controller provides more degree of freedom to stabilize the closed loop system. The increment of the rate of convergence is other advantage of the proposed controller. This paper is organized as follows: First, the stability of fractional systems is briefly discussed in Sect. 2. Section 3, presents the new type proposed controller. The stability of the controller was also discussed in this section. The proposed controller will be implemented on Genesio-Tesi system in Sect. 4. Finally, the result and the work will be concluded at Sect. 5.

2 Stability Analysis For Fractional-Order Systems

Fractional order differential equations are at least as stable as their integer orders counterparts, because systems with memory are typically more stable than their memoryless alternatives [16]. It has been shown that the autonomous dynamic $D^\alpha x = Ax, x(0) = x_0$ is asymptotically stable if the following condition is met [17]:

$$|\arg(\text{eig}(A))| > \alpha\pi/2 \quad (1)$$

where, $0 < \alpha < 1$ and $\text{eig}(A)$ represents the eigenvalues of matrix A . In this case, each component of states decays towards 0, like $t^{-\alpha}$. Furthermore, the system is stable if $|\arg(\text{eig}(A))| \geq \alpha\pi/2$ and those critical eigenvalues which satisfy have geometric multiplicity of 1. The stability region for $0 < \alpha < 1$ is shown in Fig. 1. Now, consider the following autonomous commensurate order of fractional system:

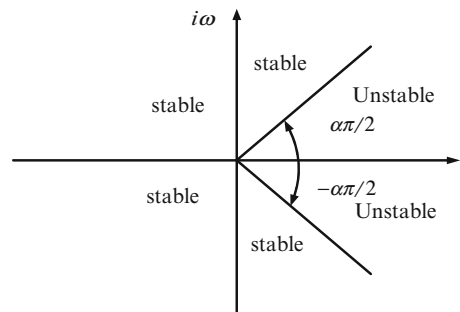


Fig. 1 Stability region of the FOLTI system with fractional order, $0 < \alpha < 1$

$$D^\alpha x = f(x). \quad (2)$$

where $0 < \alpha < 1$ and $x_2 \in \mathbb{R}^n$. The equilibrium points of system 2 are found by solving the equation:

$$f(x) = 0. \quad (3)$$

These points are locally and asymptotically stable if all eigenvalues of the Jacobian matrix $A = \frac{\partial f(x)}{\partial x}$, which are evaluated at the equilibrium points satisfy the following condition [16, 17]:

$$|\arg(\text{eig}(A))| > \alpha\pi/2 \quad (4)$$

The main advantage of fractional expression of system in stability analysis is; all parameters of system (including the region of stability) could be affected by α . This means more compactness in the system representation will be achieved rather than classic representation of systems. In other words, in comparison with integer order expression with the same resolution, fractional order expression would provide better conditions both in stability analysis and design procedure.

3 Proposed Controller Design

Before designing the controller, a brief description of the fractional controller will be presented. This controller was proposed to control such chaotic systems [9]. Applying this controller, a system of integer derivatives will be transformed to a system of fractional order. Consequently, the degree of achieving the stability will be increased. The fundamental of the designation will be described in next section.

3.1 Simple Fractional Controller

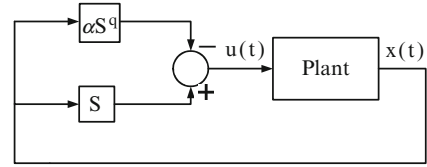
The proposed fractional controller in [9] increases the boundary of the stability. There is no need to use such a pole placement to guarantee the closed loop stability. As already mentioned this controller converts the system of integer type to a fractional one. This kind of controller will be applied on some other dynamic systems in conjunction with a state feed back controller. Let us assume the system with the following classic dynamic:

$$\dot{X} = f(X) + u(t). \quad (5)$$

where, $X \in \mathbb{R}^n$, $f(X) \rightarrow \mathbb{R}^n$ and $u(t)$ is the control signal. A schematic diagram of the closed loop system using the proposed fractional order controller is shown in Fig. 2.

As it is illustrated in Fig. 2, the control signal is of the form:

Fig. 2 Schematic diagram of the closed loop system using fractional order controller



$$u(t) = \dot{X} - \alpha D^q X. \quad (6)$$

Applying the controller in (6) (Fig. 2), the closed loop system is altered to a fractional order of the following dynamic:

$$D^q X = \alpha^{-1} f(X). \quad (7)$$

Equation 6 determines a lower boundary of the commensurate parameter q . Similarly, tuning α increases the rate of convergence.

3.2 State Feedback Controller

State feedback controller stabilizes the system or improves the performance indices by a pole placement approach. The control law uses the fed back states by $u(t) = -KX$ to satisfy the requirements. Consequently, the closed loop characteristics of the system in (5) take the form:

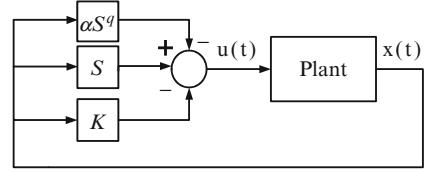
$$\dot{X} = f(X) - KX. \quad (8)$$

where, $K \in \mathbb{R}^n$ will be designed according to the required performance. Since the robustness is not guaranteed, a fractional controller is of interest to compensate possible shortcoming. In the following subsection, two aforementioned controllers will be combined to establish a new type of controller.

3.3 Combination of State Feedback and Fractional-Order Controllers

As already mentioned (Sect. 3.1), the fractional order controller improves the degree of freedom to design the controller without need for such a pole placement technique. Furthermore, the rate of convergence will be adjusted by other parameter, i.e. α . However, these designation increases the magnitude of overshoots. This in return increases the risk of the system being unstable [9]. As a partial conclusion, the proposed controller will be used when poles of the system are placed elsewhere

Fig. 3 Block diagram of the proposed controller in a closed loop system



than needed. The fractional controller satisfies the rest of important requirements. Block diagram of the proposed closed loop system is illustrated in Fig. 3.

The control law is being constructed according to Fig. 3 by

$$u(t) = \dot{X} - \alpha D^q X - KX. \quad (9)$$

Therefore, the dynamic will be altered to

$$D^q X = \alpha^{-1}(f(X) - KX). \quad (10)$$

The performance will be achieved when parameters K and α are optimally tuned.

3.4 The Stability Analysis

The stability condition of the closed loop system in (4) approaches to

$$|\arg(\text{eig}(A)) - \arg(\text{eig}(K))| > q\pi/2. \quad (11)$$

If the gain matrix K is chosen Hurwitz, the stability condition will be simplified as

$$|\arg(\text{eig}(A))| > q\pi/2. \quad (12)$$

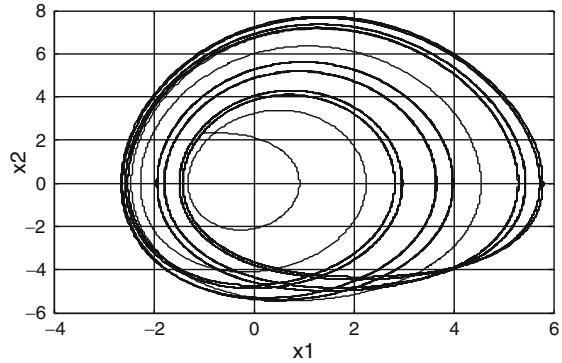
For simplicity k is chosen as $K = k_0 I_n$, where $k_0 > 0$ and I_n is the identity matrix of order n . Finally, the proposed controller will be used to stabilize Genesio-Tesi chaotic system.

4 Implementation of Proposed Controller in Genesio-Tesi Chaotic System

The proposed controller takes control over Genesio-Tesi chaotic system with the following illustrated dynamic [18]:

$$\dot{x}_1 = x_2$$

Fig. 4 Phase portrait of Genesio-Tesi system



$$\begin{aligned}\dot{x}_2 &= x_3 \\ \dot{x}_3 &= -cx_1 - bx_2 - ax_3 + x_1^2\end{aligned}\quad (13)$$

It has shown that [18] Genesio-Tesi system has a chaotic behaviour for $a = 1.2$, $b = 2.92$ and $c = 6$. The phase portrait of chaotic system is illustrated in Fig. 4. Similar to [18] initial conditions of system are considered as $x_1(0) = -1.0032$, $x_2(0) = 2.3545$ and $x_3(0) = -0.087$. In continuing, equilibrium points and Jacobian matrix will be computed. Furthermore, the matrix will be evaluated at the equilibrium points and the appropriate eigenvalues will be provided. Equilibrium points are found from dynamic in (13) as

$$(x_{1e}, x_{2e}, x_{3e}) = (0, 0, 0) \quad (x_{1e}, x_{2e}, x_{3e}) = (6, 0, 0)$$

The Jacobian matrix is as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 + 2x_{1e} & -2.92 & -1.2 \end{bmatrix}$$

These immediately result eigenvalues as, $(-1.64, 0.22 + 1.89i, 0.22 + 1.89i)$ and $(1.1, -1.15 + 2.03i, -1.15 - 2.03i)$.

Using Eq. 12 provides the stability boundary for $q < 0.92$. The simulation results considering $q = 0.9$ the same initial conditions and also $k_0 = 1$ and $\alpha = 1$ are illustrated in Fig. 5. Control inputs u_1, u_2, u_3 and controlled state x_1, x_2, x_3 are shown in this figure. The results of using the fractional controller (Fig. 5a), the state feedback (Fig. 5b), the proposed method (Fig. 5c) and all together are accordingly illustrated in Fig. 5d. It should be mentioned that controllers are activated in $t = 5s$. From the simulation results in Fig. 5a, it can be seen that states have reached to the equilibrium point approximately in 25 s, after the fractional controller have been activated. This is taken 15 s long the states to be settled when the state feedback controller was in use (Fig. 5b). Both responses are yet not satisfactory. Contrarily, the new combined controller provides a fast response. It may remain improvable

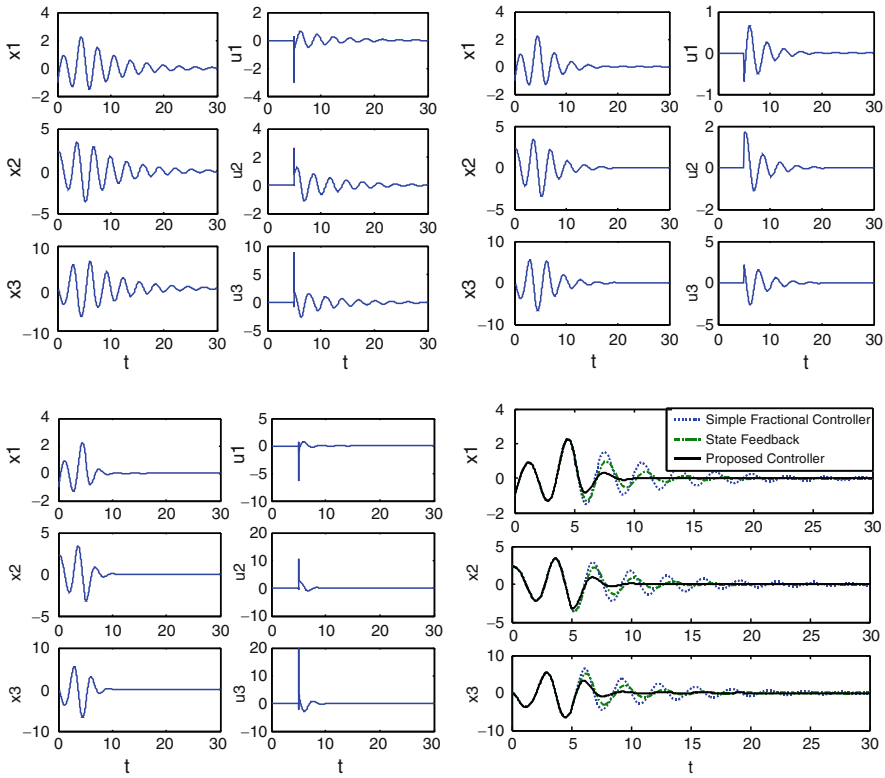


Fig. 5 Implementation of three controllers on Genesio-Tesi for the case $\alpha = 1, q = 0.9$

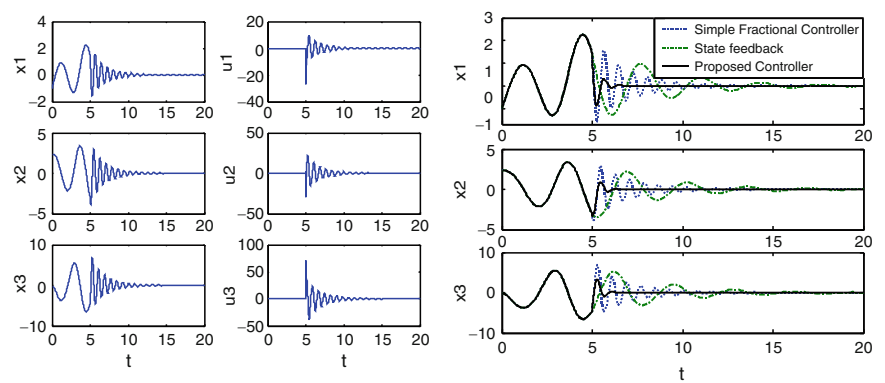


Fig. 6 Comparison of proposed controller with State feedback and simple fractional controllers

when a proper tune of increase the speed of convergence. Figure 6a is plotted for the case is properly tuned to to achieve fast response. Unfortunately, this adjustment may cause the risk of instability. Figure 6b signifies the performance of the proposed controller whilst smoothen the response.

5 Conclusion

A Fractional order controller has been proposed by combining two controllers of a fractional and state feedback controller to stabilize the chaos. The proposed controller has also increased the degree of freedom and of course the rate of convergence. The results are compared in different situations. It is also shown that this controller can be a good replacement of the pole placement technique. The significance of the proposed controller is investigated on a chaotic Genesio-Tesi system through simulation.

References

1. Hartley TT, Lorenzo CF, Qammer HK (1995) Chaos in a fractional order Chua's system. *IEEE Trans CAS-I* 42:485–490
2. Arena P, Caponetto R, Fortuna L, Porto D (1997) Chaos in a fractional order Duffing system. *Proc ECCTD*, Budapest, pp 1259–1262
3. Ahmad WM, Sprott JC (2003) Chaos in fractional-order autonomous nonlinear systems. *Chaos Soliton Fract* 16:339–351
4. Lu JG, Chen G (2006) A note on the fractional-order Chen system. *Chaos Soliton Fract* 27(3):685–688
5. Lu JG (2006) Chaotic dynamics of the fractional order $L\ddot{u}$ system and its synchronization. *Phys Lett A* 354(4):305–311
6. Li C, Chen G (2004) Chaos and hyperchaos in the fractional order Rössler equations. *Physica A* 341:55–61
7. Lu JG (2005) Chaotic dynamics and synchronization of fractional order Arneodo's systems. *Chaos Soliton Fract* 26(4):1125–1133
8. Sheu LJ, Chen HK, Chen JH, Tam LM, Chen WC, Lin KT, Kang Y (2006) Chaos in the Newton-Leipnik system with fractional order. *Chaos Soliton Fract* (in press)
9. Tavazoei MS, Haeri M (2008) Chaos control via a simple fractional-order controller. *Phys Lett A* 372(6), 798–780
10. Ge Z-M, Hsu M-Y (2007) Chaos in a generalized van der Pol system and in its fractional order system. *Chaos Soliton Fract* 33:1711–1745
11. Yau, H-T, Chen C-L (2006) Chattering-free fuzzy sliding-mode control strategy for uncertain chaotic systems. *Chaos Soliton Fract* 30:709–718
12. Zhang H, Ma X-K, Liu W-Z (2004) Synchronization of chaotic systems with parametric uncertainty using active sliding mode control. *Chaos Soliton Fract* 21:1249–1257
13. Tavazoei MS, Haeri M (2007) Determination of active sliding mode controller parameters in synchronizing different chaotic systems. *Chaos Soliton Fract* 32:583–591
14. Yau, H-T (2004) Design of adaptive sliding mode controller for chaos synchronization with uncertainties. *Chaos Soliton Fract* 22:341–347
15. Wang C, Ge SS (2001) Adaptive synchronization of uncertain chaotic systems via backstepping design. *Chaos Soliton Fract* 12:199–206

16. Ahmed E, El-Sayed AMA, El-Saka HAA (2007) Equilibrium points, stability and numerical solutions of fractional order predator-prey and rabies models. *J Math Anal Appl* 325(1): 542–553
17. Matignon D (1996) Stability results for fractional differential equations with applications to control processing. *Comput Eng Syst Appl, IEEE-SMC* 2:963–968
18. Fallahi K, Raoufi R, Khoshbin H (2008) An application of Chen system for secure chaotic communication based on extended Kalman filter and multi-shift cipher algorithm. *Comm Nonlinear Sci Num Sim* 13(4):763–781

Index

A

Abdous, F., 511–518
Abdul-Gader Jafar, M.M., 49
Adams, J.L., 257–262
Adams, M.J., 334
Adams, R.A., 216
Adolfsson, K., 277
Adomian, G., 172
Agarwal, R.P., 306
Agrawal, O.M., 171, 174
Agrawal, O.P., 277
Agrawal, P., 193, 194, 196
Ahmad, M.H., 445, 495
Ahmad, W., 213
Ahmad, W.M., 314, 511
Ahmed, E., 512, 513
Ahn, H.-S., 222, 229, 393–408, 468, 487
Aksoy, H., 334
Aktas, A.H., 334
Alam, K., 11
Al-Assaf, Y., 445, 479, 495
Albert, R., 243
Alduino, A., 26, 33–35
Aleksendric, M., 154
Allan, D., 27, 30
Allan, G., 61, 63
Alonso, J.A., 31
Altinöz, S., 333–339
Andavan, G.T.S., 10, 18
Anishchenko, V.S., 446
Antsaklis, P.J., 151
Aoun, M., 277
Apkarian, P., 75
Aquino, C.F., 81
Arango, A.C., 57
Arbuzov, A.A., 50
Arena, P., 266, 277, 314, 511
Arias, T., 27, 30
Arkun, Y., 378

Ashwell, G.J., 13
Astakhov, V.V., 446
Aström, K.J., 296, 419, 472
Atamaniuk, B., 221
Atcitty, S., 98
Atherton, D.P., 488
Atkinson, E.N., 57
Atwater, H.A., 34, 57
Aubin, J.-P., 238, 239
Auersperg, N., 57
Aviram, A., 8
Avouris, P., 27, 29, 30, 37
Axtell, M., 151, 357

B

Baaske, K., 22
Bagley, R.L., 171, 221, 363
Bagri, K., 334
Bahadir, O., 334
Balakrihsnan, V., 420
Balberg, I., 58, 63–65
Balda, J.C., 257
Baleanu, D., 193, 194, 196, 321–331, 334
Balenovic, M., 81
Balsera, I.T., 369–376
Baltog, I., 60, 62, 64
Balzani, V., 3
Bao, G., 18
Barabási, A.L., 243
Barbosa, R.S., 213, 277, 297
Bar-Cohen, Y., 263
Barnes, P.J., 423
Barnes, W., 34
Bartlett, A.C., 488
Bartosiewicz, Z., 305, 308, 310
Bashir, R., 3
Bast, R., 57
Battaglia, J.-L., 71
Baughman, R.H., 3

Beck, C., 156
 Beebe, K.R., 334
 Belhachemi, F., 257, 258
 Belli, R., 36, 37
 Belmas, R., 284
 Benson, D.A., 171
 Berggren, P., 81
 Berlinches, R.H., 411–417
 Berthon, A., 258
 Bettayeb, M., 151–160
 Beyer, H., 164, 165
 Bhaktha, S., 36, 37
 Bhattacharyya, S.P., 488, 489
 Bhowmik, P., 12
 Bianco, A., 4
 Biannic, J.-M., 75
 Bise, E.M., 151
 Bise, M.E., 357
 Blanchin, M.-G., 60, 61, 64
 Blazejczyk, O.B., 446
 Blu, T., 322
 Bode, H.W., 296, 299
 Bodson, M., 357
 Bogacki, P., 168
 Bohannan, G.W., 87
 Bohner, M., 306
 Bollen, M.H.J., 97
 Bolouki, S., 213–219
 Bonilla, B., 124
 Bonomo, C., 264, 266
 Bordóns, C., 369, 370
 Born, M., 36
 Bosch, R., 81
 Bossanyi, E., 97
 Boudjehem, B., 295–303
 Boudjehem, D., 295–303
 Bourianoff, G.I., 57
 Boustani, I., 30, 31
 Bouton, E., 284
 Bowong, S.b., 446, 448
 Bo, X., 39
 Boyd, S., 420
 Bradley, K., 4, 22
 Brewer, M., 57
 Brindley, J., 446
 Brin, I.A., 487
 Browning, R.V., 273, 274
 Bruque, N., 10, 11
 Budak, G., 17–23
 Buller, S., 258
 Bunkin, A.F., 49, 50
 Burton, T., 97

C

Cahela, D.R., 258
 Calderón, A.J., 151, 213, 295, 299, 464, 466, 480
 Calico, R.A., 171
 Camacho, E.F., 369, 370, 412, 415
 Cang, J., 495
 Cao, B.G., 295
 Cao, J., 295
 Cao, L.S., 3, 296, 301
 Cao, Y., 357, 360–362, 364, 367
 Caponetto, R., 71, 87–95, 263–271, 511
 Caputo, M., 172
 Carlen, M., 98
 Carletta, J.E., 257–262
 Carlson, G.E., 274
 Carmona Ph., 281
 Cartoixa, X., 17
 Casartelli, G.B., 81
 Cassell, A.M., 4
 Castillo, F., 97–104
 Çelebier, M., 333–339
 Çelik, V., 313–319
 Chakrabarti, R., 3, 10
 Chang, T.N., 343
 Chang, W.-D., 447, 448, 497–499
 Chapellat, H., 488
 Chapline, M.G., 4
 Charef, A., 315, 394, 464, 472
 Chatterjee, A., 277
 Chavez, J., 292
 Chen, C.-L., 479, 495, 511
 Chen, G.R., 243, 249, 445, 446, 480, 495, 511
 Cheng, Y.C., 165, 487
 Chen, H.K., 480, 481, 499, 511
 Chen, J., 10
 Chen, J.H., 480, 511
 Chen, L., 171
 Chen, N., 453–462
 Chen, Q., 28, 29
 Chen, W.C., 480, 511
 Chen, Y.Q., 71, 222, 229, 277, 357–367, 369, 393–408, 453–462, 464, 468, 472, 487
 Cheung, M., 18
 Chiang, I.W., 18
 Chiappini, A., 36, 37
 Chiasera, A., 36, 37
 Chikrii, A.A., 233–240
 Chiş, O., 193–200
 Chitlange, S.S., 334
 Chiu, M.S., 386, 391
 Cho, Y.H., 455, 458
 Chua, L.O., 446

Chutinan, A., 39
 Ciuparu, D., 30
 Ciurea, M.L., 57–66
 Clarke, D.W., 370, 371, 373, 412, 415
 Clarke, F., 239
 Clementi, E., 22
 Coimbra, C.F.M., 222
 Cois, O., 71, 151, 430
 Colbert, D.T., 12
 Collins, J.J., 319
 Connolly, J.A., 172
 Conway, L., 27
 Costa, J.S., 295
 Coutin L., 281
 Crausaz, A., 258
 Credi, A., 3
 Crooks, R.M., 18
 Cui, D.X., 18
 Cui, Y., 3
 Curran, P.F., 446
 Curtain, R.F., 164
 Cyr, P.W., 57
 Czolczynski, K., 446

D

Da Costa, J.S., 487
 Daftardar-Gejji, V., 172, 214
 Dai, H., 12
 Dai, H.J., 4, 18
 D'Andrea-Novet, B., 71, 151
 da Silva, A.J.R., 18
 Das, S., 287
 Daubechies, I., 322
 Dauphin-Tanguy, G., 107
 Davat, B., 257, 258
 Davis, J., 28, 29
 Debeljkovic, D.Lj., 154
 Debnath, L., 151
 De Doncker, R.W., 258
 de Heer, W.A., 3
 De Keyser, R., 419–426
 Dekker, C., 3
 Delavari, H., 479–485
 Delerue, C., 61, 63
 Delmaire-Sizes, F., 284
 del Toro García, X., 97–104
 De Luca, C.J., 319
 de Madrid, A.P., 369–376, 412
 Demir, Y., 313–319
 Deng, W.H., 171, 172, 174, 215, 199
 Depollier, C., 221, 222
 Derbel, N., 343–355
 Dereux, A., 34

Despi, I., 193–200
 Diethelm, D., 172
 Diethelm, K., 171, 173, 193, 199, 277
 Dinç, E., 322–331, 333–339
 Dixon, J.W., 257
 Djennoune, S., 151–160
 Do, K.M., 258
 Dongola, G., 87–95, 263–271
 Dorčák, L., 87, 151, 164, 287, 295, 464
 Dorcak, V., 98
 Doris, B., 26, 28, 29
 Dormido, S., 420
 Doyle, J., 164
 Dresselhaus, G., 29–31
 Dresselhaus, M.S., 29–31
 Drexler, K., 25
 Duan, C.K., 446
 Duan, X.F., 3
 Dubois, D., 222, 223
 Dubonos, S.V., 28, 31
 Dullerud, G., 156
 Dwyer, C., 18
 Dyakonov, V.P., 44
 Dzieliński, A., 98, 99, 152–154, 158, 287–294, 306, 307

E

Ebbesen, T., 34
 Edelman, M., 148
 Efe, M.Ö., 463–469, 471–478
 Eichhorn, F., 63
 Eidelman, S.D., 235, 237
 Eklund, P., 29, 30
 Ekstam, L., 99
 El-Khazali, R., 213, 445, 479, 495
 El-Saka, H.A.A., 512, 513
 El-Sayed, A.M.A., 512, 513
 Elsyasberg, P.E., 50
 El-Wakil, A.S., 213
 Emelyanova, N.A., 44
 Emelyanov, S.V., 464
 Enelund, M., 277
 Ensslin, K., 57
 Eslami, E., 223
 Evans, A.K., 445, 495
 Exton, H., 209, 210

F

Fagan, S.B., 18
 Fallahi, K., 515, 516
 Fang, J.Q., 446
 Fara, L., 59

Faranda, R., 258
 Farokhi, F, 171–178
 Fehske, H., 32
 Felieu, V., 295
 Feliu-Batlle, V., 464, 480
 Feliu, S., 98
 Feliu, V., 71, 97–104, 287, 295, 369, 454, 464, 480, 487
 Fellah, M., 221
 Fellah, Z.E.A., 221, 222
 Fernando, K.A.S., 4
 Ferrari, M., 25–40
 Ferreira, I.M., 277, 297
 Fert, A., 57
 Firsov, A.A., 28, 31
 Fischer, J.E., 12
 Fliess, M., 151
 Follen, M., 57
 Ford, N.J., 171–174, 277
 Fortuna, L., 71, 87–95, 263–271, 277, 314, 511
 Francis, B., 164
 Frankowska, H., 238, 239
 Freed, A.D., 171, 173, 277
 Freudenberg, J.S., 83
 Fujita, M., 31
 Fu, L., 3

G

Gabriel, J.C.P., 4, 22
 Gaburro, Z., 57
 Gafarov, F., 44
 Gafarov, F.M., 44
 Gäfvert, M., 419
 Gahinet, P., 75
 Gallay, R., 258
 Gallina, M., 258
 Gallo, A., 87–95, 263–271
 Gao, H.J., 18
 Gao, W., 464
 Garcia, G., 158
 Garcia Iturricha, A., 71
 Garcia, M., 292
 Gasparian, O., 420
 Geckeler, K.E., 19, 21
 Geering, H.P., 81
 Gehlot, K.S., 203–212
 Geim, A.K., 27, 28, 31
 Ge, S.S., 446, 496, 512
 Ge, Z.M., 480, 495, 511
 Ghaderi, R., 479–485, 495–502, 511–518
 Giannone, P., 264, 266
 Gilli, M., 314, 316, 317
 Glaeske, H.J., 203, 206, 211, 212

Glockle, W.G., 487
 Gnedenko, B.V., 125
 Goetschel, R. Jr., 225
 Göhnermeier, A., 28
 Gorenflo, R., 229, 277, 363, 365
 Goyal, S.P., 207, 209, 210
 Graybill, F., 378
 Graziani, S., 263–271
 Grigorieva, I.V., 28, 31
 Grosu, I., 244
 Gruau, C., 273–284
 Gruich, R., 258
 Gruner, G., 4
 Gualous, H., 258
 Guangming, X., 152, 154
 Guanrong, C., 223
 Guan, Z.H., 446
 Guermah, S., 151–160
 Guglielmi, M., 281
 Gu, H.-Y., 22
 Gu, L.R., 4
 Guo, K.H., 453, 455, 458
 Guo, Z.J., 18
 Gupta, K.C., 207, 209, 210
 Gupta, M.M., 223
 Gustavsson, S., 57
 Güttinger, J., 57
 Güvenċ, Z.B., 17–23

H

Haaland, D.M., 334
 Haddon, R.C., 12
 Hadid, S., 471
 Haeri, M., 171–178, 213–219, 479, 480, 496, 511–514
 Häggglund, T., 296
 Häglund, H., 419
 Hajek, O., 237
 Halijak, C.A., 274
 Hamon, M.A., 12
 Hampel, R., 223
 Hang, C.C., 296, 301
 Hanggi, P., 44
 Han, T.R., 4
 Harrison, P., 59
 Hartley, T.T., 108, 113, 171, 213, 257–262, 314, 315, 430, 464, 511
 Heitmann, J., 63
 He, J.H., 495
 Hermann, R., 310
 Hernandez, E., 30, 31
 Hernández, R., 369, 371, 412
 Hey, A., 28

Hibino, Y., 26
 Hicks, J., 258
 Hilfer, R., 365, 445
 Hollot, C.V., 488
 Holmberg, U., 419
 Hong, Y., 446
 Hortelano, M.R., 369–376
 Hory, Y., 277
 Hosseinnia, S.H., 495–502, 511–518
 Hotzel, R., 151
 Hou, E., 343
 Howe, D., 257
 Ho, W.K., 296, 301
 Hsu, M.Y., 480, 511
 Huang, J.L., 4
 Huang, W.J., 18
 Huang, Y., 3
 Hu, C.G., 18
 Hu, H.Y., 12, 458
 Hung, J.C., 464
 Hung, J.Y., 464
 Hurmuzlu, Y., 468
 Husain, I., 258
 Hwang, C., 165, 296, 487

I

Iancu, V., 57–66
 Ichise, M., 171, 274
 Jeong, M., 26, 28, 29
 Ifeachor, E., 181
 Ihn, T., 57
 Ilchmann, A., 357, 359, 364
 Ioannou, P.A., 364
 Ioele, G., 334
 Ioffe, A.D., 238
 Ionescu, C., 419–426
 Ismail-Beigi, S., 30
 Itkis, M.E., 12
 Ito, T., 18, 27
 Ivan, G.H., 193, 194
 Ivan, M., 193, 194

J

Jackson, E.A., 244
 Jafari, H., 172, 214, 479, 495
 Jallouli-Khlif, R., 343–356
 James, D.K., 10
 James, J.B., 223
 Jamshidi, M., 81
 Jemni, A., 287
 Jenkins, N., 97
 Jervis, B.W., 181

Jestin, Y., 36, 37
 Jguchit, S.B., 453, 455
 Jiang, D., 28, 31
 Jimenez, D., 17
 Jing, X.Y., 3–13
 Jinno, M., 26
 Joannopoulos, J., 27, 30, 37–39
 Johansson, M., 419
 John, S., 38, 39
 Johnson, S., 27, 37–39
 Johri, V., 18
 Jordens, C., 22
 Jory, M.J., 13
 Jo, Y.S., 22
 Junsuke, B., 453, 455

K

Kaczorek, T., 306, 504
 Kaiser, W., 28
 Kakmeni, M., 446, 448
 Kanbur, M., 321–329, 334
 Kang, Y., 480, 511
 Kapitaniak, T., 446
 Kaptouom, E., 446, 448
 Karden, E., 258
 Karimi-Ghartemani, M., 165, 167
 Kaucher, E., 225
 Kaufman, A., 223
 Kawai, H., 22
 Kaya, D., 172
 Kaynak, M.S., 334
 Kaynak, O., 472
 Kedzierski, J., 26, 28, 29
 Keel, L.H., 488
 Kelly, K.F., 18
 Kempfle, S., 164, 165
 Kendall, M.G., 50
 Ke, Y.G., 18
 Khalil, H.K., 78
 Kharitonov, V.L., 488
 Khemani, F., 107–120, 429–443
 Khoshbin, H., 515, 516
 Kilbas, A.A., 124, 126, 151, 152, 181, 233, 237, 275
 Kim, B.K., 17
 Kim, D.S., 19, 21
 Kim, J.J., 17, 455, 458
 Kim, K.H., 3
 Kim, K.J., 263, 264
 Kim, P., 4, 27, 28, 31
 Kim, S.G., 12
 Klafter, J., 125, 126
 Klamka, J., 503–509

Kleine-Ostmann, T., 22
 Klibanov, A.M., 3, 10
 Klie, R.F., 30
 Klir, G.J., 223
 Kobayashi, M.H., 222
 Koeller, R.C., 171
 Kojima, T., 171, 274
 Kok, D., 258
 Kolésnik, R., 222
 Kolmogorov, A.N., 125
 Kong, J., 4
 Kong, Y., 18
 Korean, Y.J., 20
 Kosiński, W., 221–229
 Kostial, I., 151, 164, 295, 464
 Kötz, R., 98
 Kovalev, D., 63
 Kovell, L.C., 3
 Kowalski, B.R., 334
 Kramer, R., 334
 Krasovskii, N.N., 237
 Krener, A.J., 310
 Krivonos, Yu. G., 237
 Kumar, M.J., 13
 Kumar, P., 171, 174, 277
 Kumar, R., 206
 Kunishi, W., 39
 Kunstmann, J., 30–32
 Kurths, J., 243

L

Ladaci, S., 464, 472
 Ladik, J., 22
 Lake, R., 3–5, 10
 Lake, R.K., 11
 Lall, S., 156
 Langer, R., 57
 Lannoo, M., 61, 63
 Lanusse, P., 71, 72, 171, 377–391, 454, 464
 Larsson, S., 277
 Latteux, P., 487
 Lauhon, L.J., 3
 Lauriksa, W., 222
 LaVan, D.A., 57
 Lazanu, S., 60, 62, 64
 Lazar, C., 419–426
 Lazar, M., 60, 62, 64
 Lee, J.-H., 22
 Lee Otsuka, Y., 22
 Lee, R., 12
 Lee, T.H., 394
 Lee, Y., 22
 Lee, Y.H., 12

Lei, H., 364
 Le Méhauté, A., 108
 Lepadatu, A.-M., 66
 Leu, J.F., 296
 Levina, L., 57
 Levron, F., 71, 87, 108, 277, 313, 343, 422, 424
 Lewis, J.P., 11
 Liao, S.-J., 495
 Liao, T.L., 446
 Liao, X.X., 446
 Li, C.P., 199, 215, 243, 314, 445, 495, 511
 Lieber, C.M., 3, 4
 Li, H.P., 4, 17
 Li, J.Q., 17, 480
 Lin, B.J., 3
 Lin, H., 488
 Lin, J., 108
 Lin, K.T., 480, 511
 Lin, R., 174
 Lin, W., 364
 Lin, Y., 4
 Liu, F., 10, 18, 174
 Liu, J., 18
 Liu, W.-Z., 511
 Liu, W.-Z.C., 495
 Liu, Y., 3, 18
 Li, X.J., 57
 Li, Y., 57, 357–367
 Loannou, P.A., 364
 Loh, E.J., 386, 391
 Longchamp, R., 419
 Long, W., 152, 154
 Looze, D.P., 83
 Lorenzo, C.F., 108, 113, 171, 213, 257–262, 314, 315, 430, 464, 511
 Lubich, Ch., 171, 174
 Lu, J.G., 479, 480, 511
 Lü, J.H., 243
 Lukyanchenko, V.A., 49, 50
 Lundstrom, M., 26, 29
 Lu, W., 3

M

Ma, C., 277
 Machado, J.A.T., 213, 221, 297, 487
 Maciejowski, J.M., 369, 370, 412, 415
 Magin, R.L., 151
 Mahmoudian, M., 445–451
 Maier, S., 34
 Mainardi, F., 229
 Mainardi, R., 363, 365
 Malek-Zavarei, M., 81

- Malti, R., 107–120, 277, 281
 Manabe, S., 274, 297, 487
 Mañoso, C., 369, 371, 412
 Maraoui, S., 158
 Marichev, O.I., 124, 126, 151, 152, 181, 233, 237, 275
 Marquardt, D.W., 267
 Martinez–Morales, A.A., 3–13
 Martin, R.B., 18
 Mathieu, B., 71, 72, 74, 75, 87, 313, 422, 424
 Matichin, I.I., 236
 Matignon, D., 71, 151, 214, 215, 431, 449, 471, 512, 513
 Matychyn, I.I., 233–240
 Ma, X.-K., 495, 511
 McBride, A.C., 126, 203–205
 McDonald, S.A., 57
 McGuire, T., 57
 Mead, C., 27, 28
 Meade, R., 27, 37–39
 Meerschaert, M.M., 171
 Meindl, J., 28, 29
 Melchior, P., 343, 344
 Mellor, P.H., 257
 Merrikh-Bayat, F., 163–169
 Merveillaut, M., 273–284
 Messaoud, H., 158
 Metzger, R.M., 8, 10, 13
 Metzler, R., 125, 126
 Michel, A.N., 151
 Mickelson, E.T., 18
 Mikusiński, J., 287
 Miller, D., 33, 34
 Miller, J.R., 258, 259
 Miller, K.S., 181, 221, 234, 237, 275, 322, 365
 Miranda, E., 17
 Mitrani, D., 292
 Mitroi, M.R., 66
 Mittag-Leffler, G., 151
 Miyai, E., 39
 Miyamoto, Y., 26
 Mohan, N., 97, 102
 Mohtadi, C., 370, 371, 412, 415
 Molitor, F., 57
 Momani, S., 172, 471, 479–485, 495, 511–518
 Monje, C.A., 71, 151, 295, 299, 369, 454, 464, 487
 Montroll, E.W., 125
 Moon, F.C., 314
 Moore, K.L., 71, 277, 487
 Moreau, X., 107–120
 Moreno, J., 257
 Morman, Jr K.N., 364
 Moroz, A., 49, 50, 53
 Moser, E., 36, 37
 Mota, R., 18
 Motter, A.E., 243
 Moukam, F.M., 446, 448
 Moze, M., 71–85, 281
 Mozorov, S.V., 28, 31
 Mozyrska, D., 305, 308
 Mroccka, J., 423
 Mukhopadhyay, S., 367
 Müller, F., 63
 Müller, T., 57
 Munro, N., 488
 Myers, D., 258
- N**
- Nagayanagi, Y., 171, 274
 Nakada, K., 31
 Nakagava, N., 487
 Namio, I., 453, 455
 Nanot, F.M., 87, 313, 422, 424
 Narin, I., 321–329
 Natarj, P.S.V., 295
 Nelms, R.M., 258
 Nelson-Gruel, D., 377–391
 Nelson, S.O., 50
 Nemat-Nasser, S., 264
 Nepal, D., 19, 21
 Ng, K.K., 4
 Nguyen, H.T., 222, 223
 Niemann, J., 18
 Nigmatullin, R.R., 43–55, 108
 Nikolaev, P., 4, 12
 Nimmo, S., 445, 495
 Nise, N., 420
 Nistor, L.C., 60, 61
 Noda, S., 39
 Nonnenmacher, T.F., 487
 Norris, D., 39
 Novoselov, K., 28, 31
 Nuez, I., 98, 99
- O**
- Odibat, Z., 172, 495
 Odom, T.W., 4
 Ohnishi, D., 39
 Okano, T., 39
 Okazaki, N., 13
 Okazaki, S., 27
 Oldham, K.B., 151, 152, 221, 226, 275, 288, 296, 213, 306, 313, 314, 322, 369, 411, 471
 Oldja, A., 258

O'Leary, P., 87
 Onaral, B., 315
 Onder, C.H., 81
 Oprea, D., 193–200
 Ordejon, P., 31
 Orsoni, B., 343
 Ortigueira, M.D., 464
 Ortuzar, M.E., 257
 O'Shaughnessy, L., 124
 Ostalczyk, P., 181–191
 Oton, C.J., 57
 Otto, P., 22
 Ou, C.-Y., 495
 Oustaloup, A., 71, 72, 74, 75, 87, 108, 110,
 111, 151, 171, 181, 273–284, 313,
 343–345, 351, 352, 357, 377–391,
 422, 424, 430, 441, 460, 487
 Oustaloup, L., 71
 Oustaloup, O., 71–85
 Ozbay, E., 34
 Ozdemir, A., 334
 Özdoğan, C., 32
 Özgüner, Ü., 464
 Ozkan, C.S., 3–13, 17–23
 Ozkan, M., 3–13, 17–23
 Ozturk, N., 168, 487
 Ozyetkin, M.M., 487–493

P

Pandey, R.R., 3–5, 10, 11
 Pands, S.K., 394
 Panetta, C.A., 13
 Paniccia, M., 26, 33–35
 Pao, L.Y., 343
 Parreño, A., 97–104
 Pauling, L., 30
 Pavesi, L., 57
 Payne, M., 27, 30
 Pedro, H.T., 222
 Percy, P., 26, 29
 Penchev, M., 3–13
 Peng, G., 314
 Pentia, E., 60, 62, 64
 Pen, Y., 152, 154
 Pereira, J.M.C., 222
 Perkovic, A., 81
 Pershin, S.M., 49, 50
 Peterson, A., 306
 Petit, P., 12
 Petrás, I., 87, 98, 151, 287, 295, 369, 464, 472,
 487
 Pfefferle, L., 30
 Picart, D., 284

Piguet, Y., 419
 Pi, Y., 394, 395, 397, 403
 Podlubny, I., 71, 87, 164, 181, 193, 197, 214,
 221, 226, 227, 229, 234, 275, 277,
 288, 295, 306, 313, 322, 357, 358,
 369, 371, 411, 414, 445, 454, 468,
 471, 472, 487
 Podlubny, L., 98, 101
 Poinot, T., 108, 287
 Polak, A., 423
 Pommier, V., 71
 Porto, D., 71, 87, 266, 277, 314, 511
 Post, E.L., 124
 Poty, A., 343, 344
 Poullain, S., 487
 Prade, H.H., 222, 223
 Prato, M., 4
 Prokopowicz, P., 222–225
 Puissegur, L., 71
 Puta, M., 193, 196, 198

Q

Qammer, H.K., 171, 213, 314, 315, 511
 Quandt, A., 25–40
 Quate, C.F., 4
 Quintana, J., 98, 99
 Qu, L.W., 4, 18

R

Radwan, A.G., 213
 Rael, S., 257, 258
 Rafik, R., 258
 Ragno, G., 334
 Ramos, A., 98, 99
 Ranjbar, A., 479–485, 495–502, 511–518
 Raoufi, R., 515, 516
 Rappoport, I.S., 237
 Ratner, M.A., 8
 Rawlett, A.M., 10
 Raychowdhury, A., 17
 Raynaud, H.F., 151
 Reed, M.A., 10
 Reeves, D., 378
 Ren, Y., 446
 Righini, G.C., 25–40
 Rim, K., 26, 28, 29
 Rinzler, A.G., 3, 12, 18
 Rivero, M., 123–149
 Robbins, W.P., 97, 102
 Robert, J., 12
 Roduner, M.R., 81
 Roh, Y., 22

Romero, M., 371–376, 411–423
 Roncero-Sánchez, P., 97–104
 Rosenstein, M.T., 319
 Ross, B., 181, 221, 234, 237, 275, 322
 Rossiter, J.M., 369, 370, 412, 415
 Rothmund, P.W.K., 18
 Roy, K., 17
 Rubio, A., 30, 31
 Rutman, R.S., 277

S

Sabatier, J., 71–85, 151, 171, 273–284, 430,
 454, 464, 487
 Sa da Costa, J., 277, 463
 Sadati, J., 495–502
 Sadler, P.J., 18
 Şahin, S., 334
 Saigo, M., 203, 206, 211, 212
 Sakai, K., 39
 Sakarkar, D.N., 334
 Salazar, J., 292
 Saleh, B., 26, 27, 34–37
 Sambles, J.R., 13
 Samko, S.G., 124, 126, 151, 152, 181, 233,
 237, 275, 365
 Sanos, R.S., 453, 455
 Sargent, E.H., 57
 Satterfield, W., 57
 Scammon, R.J., 273, 274
 Schneider, J.W., 3
 Schneider, V., 223
 Schnez, S., 57
 Schofield, N., 257
 Schröder, D., 364
 Schupbach, R.M., 257
 Schuster, H., 364
 Schweppe, F.C., 314
 Scuseria, G.E., 12
 Sebaa, N., 222
 Seering, W.P., 343
 Serrier, P., 107–120
 Seybold, H.J., 365
 Shabunin, A.V., 446
 Shahinpoor, M., 263, 264
 Shampine, L.F., 168
 Shankar, S., 357
 Shan, X., 446
 Sharpe, D., 97
 Shawagfeh, N., 172
 Sheu, L.J., 480, 511
 Shields, A., 57
 Shunji Manabe, 454, 459
 Siami, M., 213–219

Sierociuk, D., 98, 99, 152–154, 158, 287–294,
 306, 307
 Silva-Ortigoza, R., 102
 Simons, C.A., 81
 Simpson, A.C., 171, 172, 174
 Simpson, J.O., 263
 Singer, N.C., 343
 Singh, K.V., 3–13
 Singhose, W., 343
 Sira-Ramírez, H., 102, 464, 480
 Ślęzak, D., 222–225
 Smalley, R.E., 3, 12
 Smith, G., 49, 50, 53
 Smith, J., 263
 Sneddon, I. N., 126
 Soh, H.T., 4
 So, H.M., 17
 Soliman, A.M., 213
 Son, D.T., 258
 Sorimachi, K., 487
 Soylemez, M.T., 488
 Spainer, J., 313, 314
 Spanier, J., 151, 152, 221, 226, 275, 288, 296,
 306, 322, 369, 411, 471
 Sprott, J.C., 314, 511
 Srivastava, H.M., 126, 151, 152, 207, 209, 210
 Stampfer, C., 57
 Stanislavsky, A.A., 148
 Star, A., 4, 18, 22
 Stavarache, I., 61û63
 Strazzeri, S., 264, 266
 Strogatz, S.H., 243
 Stuart, A., 50
 Sturm, J., 39
 Subbotin, A.I., 237
 Sun, H.H., 315
 Sun, L., 18
 Sun, W.G., 243
 Sun, Y.P., 4
 Suykens, J.A.K., 446
 Sze, S.M., 4

T

Tabata, T., 22
 Tagmatarchis, N., 4
 Tam, L.M., 480, 511
 Tanaka, K-H., 22
 Tang, H., 30
 Tan, N., 487–493
 Tannenbaum, A., 164
 Tan, Y., 495

Tarasov, V.E., 193
 Tas, A., 334
 Tasis, D., 4
 Tatarchuk, B.J., 258
 Tat, P.T., 223
 Tavazoei, M.S., 171–178, 213–219, 479, 480,
 496, 511–514
 Taylor, S., 4
 Tchawoua, C., 446, 448
 Tebbikh, H., 295–303
 Teich, M., 26, 27, 34–37
 Tenreiro Machado, J.A., 277, 460
 Teodorescu, V.S., 58, 60–65
 Terpak, J., 464
 Teter, M., 27, 30
 Tharewal, S., 295
 Thess, A., 12
 Thomas, E.V., 334
 Thomas, J.L., 487
 Tikhomirov, V.M., 238
 Timashev, S.F., 44
 Tomanek, D., 12
 Tome, J., 292
 Torvik, P.J., 221, 363
 Tosello, C., 36, 37
 Tosun, A., 334
 Totzeck, M., 28
 Tour, J.M., 10
 Townley, S., 364
 Trigeassou, J.C., 108, 287
 Trujillo, J.J., 123–149, 151, 152
 Trumel, H., 273–284
 Tsallis, C., 50
 Tsang, S.C., 18
 Tsao, Y.Y., 315
 Tsay, S.Y., 296
 Tu, E., 18
 Tuffs, P.S., 370, 371, 412, 415
 Turin, R.C.E., 81
 Turo, A., 292
 Turska, E., 221
 Turski, A.J., 221

U

Ulrich, W., 28
 Undeland, T.M., 97, 102
 Unser, M., 322
 Uraz, A., 168, 487
 Üstündag, Ö., 322, 334
 Utkin, V.I., 464
 Utzinger, U., 57

V

Valério, D., 103, 277, 295, 463
 Valrio, D., 487
 Vandewalle, J., 446
 Vanloan, C.F., 171, 172
 Vardulkis, A., 382
 Veillette, R.J., 257–262
 Velasco, M.P., 123–149
 Venema, L.C., 3
 Venturi, M., 3
 Vernille, J.P., 3
 Vichnevetsky, R., 82
 Vinagre, B.M., 71, 87, 98, 151, 213, 287, 295,
 299, 369–376, 412, 454, 464, 466,
 472, 480, 487
 Vlasov, Y., 39
 Voxman, W., 225
 Vyas, D.N., 203–212

W

Wagenknecht, M., 223
 Wagner, D.A., 364
 Wahi, P., 277
 Wajdi, A.M., 445, 479, 480, 495
 Walczak, B., 322
 Walters, R.J., 57
 Wang, C., 446, 496, 512
 Wang, H., 11
 Wang, K.L., 3–5, 10
 Wang, W., 4
 Wang, X.F., 3–5, 10, 17–23, 243, 249
 Wang, Y., 257–262, 446
 Wankhede, S.B., 334
 Watts, D.J., 243
 Wazwaz, A.M., 172
 Weiss, G.H., 125
 Wen, X., 446
 Westerlund, S., 99, 487
 Westermaier, C., 364
 Wheatcraft, S.W., 171
 Wiberg, D.M., 378
 Wildoer, J.W.G., 3
 Winn, J., 27, 37–39
 Wittenmark, B., 419, 472
 Wolf, E., 36
 Woodcroft, B., 25, 40
 Wu, C.W., 446
 Wu, X., 480
 Wu, Z., 458

X

Xu, C., 12
 Xu, D.L., 243

Xue, D., 222, 229, 464
Xu, H., 495
Xu, J.-X., 394

Y

Yablonovitch, E., 38
Yang, B., 364
Yang, M., 26, 28, 29
Yang, T., 446
Yang, X.S., 446
Yan, J.-J., 447, 448, 497–499
Yau, H.-T., 479, 495, 496, 499, 511, 512
Yazdanpanah, V.R., 10
Yeroglu, C., 487–493
Yilmaz, C., 468
Yi, L.X., 63
Yin, X., 446
Yi-Yong, N., 154
Yoo, J-O., 22
Yoon, T.W., 373
Young, K.D., 464
Yulmetyeva, D.G., 44
Yulmetyev, R.M., 44
Yu, X.H., 243

Z

Zacharias, M., 63
Zadeh, L.A., 223
Zakhidov, A.A., 3
Zaslavsky, G.M., 148
Zemanian, A.H., 203, 204
Zergainoh, A., 151
Zhang, H., 495, 511
Zhang, J., 364
Zhang, J.P., 18
Zhang, Q.L., 17, 154
Zhang, Y., 28, 31
Zhang, Y.H., 57
Zhang, Y.Y., 18
Zhao, B., 12
Zhao, C.N., 394, 408
Zhao, D., 171
Zhong, G.Q., 446
Zhou, B., 4
Zhou, C.S., 243
Zhu, D., 3
Zhu, Y., 30
Zwart, H.J., 164